Math 2270 § 2. **Third Midterm Part 1** Name: Solutions
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1. *Find the inverse matrix A^{-1} using Cramer's Rule. Other methods will receive zero credit.*

$$A = \begin{bmatrix} 6 & 0 & 0 \\ 4 & 5 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

The determinant of a lower triangular matrix is the product of the diagonals.

$$\begin{vmatrix} 6 & 0 & 0 \\ 4 & 5 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 6 \cdot 5 \cdot 3 = 90.$$

Cramer's rule for the (i, j) entry of the inverse matrix is

$$(A^{-1})_{ij} = \frac{(-1)^{i+j} \det A_{ji}}{\det A}$$

where A_{ij} is the cofactor matrix, the one obtained from A by striking out the ith row and jth column. Thus

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} + \begin{vmatrix} 5 & 0 \\ 2 & 3 \end{vmatrix} & - \begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} & + \begin{vmatrix} 0 & 0 \\ 5 & 0 \end{vmatrix} \\ - \begin{vmatrix} 4 & 0 \\ 1 & 3 \end{vmatrix} & + \begin{vmatrix} 6 & 0 \\ 1 & 3 \end{vmatrix} & - \begin{vmatrix} 6 & 0 \\ 4 & 0 \end{vmatrix} \\ + \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} & - \begin{vmatrix} 6 & 0 \\ 1 & 2 \end{vmatrix} & + \begin{vmatrix} 6 & 0 \\ 4 & 5 \end{vmatrix} \end{pmatrix} = \frac{1}{90} \begin{pmatrix} 15 & 0 & 0 \\ -12 & 18 & 0 \\ 3 & -12 & 30 \end{pmatrix}.$$

2. (a) *Find the determinant*

Expand on second column.

$$\begin{vmatrix} 1 & 0 & 3 & 2 \\ 2 & 5 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{vmatrix} = +5 \begin{vmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 5 \cdot 0 - 2(2 + 3 + 4 - 1 - 6 - 4) = 4.$$

(b) *Suppose elementary row operations reduce the matrix A to R. What is the determinant $\det A$? Why?*

$$R = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Note that R is upper triangular so its determinant is the product of diagonals $\det R = 0$. If the reduction is $E_k \cdot E_{k-1} \cdots E_1 \cdot A = R$ where the E_i are elementary matrices, then by taking determinants we have

$$\det(E_k) \cdot \det(E_{k-1}) \cdots \det(E_1) \cdot \det(A) = \det(R) = 0.$$

since the determinant of a product is the product of determinants. Now since the determinants of the elementary matrices don't vanish, $\det(E_i) \neq 0$, it follows that $\det(A) = 0$.

Another answer is to notice that R has a free variable, so A is not invertible. Thus $\det A = 0$.

(c) *What is the area of the ellipse E? [Hint: think a little.]*

$$E = \{(x,y) \in \mathbf{R}^2 : (2x+3y)^2 + (-3x+y)^2 \le 3^2\}.$$

Define the matrix transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Then the circular region of radius 3

$$C = \left\{(u,v) : u^2 + v^2 \le 3^2\right\}$$

has area $\pi r^2 = 9\pi$. We have $T(E) = C$. Using the fact that the area of a matrix transformation of a domain changes by a factor determinant of the matrix

$$9\pi = \text{Area}(C) = \text{Area}(T(E)) = \left|\det\begin{pmatrix} 2 & 3 \\ -3 & 1 \end{pmatrix}\right| \text{Area}(E) = 11\,\text{Area}(E)$$

we see that

$$\text{Area}(E) = \frac{9\pi}{11}.$$

3. *Find bases for* $\text{Nul}\,A$ *and* $\text{Col}\,A$ *where*

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 & 3 \\ 3 & 6 & 6 & 6 & 3 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix}$$

Doing row operations we find

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 & 3 \\ 3 & 6 & 6 & 6 & 3 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 0 & -2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 0 & -2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Solving for $\text{Nul}\,A = \{\mathbf{x} \in \mathbf{R}^5 : A\mathbf{x} = \mathbf{0}\}$ which is the same as solving $R\mathbf{x} = \mathbf{0}$, we have free variables x_4 and x_4. Thus $x_5 = 0$, $-2x_2 = x_5 + x_4 + x_3$, so $x_2 = -\frac{1}{2}x_3 - \frac{1}{2}x_4$ and $x_1 = -2x_5 - 2x_4 - 2x_3 - 2x_2 = -2x_4 - 2x_3 + (x_3 + x_4) = -x_3 - x_4$. Hence

$$\text{Nul}\,A = \left\{\begin{bmatrix} -x_3 - x_4 \\ -\frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} : x_3, x_4 \in \mathbf{R}\right\} = \left\{x_3\begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} : x_3, x_4 \in \mathbf{R}\right\}$$

$$= \text{span}\,(\mathcal{B}), \qquad \mathcal{B} = \left\{\begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

Since the vectors are not multiples of one another, $\mathcal{B}$ is linearly independent, and since it also spans $\text{Nul}\,A$ it is a basis for $\text{Nul}\,A$.

The pivot columns 1, 2 and 5 of A form the basis of $\text{Col}\,A$. Thus the basis of $\text{Col}\,A$ is

$$\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix}\right\}.$$

4. (a) *Determine if* $S = \{2+3t+4t^2, 1+2t+3t^2, t+2t^2\}$ *is a basis for* $\mathbb{P}_2$*, the space of polynomials of degree two or less.*

   Write the coordinates of the polynomials in the basis $\mathcal{B} = \{1, t, t^2\}$

$$[2+3t+4t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \qquad [1+2t+3t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \qquad [t+2t^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

   Put the columns in a matrix and row reduce to see if the columns are a basis.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   There is a free variable, so the columns are dependent. It follows that the polynomial set S, being isomprphic to $\operatorname{Col} A$ under the coordinate transformation, is dependent, so S is not a basis of $\mathbb{P}_2$.

   (b) *For the basis* $\mathcal{B}$ *of* $\mathbf{R}^3$*, find the coordinates* $[\mathbf{w}]_{\mathcal{B}}$ *of the vector* $\mathbf{w}$ *in the* $\mathcal{B}$ *basis.*

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}, \qquad \mathbf{w} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

   Form the matrix

$$P_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 2 \\ 6 & 5 & 3 \end{bmatrix}.$$

   The coordinates $\mathbf{c} = [\mathbf{w}]_{\mathcal{B}}$ solve the equation $P_{\mathcal{B}}\mathbf{c} = \mathbf{w}$. Back-substitution, we find $c_3 = 6$, $4c_2 = 6 - 2c_3 = 6 - 12 = -6$ so $c_2 = -\frac{3}{2}$ and $6c_1 = 6 - 5c_2 - 3c_3 = 6 + \frac{15}{2} - 18 = -\frac{9}{2}$. Hence $c_1 = -\frac{3}{4}$.

5. *Determine whether the following statements are true or false. If true, give a short explanation. If false, find matrices for which the statement fails.*

   (a) STATEMENT. *If A and B are 2×2 matrices then* $\det(A+B) = \det A + \det B$.

   FALSE. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ but $\det(A+B) = 1 \neq 0 + 0 = \det A + \det B$.

   (b) *(Version 1 from PDF of Part 2.)* STATEMENT. it For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in a vector space $\mathbb{V}$, if $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$ then $\mathbf{u} = \mathbf{v}$.

   TRUE. The statement follows from the axioms of a vector space. A casual answer invoking the axioms would receive full credit. Here is a strict proof.

$\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$	Assumption.
$(\mathbf{u} + \mathbf{w}) + (-\mathbf{w}) = (\mathbf{v} + \mathbf{w}) + (-\mathbf{w})$	V5. Additive inverse $-\mathbf{w}$ of $\mathbf{w}$ exists. Post-add it to both sides.
$\mathbf{u} + (\mathbf{w} + (-\mathbf{w})) = \mathbf{v} + (\mathbf{w} + (-\mathbf{w}))$	V3. Associativity of addition
$\mathbf{u} + \mathbf{0} = \mathbf{v} + \mathbf{0}$	V5. Property of additive inverse
$\mathbf{u} = \mathbf{v}$	V4. Property of zero

   (b) *(Version 2 from Canvas of Part 2.)* STATEMENT. it For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in a vector space $\mathbb{V}$, if $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$ then $\mathbf{v} = \mathbf{w}$.

   FALSE. The best answer is a counterexample. In the vector space $\mathbf{R}^2$ take $\mathbf{u} = \mathbf{v} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, $\mathbf{w} = \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$. Then $\mathbf{u} + \mathbf{w} = \left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right] = \mathbf{v} + \mathbf{w}$ but $\mathbf{v} \neq \mathbf{w}$.

(c) STATEMENT. *Let $\mathcal{F} = \{f : \mathbf{R} \to \mathbf{R}\}$ be the vector space of functions of the real numbers. Then the transformation $T : \mathcal{F} \to \mathcal{F}$ defined by $T[f](x) = \cos[f(x)]$ is a linear transformation.*

FALSE. Some students confused $\mathcal{F}$, a space of functions with $\mathbf{R}$, the numbers. The easiest way to see that T is not linear is to test the property of linear transformations $L : \mathcal{F} \to \mathcal{F}$ that $L[z](x) = z(x)$, where $z(x) = 0$, the constant function, is the zero vector of $\mathcal{F}$. For this transformation, $T[z](x) = \cos(z(x)) = \cos(0) = 1$ wich is not $z(x)$, hence T is not linear.

6. (a) *Let $\mathcal{M}$ be the vector space of $n \times n$ matrices. Let $A \in \mathcal{M}$ be any matrix. Show that $H = \{X \in \mathcal{M} : AXA = 0\}$ is a subspace of $\mathcal{M}$.*

Show that H satisfies the conditions of a subspace. First, we have $A0A = 0$ so $0 \in H$. Second, if $X, Y \in H$ then $AXA = 0$ and $AYA = 0$. It follows that $A(X+Y)A = 0$ so $X + Y \in H$. Third, if X in H and $c \in \mathbf{R}$ Then $AXA = 0$ so $A(cX)A = cAXA = 0$ so $cX \in H$. Since H contains zero and is closed under addition and scalar multiplication, it is a subspace of M.

The reason this works is because $T(X) = AXA$ is a linear transformation $T : M \to M$ and $H = \ker T$, which is a linear subspace.

(b) *Let $H = \operatorname{span}(S)$ be a subspace of $\mathbf{R}^4$. Find a subset of S that is a basis of H.*

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

We insert the vectors as columns of A and reduce. Since $H = \operatorname{Col} A$, the pivot columns form the basis.

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first, second and fourth columns are pivot columns. Thus the basis for H is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$