Math 2270 § 2. **Second Midterm Part 1** Name: Solutions
Treibergs $\sigma\tau$ February 24, 2021

1. *Find the inverse matrix A^{-1} where*

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

Augment the matrix by the identity and do row reductions.

$$\begin{bmatrix} 2 & 2 & 4 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 4 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & -2 & -2 & 1 & -2 & 0 \\ 0 & -1 & -2 & 0 & -2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 2 & -1 \\ 0 & -2 & -2 & 1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & -3 & 2 \\ 0 & 1 & 2 & 0 & 2 & -1 \\ 0 & 0 & 2 & 1 & 2 & -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 & 0 & -3 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & -2 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & -1 \end{bmatrix}$$

So

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -2 & 1 \\ -1 & 0 & 1 \\ \frac{1}{2} & 1 & -1 \end{bmatrix} \qquad \text{check} \qquad AA^{-1} = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -2 & 1 \\ -1 & 0 & 1 \\ \frac{1}{2} & 1 & -1 \end{bmatrix} = I.$$

2. (a) *Detemine whether each of the given matrices is invertible. Use as few calculations as possible. Justify your answer.*

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 \end{bmatrix} \sim \begin{bmatrix} 7 & 8 & 9 & 10 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

INVERTIBLE. By swapping rows, A is row equivalent to a matrix which has $n = 4$ pivots, hence is row equivalent to I.

$$B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 \\ 2 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 3 & 4 \end{bmatrix}$$

NOT INVERTIBLE. For example, the third row is double the second row. Hence row reduction yields a zero row and B is not row equivalent to the identitiy matrix.

$$C = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 4 & 5 & 2 & 7 \\ 6 & 5 & 3 & 3 \\ 8 & 0 & 4 & 0 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

NOT INVERTIBLE. For example, the first column is double the third column. Thus, this $\mathbf{x}$ satisfies $C\mathbf{x} = \mathbf{0}$ so is a nontrivial solution of the homogeneous equation.

(b) *Let A be an $n \times n$ matrix. Without quoting Theorems 7 or 8, argue carefully why if A is row equivalent to an invertible matrix B then A is invertible.*

We follow the argument of Theorem 7. To be row equivalent means that there is a sequence of row operations given by elementary matrices $E_1, \ldots, E_k$ which reduce A to B

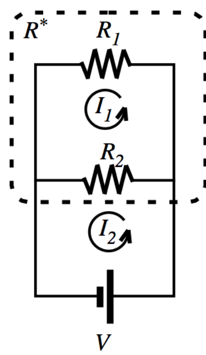
$$E_k \cdot E_{k-1} \cdots E_2 \cdot E_1 \cdot A = B.$$

By premultiplying be their inverses we find

$$A = E_1^{-1} \cdot E_2^{-1} \cdots E_{k-1}^{-1} \cdot E_k^{-1} \cdot B.$$

But all the matrices on the right are invertible because the E_i^{-1} are elementary matrices themselves. Thus A is the product of invertible matrices, thus is invertible.

3. *Write a matrix equation that determines the loop currents. Using an inverse matrix solve for the currents in terms of R_1, R_2 and V. Compute the effective resistance $R^* = V/I_2$ of the dotted box (resistors in parallel.)*



Let I_1 and I_2 be loop currents. By Kerchhoff's Loop Law, the oriented sum of voltage drop going around each loop equals the oriented sum of voltage sources. The loop equations are

$$\begin{aligned} R_1 I_1 + R_2(I_1 - I_2) &= (R_1 + R_2) I_1 - R_2 I_2 &&= 0 \\ R_2(I_2 - I_1) &= -R_2 I_1 + R_2 I_2 &&= V. \end{aligned}$$

Rewriting as a matrix equation,

$$\begin{bmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ V \end{bmatrix}$$

Using the inverse matrix we find the solution

$$\begin{aligned} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} &= \begin{bmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ V \end{bmatrix} \\ &= \frac{1}{(R_1 + R_2)R_2 - R_2{}^2} \begin{bmatrix} R_2 & R_2 \\ R_2 & R_1 + R_2 \end{bmatrix} \begin{bmatrix} 0 \\ V \end{bmatrix} \\ &= \frac{V}{R_1 R_2} \begin{bmatrix} R_2 \\ R_1 + R_2 \end{bmatrix} \end{aligned}$$

Thus the effective resistance of the dotted box is

$$R^* = \frac{V}{I_2} = \frac{R_1 R_2}{R_1 + R_2}$$

which, of course, is the resistance of a pair of resistors in parallel.

4. (a) *Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be a linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Is T one-to-one? Why or why not? Is T onto? Why or why not?*

$$A = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 4 & 2 & 0 \\ 9 & 6 & 3 & 5 \end{bmatrix}$$

Applying row operations,

$$\begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 4 & 2 & 0 \\ 9 & 6 & 3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 2 & 1 & 1 \\ 9 & 6 & 3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -6 & -3 & 1 \\ 0 & -30 & -15 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 6 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The map $T(\mathbf{x}) = A\mathbf{x}$ is not one-to-one because there is a free variable for A. It is not onto because each row does not have a pivot.

(b) *For this A, is $B = A^T A$ invertible? Explain. [Hint: Find an easy argument that doesn't involve long computations.]*

We find a nonzero solution of $A\mathbf{x} = \mathbf{0}$ by setting the free variables equal to $x_3 = 1$, $x_4 = 3$ so $x_2 = 0$ and $x_1 = -2$ or $\mathbf{x} = [-2, 0, 1, 3]^T \neq \mathbf{0}$. It follows that

$$B\mathbf{x} = A^T A\mathbf{x} = A^T\mathbf{0} = \mathbf{0}.$$

Thus B has a nontrivial solution of $B\mathbf{x} = \mathbf{0}$, so B is not invertible.

5. (a) *Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation that first reflects points through the vertical x_2 axis and then rotates points $\frac{\pi}{6}$ radians. Find the standard matrix of T. Hint:* $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

The standard matrix is determined by what the transformation does to the basic vectors, namely, $T(\mathbf{x}) = M\mathbf{x}$ where

$$M = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix}, \qquad \text{where} \qquad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let R denote the reflection and S the rotation, so $T = S \circ R$. Applying to $\mathbf{e}_1$ we see that

$$R(\mathbf{e}_1) = R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad S(R(\mathbf{e}_1)) = S\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix},$$

$$R(\mathbf{e}_2) = R\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad S(R(\mathbf{e}_2)) = S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

Hence

$$M = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

(b) *Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a one-to-one linear transformation. Suppose that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbf{R}^n$ are linearly independent. Show that $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$ and $T(\mathbf{v}_3)$ are linearly independent.*

We show that the only linear combination which equals zero is the zero combination. Suppose there are x_1, x_2 and x_3 such that

$$x_1 T(\mathbf{v}_1) + x_2 T(\mathbf{v}_2) + x_3 T(\mathbf{v}_3) = \mathbf{0}.$$

By linearity (superposition),

$$T(x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) = \mathbf{0}.$$

But T is one-to-one linear transformation so $T(\mathbf{z}) = \mathbf{0}$ implies $\mathbf{z} = \mathbf{0}$. Hence

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}.$$

But we assumed that the vectors are linearly independent, thus

$$x_1 = x_2 = x_3 = 0$$

as desired.

6. *Determine whether the following statements are true or false. If true, give a short explanation. If false, find matrices for which the statement fails.*

(a) STATEMENT. *For 2×2 nonzero matrices A, B and C, if $AB = AC$ then $B = C$.*

FALSE. The matrix A cannot be cancelled unless it is invertible. Just about any matrices will provide a counterexample. For one, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \neq C = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}, \; AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = AC.$$

(b) STATEMENT. *If the 2×2 matrix A is invertible then it has only one inverse.*

TRUE. Suppose B and C are two inverses of A. Then $AB = BA = I$ and $AC = CA = I$. Thus $AB = AC$ since both equal I. Pre-multiplying both sides by the inverse B we see from associativity of multiplication and inverse properties that

$$B = IB = (BA)B = B(AB) = B(AC) = (BA)C = IC = C.$$

Thus any two inverses are equal so the inverse is unique.

(c) STATEMENT. *For 2×2 matrices A and B, their transposes satisfy $(AB)^T = A^T B^T$.*

FALSE. The transpose of a product is the product of transposes in the opposite order. Again, we need two matrices such that $AB \neq BA$. One counterexample is given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, AB = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = (AB)^T \neq A^T B^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$