Math 2270 § 2. **First Midterm** Name: Solutions
Treibergs $\sigma\tau$ February 3, 2021

1. *Describe the solution of this system in parametric vector form. You must show all steps to receive full credit.*

$$\begin{aligned} x_2 + 2x_3 + \ x_4 + 3x_5 &= 3 \\ x_1 + 2x_2 + \ x_3 \qquad + 3x_5 &= 5 \\ 3x_1 + 4x_2 \ - x_3 - 2x_4 + 7x_5 &= 5 \end{aligned}$$

Write the augmented matrix and perform row operations.

$$\begin{pmatrix} 0 & 1 & 2 & 1 & 3 & 3 \\ 1 & 2 & 1 & 0 & 3 & 5 \\ 3 & 4 & -1 & -2 & 7 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 0 & 3 & 5 \\ 0 & 1 & 2 & 1 & 3 & 3 \\ 3 & 4 & -1 & -2 & 7 & 5 \end{pmatrix} \quad \text{Swap } R_1 \text{ and } R_2$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 0 & 3 & 5 \\ 0 & 1 & 2 & 1 & 3 & 3 \\ 0 & -2 & -4 & -2 & -2 & -10 \end{pmatrix} \quad \text{Replace } R_3 \text{ by } R_3 - 3R_1$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 0 & 3 & 5 \\ 0 & 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 & -4 \end{pmatrix} \quad \text{Replace } R_3 \text{ by } R_3 + 2R_2$$

x_3 and x_4 are free variables and can be set to any real value. Solving we find

$$\begin{aligned} &4x_5 = -4 \quad \text{so } x_5 = -1 \\ &x_2 = 3 - 2x_3 - x_4 - 3x_5 = 6 - 2x_3 - x_4 \\ &x_1 = 5 - 2x_2 - x_3 - 3x_5 = 5 - 2(6 - 2x_3 - x_4) - x_3 + 3 = -4 + 3x_3 + 2x_4. \end{aligned}$$

Thus the set of solutions is

$$\mathcal{S} = \left\{ \begin{bmatrix} -4 + 3x_3 + 2x_4 \\ 6 - 2x_3 - x_4 \\ x_3 \\ x_4 \\ -1 \end{bmatrix} : \text{where } x_3 \text{ and } x_4 \text{ are any real numbers.} \right\}$$

or in parametric vector form

$$\mathcal{S} = \left\{ \begin{bmatrix} -4 \\ 6 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} : \text{where } x_3 \text{ and } x_4 \text{ are any real numbers.} \right\}.$$

2. *Describe geometrically those* $\mathbf{c}$ *for which you can solve* $A\mathbf{x} = \mathbf{c}$. *Describe geometrically those* $\mathbf{c}$ *for which you can solve* $B\mathbf{x} = \mathbf{c}$, *where*

$$A = \begin{bmatrix} 0 & 0 & 13 \\ 0 & 12 & 11 \\ 10 & 9 & 8 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & 2 \\ 1 & 8 & 7 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

By swapping the first and last rows, we see that the matrix A is row equivalent to

$$\begin{pmatrix} 0 & 0 & 13 \\ 0 & 12 & 11 \\ 10 & 9 & 8 \end{pmatrix} \to \begin{pmatrix} 10 & 9 & 8 \\ 0 & 12 & 11 \\ 0 & 0 & 13 \end{pmatrix}$$

This matrix has a pivot in each column, so that $A\mathbf{x} = \mathbf{c}$ may be solved for every $\mathbf{c}$.

Perform row operations on the augmented matrix $(B|\mathbf{c})$.

$$\begin{pmatrix} 1 & 3 & 3 & c_1 \\ 2 & 1 & 2 & c_2 \\ 1 & 8 & 7 & c_3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 3 & c_1 \\ 0 & -5 & -4 & c_2 - 2c_2 \\ 0 & 5 & 4 & c_3 - c_1 \end{pmatrix} \begin{matrix} \\ \text{Replace } R2 \text{ by } R2 - 2R_1 \\ \text{Replace } R3 \text{ by } R3 - R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 3 & 3 & c_1 \\ 0 & -5 & -4 & c_2 - 2c_2 \\ 0 & 5 & 4 & c_3 - c_1 + (c_2 - 2c_1) \end{pmatrix} \text{ Replace } R3 \text{ by } R3 + R_2$$

A geometric condition for $\mathbf{c}$ to be able to solve $B\mathbf{x} = \mathbf{c}$is that the equations be consistent. This requires that the bottom row of the REF be zero for $\mathbf{c}$, or $c_3 - c_1 + (c_2 - 2c_1) = c_3 + c_2 - 3c_1 = 0$.

3. (a) *Let* A *be a matrix and* $\mathbf{b}$, $\mathbf{p}$ *be vectors such that* $A\mathbf{p} = \mathbf{b}$. *Show that if* $\mathbf{x}$ *is any solution of* $A\mathbf{x} = \mathbf{b}$ *then there is a vector* $\mathbf{v}$ *such that* $A\mathbf{v} = \mathbf{0}$ *and* $\mathbf{x} = \mathbf{p} + \mathbf{v}$.

   Let $\mathbf{x}$ be a solution of $A\mathbf{x} = \mathbf{b}$. Let $\mathbf{v} = \mathbf{x} - \mathbf{p}$. Then $\mathbf{x} = \mathbf{p} + \mathbf{v}$ and $A\mathbf{v} = A(\mathbf{x} - \mathbf{p}) = A\mathbf{x} - A\mathbf{p}$ by linearity of matrix multiplication. But this equals $A\mathbf{v} = \mathbf{b} - \mathbf{b} = \mathbf{0}$. Thus we have shown that every solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{x} = \mathbf{p} + \mathbf{v}$ where $A\mathbf{v} = \mathbf{0}$.

   (b) *Let* A *be a* 2×2 *matrix such that* $A \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   Solve $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$. *Find* $A \begin{bmatrix} 4 \\ -6 \end{bmatrix}$.

   First we solve $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$. The augmented matrix row reduces

$$\begin{pmatrix} 2 & 1 & 4 \\ 2 & 3 & -6 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 4 \\ 0 & 2 & -10 \end{pmatrix}$$

   so $x_2 = -5$ and $2x_1 = 4 - x_2 = 9$ so $x_1 = \frac{9}{2}$. This expresses the last vector as a linear combination

$$\begin{bmatrix} 4 \\ -6 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{9}{2} \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

   It follows by linearity of matrix multiplication that

$$A \begin{bmatrix} 4 \\ -6 \end{bmatrix} = A \left(\frac{9}{2} \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) = \frac{9}{2} A \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 5A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{9}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} \\ -5 \end{bmatrix}.$$

4. (a) *Define the* span *the set of vectors* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$. *For the given set of vectors* $\mathcal{S}$, *is* $\mathbf{R}^3 = \operatorname{span}\{\mathcal{S}\}$? *Explain. Is* $\mathbf{b}$ *in* $\operatorname{span}\mathcal{S}$? *Explain.*

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 18 \\ 1 \end{bmatrix} \right\}, \qquad \mathbf{b} = \begin{bmatrix} 13 \\ -3 \\ 7 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

The *span* of the set of vectors $\mathcal{Q} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the set of all linear combinations of these vectors. In symbols,

$$\operatorname{span}\mathcal{Q} = \{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p : \ x_1, x_2, \ldots, x_p \text{ may be any real numbers}\}$$

$\mathbf{R}^3 = \operatorname{span}\mathcal{S}$ all elements $\mathbf{c} \in \mathbf{R}^3$ are linear combinations of $\mathcal{S}$. In other words, if A is the matrix whose column vectors are from $\mathcal{S}$ then $A\mathbf{x} = \mathbf{c}$ may be solved for any $\mathbf{c} \in \mathbf{R}^3$. Forming the augmented matrix $(A|\mathbf{b}|\mathbf{c})$ and row reducing, we find

$$\begin{pmatrix} 1 & 3 & -1 & 13 & c_1 \\ 4 & 1 & 18 & -3 & c_2 \\ 1 & 2 & 1 & 7 & c_3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 & 13 & c_1 \\ 0 & -11 & 22 & -55 & c_2 - 4c_1 \\ 0 & -1 & 2 & -6 & c_3 - c_1 \end{pmatrix} \quad \begin{matrix} \text{Replace } R_2 \text{ by } R_2 - 4R_1 \\ \text{Replace } R_3 \text{ by } R_3 - R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 3 & -1 & 13 & c_1 \\ 0 & 1 & -2 & 5 & -\frac{1}{11}c_2 + \frac{4}{11}c_1 \\ 0 & -1 & 2 & -6 & c_3 - c_1 \end{pmatrix} \quad \text{Replace } R_2 \text{ by } -\tfrac{1}{11}R2$$

$$\to \begin{pmatrix} 1 & 3 & -1 & 13 & c_1 \\ 0 & 1 & -2 & 5 & -\frac{1}{11}c_2 + \frac{4}{11}c_1 \\ 0 & 0 & 0 & -1 & c_3 - \frac{1}{11}c_2 - \frac{7}{11}c_1 \end{pmatrix} \quad \text{Replace } R_3 \text{ by } R_3 + R2$$

The row echelon form of $A\mathbf{x} = \mathbf{b}$ has a "zero equals nonzero" row, so is inconsitent. $\mathbf{b}$ is not in the span of $\mathcal{S}$. To avoid this bad row in the row echelon form of $A\mathbf{x} = \mathbf{c}$, we see that the geometric condition to be able to solve the system is that the lower corner vanish, namely, $c_3 - \frac{1}{11}c_2 - \frac{7}{11}c_1 = 0$.

5. *Suppose that the row echelon form of the matrix* A *is* R. *Is the solution of* $A\mathbf{x} = 0$ *unique? Explain why or why not. Is the solution of* $A\mathbf{x} = 0$ *unique? Explain why or why not. Does the equatiomn* $B\mathbf{x} = \mathbf{c}$ *always have a unique solution? Explain why or why not. For "$*$" representing any number, the matrices are given by*

$$R = \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 2 & * & * & * \\ 0 & 0 & 0 & 0 & 3 & * \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & * & * & * \\ 0 & 2 & * & * \\ 0 & 0 & 3 & * \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ ]0 \end{bmatrix}.$$

The row echelon form of $A\mathbf{x} = \mathbf{0}$ is $R\mathbf{x} = \mathbf{0}$. Since it has x_2 and x_4 as free variables, the solution space depends on two arbitrary constants so $\mathbf{x} = \mathbf{0}$ is not the only solution. The solution is not unique.

The row echelon form of $B\mathbf{x} = \mathbf{c}$ is $R\mathbf{x} = \tilde{\mathbf{c}}$, where $\tilde{\mathbf{c}}$ is the result of applying row operations that reduce B to T to the vector $\mathbf{c}$. Since there are no free variables, the solution is unique if it exists. However, since T has zero rows, then for some bad matrices B, the row operations may introduce nonzero values in the last two tows of $\tilde{\mathbf{c}}$. Thus, no, the equatiomn $B\mathbf{x} = \mathbf{c}$ does not always have a unique solution. Its solution may not exist.

6. *The Cambridge Diet Formula is a mixture of foodstuffs that contain essential nutrients with few calories. 100g of soy flour contains 50g of protein, 40g of carbohydrate and 10g of fat. 100g of whey contains 30g of protein, 60g of carbohydrate and 10g of fat. Write a vector equation for the unknown amounts in grams of soy flour and whey to be mixed in order to get a formula with 36g of protein, 54g of carbohydrates and 10g of fat. If possible, solve for how much soy flour and whey are needed for this mixture.*

   We are given vectors of grams of protein, carbohydrates and fat per 100g of foodstuff. Their linear combination with weights x_1 and x_2, in hundreds of grams soy flour and whey, respectively, gives the nutrition vector of the mixture which is to equal the specified nutrition vector. The vector equation is

$$\begin{bmatrix} \text{g. protein} \\ \text{g. carbohydrate} \\ \text{g. fat} \end{bmatrix} = x_1 \begin{bmatrix} 50 \\ 40 \\ 10 \end{bmatrix} + x_2 \begin{bmatrix} 30 \\ 60 \\ 10 \end{bmatrix} = \begin{bmatrix} 36 \\ 54 \\ 10 \end{bmatrix}$$

   For an equation in grams, we would replace x_i by y_i, where $y_i = 100x_i$ is the number of grams. Write this equation, $A\mathbf{x} = \mathbf{b}$, as an augmented matrix which we row reduce.

$$\begin{pmatrix} 50 & 30 & 36 \\ 40 & 60 & 54 \\ 10 & 10 & 10 \end{pmatrix} \to \begin{pmatrix} 10 & 10 & 10 \\ 40 & 60 & 54 \\ 50 & 30 & 36 \end{pmatrix} \quad \text{Swap } R1 \text{ and } R3$$

$$\to \begin{pmatrix} 10 & 10 & 10 \\ 0 & 20 & 14 \\ 0 & -20 & -14 \end{pmatrix} \quad \begin{matrix} \text{Replace } R_2 \text{ by } R2 - 4R1 \\ \text{Replace } R_2 \text{ by } R2 - 5R1 \end{matrix}$$

$$\to \begin{pmatrix} 10 & 10 & 10 \\ 0 & 20 & 14 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Replace } R_3 \text{ by } R3 + R2$$

   The system of three equations in two unknowns is consistent because there is no bad row. Solving, we see that $x_2 = \frac{14}{20} = .7$ and $10x_1 = 10 - 10x_2 = 10 - 7 = 3$ so that $x_1 = .3$. Thus the grams of soy flour and whey that satisfy the vector equation are $y_1 = 100x_1 = 30$ and $y_2 = 100x_2 = 70$, respectively.