Math 2270 § 2. **Third Midterm Part 1** Name: Solutions
Treibergs *at* March 17, 2021

1.	/17
2.	/17
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5.	/18
6.	/18
Total	/70

This is an open book test. You may use your text and notes. Do not use a a calculator or consult with another person or use the internet. Write clearly, justify your answers and show your work to receive credit. PDF answer files must be upload to canvas within the time provided. There are [34] total points for Part 1 of the exam. When you complete Part 1 you may go on to Part 2. **DO ONLY TWO PROBLEMS FROM PART ONE 1,2,3 AND ONLY TWO PROBLEMS FROM PART TWO 4,5,6.**

1. *Find x_3 using Cramer's Rule. Other methods will receive zero credit.*

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= -2 \\ 2x_1 + 3x_2 + 4x_3 &= -1 \\ 3x_1 \qquad\quad + 5x_3 &= \ \ 8 \end{aligned}$$

The equation is equivalent to $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 0 & 5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -2 \\ -1 \\ 8 \end{bmatrix}$$

Cramer's rule for x_3 is

$$x_3 = \frac{\det(A_3(\mathbf{b}))}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & -2 \\ 2 & 3 & -1 \\ 3 & 0 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 0 & 5 \end{vmatrix}} = \frac{24 - 6 + 0 - 0 - 32 + 18}{15 + 24 + 0 - 0 - 20 - 27} = \frac{4}{-8} = -\frac{1}{2}.$$

2. (a) *Find* $\operatorname{rank}(A)$ *and bases for* $\operatorname{Nul} A$ *and* $\operatorname{Col} A$ *where*

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 6 & 6 & 6 \\ 0 & 2 & 1 & 1 \\ 1 & 4 & 3 & 3 \end{bmatrix}$$

Reduce the matrix.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 6 & 6 & 6 \\ 0 & 2 & 1 & 1 \\ 1 & 4 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There first two columns are the pivots so $\operatorname{rank}(A) = 2$. The pivot columns give a basis for the column space.

$$\text{A basis for } \operatorname{Col} A \text{ is } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \\ 2 \\ 4 \end{bmatrix} \right\}.$$

x_3 and x_4 are free variables and can take any values. Hence $x_2 = -\frac{1}{2}x_3 - \frac{1}{2}x_4$ and $x_1 = -2x_2 - 2x_3 - 2x_4 = -x_3 - x_4$. The null space is the space of solutions of $A\mathbf{x} = \mathbf{0}$ and $\mathcal{B}$ is its basis.

$$\operatorname{Nul} A = \left\{ \begin{bmatrix} -x_3 - x_4 \\ -\frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} : x_3, x_4 \in \mathbf{R} \right\} = \operatorname{span}(\mathcal{B}), \quad \mathcal{B} = \left\{ \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) *Let $P = \{s\mathbf{a} + t\mathbf{b} : 0 \le s, t \le 1\}$. What is the area of the image $T(P)$, where*

$$\mathbf{a} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}, \qquad T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The transformation $T(\mathbf{x}) = M\mathbf{x}$ multiplies the area of P by the determinant of the transformation. The area of the parallelogram is the determinant of the matrix with generators as columns. Thus

$$\operatorname{Area}(T(P)) = |\det M| \operatorname{Area}(P) = \left|\det\left(\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\right)\right| \left|\det\left(\begin{bmatrix} 5 & 8 \\ 7 & 6 \end{bmatrix}\right)\right| = 2 \cdot 26 = 52.$$

3. (a) *Find the determinant in two different ways.*

$$\Delta = \begin{vmatrix} 2 & 2 & 4 & 4 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 2 \\ 1 & 3 & 1 & 0 \end{vmatrix}$$

First way is to expand on the second row (or fourth column).

$$\Delta = +1 \cdot \begin{vmatrix} 2 & 4 & 4 \\ 1 & 3 & 2 \\ 1 & 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 3 & 0 \end{vmatrix} = (8 + 4 - 4 - 12) - (4 + 12 - 12 - 4) = -4$$

The second way is by row operations and multiplying the diagonals of an upper triangular matrix.

$$\begin{vmatrix} 2 & 2 & 4 & 4 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 2 \\ 1 & 3 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 2 & 4 & 4 \\ 1 & 3 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -4 & -2 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = -(1)(1)(-2)(-2) = -4$$

(b) *Suppose that row replacement operations reduce the matrix A to the matrix A_2 where $a_{11} \ne 0$ and $b_{22} \ne 0$. Find the LU decomposition of A.*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix} = A_2$$

We complete the reduction to find the $R = U$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix} \to \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} - \dfrac{b_{32}b_{22}}{b_{22}} \end{bmatrix} = R = U.$$

The L is found from quotients of the first column of A and the second column of A_2.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{a_{21}}{a_{11}} & 1 & 0 \\ \dfrac{a_{21}}{a_{11}} & \dfrac{b_{32}}{b_{22}} & 1 \end{bmatrix}$$

4. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \qquad \mathbf{y} = \begin{bmatrix} 5 \\ 5 \\ 4 \\ 7 \end{bmatrix}$$

Let $\mathcal{H} = \operatorname{span}(\mathcal{B})$. Show that $\mathcal{B}$ is a basis for $\mathcal{H}$. Show that $\mathbf{y} \in \mathcal{H}$. Find the coordinates $[\mathbf{y}]_{\mathcal{B}}$. Find a vector $\mathbf{z} \in \mathcal{H}$ whose coordinates $[\mathbf{z}]_{\mathcal{B}} = \mathbf{c}$ satisfy $c_i = 1$ for all i.

The set $\mathcal{B}$ already spans $\mathcal{H}$ by its very definition. $\mathcal{B}$ is a basis if its vectors are linearly independent. Let B be the matrix whose columns are the vectors of $\mathcal{B}$. The columns are independent if the REF of B has no free variables. If $\mathbf{y}$ is in $\mathcal{H}$, it is linear combination of the columns of B. In other words, there is $\mathbf{c} \in \mathbf{R}^3$ such that $B\mathbf{c} = \mathbf{y}$ is consistent. Operating on the augmented system $[B|\mathbf{y}]$,

$$\begin{bmatrix} 2 & 1 & 0 & 5 \\ 2 & 1 & 0 & 5 \\ 2 & 0 & 1 & 4 \\ 3 & 0 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & -\frac{3}{2} & 1 & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The B part has three pivots and there are no free variables. Thus $\mathcal{B}$ is also linearly independent and so it is a basis. There are no zero equals nonzero rows so the system is consistent. Hence $\mathbf{y}$ is a linear combination of the vectors in $\mathcal{B}$ so $\mathbf{y} \in \mathcal{H}$. Solving for $\mathbf{c}$, we see that $c_3 = -2$, $c_2 = 1 + c_3 = -1$ and $2c_1 = 5 - c_2 = 6$ so $c_1 = 3$. Thus the coordinates satisfy $B\mathbf{c} = \mathbf{y}$, so

$$[\mathbf{y}]_{\mathcal{B}} = \mathbf{c} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

The vector whose coordinates $\mathbf{c}$ are all ones is then given by $B\mathbf{c} = \mathbf{z}$ or

$$\mathbf{z} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 4 \end{bmatrix}.$$

5. (a) *Let* $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ *be an* $n \times n$ *matrix whose columns are* $\mathbf{a}_i$. *Let*

$$T(\mathbf{x}) = \det([\mathbf{a}_1\ \cdots \mathbf{a}_{n-1}\ \mathbf{x}])$$

be the function of the vector $\mathbf{x} \in \mathbf{R}^n$ *that evaluates the determinant of* A *with* $\mathbf{x}$ *replacing the last column. Show that* $T(\mathbf{x})$ *is a linear transformation.*

We have $T : \mathbf{R}^n \to \mathbf{R}$. The column expansion is linear in $\mathbf{x}$. To see this, expanding the determinant on the last column,

$$T(\mathbf{x}) = \sum_{k=1}^{n} b_k x_k, \qquad \text{where } b_k = (-1)^{k+n} \det(A_{k,n}),$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix gotten by crossing out the ith row and jth column of A. This way we see that T is a matrix transformation where M is the $1 \times n$ matrix

$$T(\mathbf{x}) = M\mathbf{x}, \qquad \text{where} \qquad M = [b_1, b_2, \ldots, b_n].$$

Hence T is linear. To verify, we may continue to check the two conditions for linearity. For any $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ and any $c \in \mathbf{R}$ we have

$$T(\mathbf{u} + \mathbf{v}) = M(\mathbf{u} + \mathbf{v}) = M\mathbf{u} + M\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v}),$$
$$T(c\mathbf{u}) = M(c\mathbf{u}) = cM\mathbf{u} = cT(\mathbf{u}).$$

(b) *For the* $n \times n$ *consumption matrix* C, *let* $X_k = I + C + C^2 + \cdots + C^k$. *Show that* X_k *satisfies* $X_0 = I$ *and the recursion*

$$X_{k+1} = CX_k + I, \qquad \textit{for } k = 0, 1, 2, 3, \cdots$$

Compute X_0, X_1, X_2 *for* $C = \begin{bmatrix} .6 & .1 \\ .2 & .7 \end{bmatrix}$. *Determine* $\lim_{k\to\infty} X_k$.

When $k = 0$, the series is just $X_0 = I$ with no other terms. We see that the formula applied to X_k,

$$I + CX_k = I + C\left(I + C + C^2 + \cdots + C^k\right) = I + \left(C + C^2 + \cdots + C^{k+1}\right) = X_{k+1}.$$

Computing the first three terms,

$$X_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$X_1 = X_0 + C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} .6 & .1 \\ .2 & .7 \end{bmatrix} = \begin{bmatrix} 1.6 & .1 \\ .2 & 1.7 \end{bmatrix}$$
$$X_2 = X_1 + C^2 = \begin{bmatrix} 1.6 & .1 \\ .2 & 1.7 \end{bmatrix} + \begin{bmatrix} .38 & .13 \\ .26 & .51 \end{bmatrix} = \begin{bmatrix} 1.98 & .23 \\ .46 & 2.21 \end{bmatrix}$$

Since C is a consumption matrix, the infinite sum converges. The limit is given by the formula for the geometric sum

$$\lim_{k\to\infty} X_k = \sum_{k=0}^{\infty} C^k = (I - C)^{-1} = \begin{bmatrix} .4 & -.1 \\ -.2 & .3 \end{bmatrix}^{-1}$$
$$= \frac{1}{((.4)(.3) - (-.1)(-.2))} \begin{bmatrix} .3 & .1 \\ .2 & .4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}.$$

6. *Determine whether the following statements are true or false. If true, give a short explanation. If false, find matrices for which the statement fails.*

   (a) STATEMENT. *If a* 2×2 *matrix satisfies* $A^2 = I$ *then a basis for* $\mathrm{Col}(A)$ *is* $\{\binom{1}{0}, \binom{0}{1}\}$.
   TRUE. $A^2 = I$ implies that A is invertible. (Either $D = A$ is a right inverse of A or $\det(A)^2 = \det(I) = 1$ so $\det(A) \neq 0$.) Hence $\mathrm{Col}\, A = \mathbf{R}^2$ for which $\{\binom{1}{0}, \binom{0}{1}\}$ is a basis.

   (b) STATEMENT. *For vectors* $\mathbf{u}, \mathbf{v}$ *in a vector space* $\mathbb{V}$, *the equation* $\mathbf{u} + \mathbf{x} = \mathbf{v}$ *has a solution* $\mathbf{x}$.

   TRUE. The easiest way to show the existence of a vector is to give a formula for it and check that it works. Since $\mathbf{u} \in \mathbb{V}$, by Axiom V5 it has an additive inverse $-\mathbf{u}$. We claim $\mathbf{x} = (-\mathbf{u}) + \mathbf{v}$ solves the equation. A casual answer invoking the axioms would receive full credit. Here is a strict proof that this $\mathbf{x}$ satisfies the equation. Start from the left side and replace by justifiably equal expressions until you reach the right side.

$$\begin{aligned} \mathbf{u} + \mathbf{x} &= \mathbf{u} + ((-\mathbf{u}) + \mathbf{v}) && \text{Insert the claim.} \\ &= (\mathbf{u} + (-\mathbf{u})) + \mathbf{v} && \text{V3. Associativity of addition.} \\ &= \mathbf{0} + \mathbf{v} && \text{V5. Additive inverse} \\ &= \mathbf{v} + \mathbf{0} && \text{V2. Commutivity of addition} \\ &= \mathbf{v} && \text{V4. Property of zero} \end{aligned}$$

   Another line of argument is to assume that a solution exists and then deduce the formula starting from pre-adding $-\mathbf{u}$ to the equation. This argument is incomplete: it only says if a solution existed, that's what it should be. You would still have to check that the expression actually solves the equation.

   (c) STATEMENT. *Let* $\mathcal{F} = \{f : \mathbf{R} \to \mathbf{R}\}$ *be the vector space of functions of the real numbers. Then the subset* $\mathcal{H} = \{g \in \mathcal{F} : g(t) = 0$ *for all* $t \leq 0\}$ *is a subspace of* $\mathcal{F}$.

   TRUE. One checks the three conditions satisfied by a subspace. First, the zero function $z(t) = 0$ for all $t \in R$ satisfies $z(t) = 0$ for all $t \leq 0$, thus is in $\mathcal{H}$. Second, if $f, g \in \mathcal{H}$ then $f(t) = g(t) = 0$ for all $t \leq 0$. Then the sum satisfies $(f+g)(t) = f(t)+g(t) = 0+0 = 0$ for all $t \leq 0$, thus $f + g$ in $\mathcal{H}$. Third, if $f \in \mathcal{H}$ and $c \in \mathbf{R}$ then $f(t) = 0$ for all $t \leq 0$. Then the multiplication by scalar satisfies $(cf)(t) = cf(t) = c \cdot 0 = 0$ for all $t \leq 0$, thus cf in $\mathcal{H}$.