Math 2270 § 2. **Fourth Midterm Part 1** Name: Solutions
Treibergs *at* November 18, 2020

1. *Find all eigenvalues and eigenvectors. Show your work to get credit.*

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

First, find the characteristic polynomial using the $3 \times 3$ ↘↘↘↙↙↙ determinant formula

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} \begin{aligned} &= (2-\lambda)^2(1-\lambda) - (1-\lambda) = (1-\lambda)[(2-\lambda)^2 - 1] \\ &= (1-\lambda)[3 - 4\lambda + \lambda^2] = (1-\lambda)(1-\lambda)(3-\lambda) \end{aligned}$$

Thus the eigenvalues are the roots $\lambda = 1, 1, 3$. Solving for independent eivenvectors, for $\lambda_1 = 1$

$$0 = (A - \lambda_1 I)(\mathbf{v}_1, \mathbf{v}_2) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For $\lambda_2 = 3$

$$0 = (A - \lambda_2 I)\mathbf{v}_3 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus the $\lambda_1$ eigenvectors are linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ and all $\lambda_2$ eigenvectors are linear combinations of $\mathbf{v}_3$.

2. *Find a matrix that diagonalizes $A$. Show that your matrix does the job.*

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

Find the characteristic polynomial to find the eigenvalues.

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = -5 - 4\lambda + \lambda^2 = (-5+\lambda)(1+\lambda)$$

whose roots are eigenvalues $\lambda_1 = 5$, $\lambda_2 = -1$. Solving for eigenvectors,

$$0 = (A - \lambda_1 I)\mathbf{v}_1 = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad 0 = (A - \lambda_2 I)\mathbf{v}_1 = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Thus a diagonalizing matrix $P$ and diagonal matrix $D$ are

$$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

To check that $P$ diagonalizes $P^{-1}AP = D$, it is easier see that

$$AP = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 10 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = PD.$$

$=$

3. *Prove that* $\lim_{n\to\infty} A^n \mathbf{v} = \mathbf{w}$, *where*

$$A = \begin{bmatrix} .7 & .4 \\ .3 & .6 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}.$$

We find eigenvectors and express $\mathbf{v}$ as their linear combination. Then the powers are easy. Find the characteristic polynomial to find the eigenvalues.

$$\begin{vmatrix} .7-\lambda & .4 \\ .3 & .6-\lambda \end{vmatrix} = (.7-\lambda)(.6-\lambda) - .12 = .3 - 1.3\lambda + \lambda^2 = (.3-\lambda)(1-\lambda)$$

whose roots are eigenvalues $\lambda_1 = .3$, $\lambda_2 = 1$. Solving for eigenvectors,

$$0 = (A - \lambda_1 I)\mathbf{v}_1 = \begin{bmatrix} .4 & .4 \\ .3 & .3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad 0 = (A - \lambda_2 I)\mathbf{v}_1 = \begin{bmatrix} -.3 & .4 \\ .3 & -.4 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Now

$$\mathbf{v} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \mathbf{v}_1 + 2\mathbf{v}_2.$$

We apply $A^n$ and use $A^n \mathbf{v}_i = \lambda_i^n \mathbf{v}$,

$$A^n\mathbf{v} = A^n(\mathbf{v}_1 + 2\mathbf{v}_2) = A^n\mathbf{v}_1 + 2A^n\mathbf{v}_2 = (.3)^n\mathbf{v}_1 + 2(1)^n\mathbf{v}_2 \to 2\mathbf{v}_2 = \mathbf{w}$$

as $n \to \infty$.

4. *Let*

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 4 & -1 \\ 1 & -4 & 7 \\ 2 & 4 & 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

*Find a basis for* Row $A$. *Is the vector* $\mathbf{v} \in (\text{Row } A)^\perp$? *Why?*

A basis may be found by doing row operations on $A$.

$$\begin{bmatrix} 1 & 0 & 3 \\ 1 & 4 & -1 \\ 1 & -4 & 7 \\ 2 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & -4 \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for Row $A$ consists of pivot rows of REF

$$\{\mathbf{b}_1, \mathbf{b}_2\} = \{(1,0,3), (0,4,-4)\}.$$

$\mathbf{v} \in (\text{Row } A)^\perp$ if $\mathbf{v}\cdot\mathbf{w} = 0$ for all $\mathbf{w} \in \text{Row } A$. But since such $\mathbf{w} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ are linear combinations of the basis vectors, it suffices to check that $\mathbf{v}$ is perpendicular to each basis vector. Now

$$\mathbf{b}_1\cdot\mathbf{v} = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = 3 - 3 = 0, \qquad \mathbf{b}_2\cdot\mathbf{v} = \begin{bmatrix} 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = -4 + 4 = 0.$$

Hence $\mathbf{w}\cdot\mathbf{v} = (c_1\mathbf{b}_1 + c_2\mathbf{b}_2)\cdot\mathbf{v} = c_1\mathbf{b}_1\cdot\mathbf{v} + c_2\mathbf{b}_2\cdot\mathbf{v} = 0 + 0 = 0$. Thus $\mathbf{w}\cdot\mathbf{v} = 0$ for all $\mathbf{w} \in \text{Row } A$ so $\mathbf{v} \in (\text{Row } A)^\perp$.

5. *Determine whether the following statements are true or false. If true, give a short explanation. If false, find matrices for which the statement fails.*

   (a) STATEMENT. $A = \begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix}$ *is similar to* $B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$.

   TRUE. The characteristic polynomial of the matrix $A$ is

   $$\begin{vmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda)+2 = 10-6\lambda+\lambda^2 = 1+(3-\lambda)^2,$$

   whose roots are complex eigenvalues, $\lambda = 3 \mp i$. The matrix is similar to a dilation and rotation which for $\lambda = 3-i$ takes the form $B$.

   (b) STATEMENT. *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be subspaces of* $\mathbf{R}^n$ *such that* $\mathbb{V} \subset \mathbb{W}$. *Then* $\mathbb{W}^\perp \subset \mathbb{V}^\perp$.

   TRUE. We show $\mathbf{x} \in \mathbb{W}^\perp$ implies $\mathbf{x} \in \mathbb{V}^\perp$. $\mathbf{x} \in \mathbb{W}^\perp$ means $\mathbf{x} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in \mathbb{W}$. Hence $\mathbf{x} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in \mathbb{V}$ because $\mathbf{v} \in \mathbb{V} \subset \mathbb{W}$. Thus $\mathbf{x} \in \mathbb{V}^\perp$.

   (c) STATEMENT. *Let $A$ be a $2 \times 2$ matrix with an eigenvalue $\lambda$ of multiplicity two. Then the $\lambda$ eigenspace of $A$ is two dimensional.*

   FALSE. For example, the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has double eigenvalue $\lambda = 0$, but $A - \lambda I = A$ has rank one so its null space (0 eigenspace of $A$) has dimension one.

6. (a) *Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most two. The linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ that moves a polynomial one unit to the left is defined by $T[\mathbf{f}](t) = \mathbf{f}(t+1)$. Find the matrix of the transformation $T$ in the basis $\mathcal{B} = \{1, t, t^2\}$.*

   Since we have

   $$T[1](t) = 1, \quad T[t](t) = t+1, \quad T[t^2](t) = (t+1)^2 = t^2+2t+1$$

   the matrix of the transformation is

   $$M = \begin{bmatrix} \begin{bmatrix} T[1] \end{bmatrix}_{\mathcal{B}} & \begin{bmatrix} T[t] \end{bmatrix}_{\mathcal{B}} & \begin{bmatrix} T[t^2] \end{bmatrix}_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (b) *Find $\mathcal{T}(\mathbf{v})$ where $\mathcal{T} : \mathbf{R}^2 \to \mathbf{R}^2$ is the linear transformation whose matrix in the $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ basis is given by*

   $$[\mathcal{T}(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} [\mathbf{x}]_{\mathcal{B}} \text{ for all } \mathbf{x} \in \mathbf{R}^2, \text{ where } \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

   Solving for $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$,

   $$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$$

   so $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$. Thus

   $$[\mathcal{T}(\mathbf{v})]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{9}{2} \end{bmatrix}.$$

   It follows that

   $$\mathcal{T}(\mathbf{v}) = \frac{3}{2}\mathbf{b}_1 + \frac{9}{2}\mathbf{b}_2 = \frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{9}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}.$$