Math 2270 § 2. **Second Midterm Part 1** Name: Solutions

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1. *Find the inverse matrix A^{-1} where*

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & -2 & 0 \end{bmatrix}$$

Augment the matrix by the identitiy and do row reductions.

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & -2 & 0 \\ 0 & 0 & 2 & 0 & -2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & -1 & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -1 & \frac{1}{2} \end{bmatrix}$$

So

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$$

2. (a) *Detemine whether each of the given matrices is invertible. Use as few calculations as possible. Justify your answer.*

$$A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 3 & 4 & 9 \\ 3 & 4 & 5 & 12 \\ 4 & 5 & 6 & 15 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 4 & 4 & 6 \\ 2 & 2 & 3 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

A is not invertible. The fourth column is the sum of the first three columns, so the columns are not independent.

B is not invertible. The third row is twice the second, so row reduction will yield a zero row. B is not row equivalent to the identity so is not invertible.

C is invertible. The columns are positive multiplies of the four standard basis vectors, so span $\mathbf{R}^4$. Thus C is invertible.

(b) *Let A be an $n \times n$ matrix. Without quoting Theorems 7 or 8, argue carefully why if A is row equivalent to the identity matrix then there is a matrix D such that $AD = I$.*

Being row equivalent to the identity matrix, there are finitely many elementary row operations that reduce A to I, in other words, there are elementary matrices so that

$$E_k E_{k-1} \cdots E_1 A = I = DA.$$

where $D = E_k E_{k-1} \cdots E_1$ is a left inverse of A. But the elementary matrices are invertible since the elementary row operations can be undone. By premultiplying by the inverse matrices, we see that

$$A = E_1^{-1} F_2^{-1} \cdots E_k^{-1} I = E_1^{-1} F_2^{-1} \cdots E_k^{-1}.$$

It follows that by cancelling inner matrices one by one,

$$AD = E_1^{-1} \cdots E_k^{-1} E_k \cdots E_1 = E_1^{-1} \cdots E_{k-1}^{-1} E_{k-1} \cdots E_1 = \cdots = E_1^{-1} E_1 = I.$$

In other words, D is also a right inverse of A, so A is invertible.

3. *Consider an input/output model of an economy with three sectors whose production vector is* $\mathbf{x}$. *The intermediate demand needed by the sectors for producing this output is* $C\mathbf{x}$ *where* C *is the given consumption matrix. Do* C *and* $\mathbf{d}$ *satisfy the conditions of Leontief's Theorem (Theorem 11)? How would you tell if the inverse matrix given above is correct? (Don't do it.) Write the equation and solve it for the level of production* $\mathbf{x}$ *needed to meet the final demand* $\mathbf{d}$ *above. Descrbe an iterative approximation procedure involving* C *to find the production vector* $\mathbf{x}$. *Compute at least two iterations of your approximation procedure and check that it is tending toward* $\mathbf{x}$.

$$C = \begin{bmatrix} .6 & .3 & .1 \\ .1 & .3 & .1 \\ .1 & .3 & .6 \end{bmatrix}, \qquad (I-C)^{-1} = \frac{1}{3}\begin{bmatrix} 10 & 6 & 4 \\ 2 & 6 & 2 \\ 4 & 6 & 10 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$$

C and $\mathbf{d}$ do satisfy Leonteiff's conditions for Theorem 11. All entries of both are nonnegative, and the column sums of C

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} .6 & .3 & .1 \\ .1 & .3 & .1 \\ .1 & .3 & .6 \end{bmatrix} = \begin{bmatrix} .8 & .9 & .8 \end{bmatrix}$$

are all strictly less than one. To check if the given matrix is truly the inverse, one should compute $(I-C)(I-C)^{-1}$ from above and verify that the product is I.

The equation for the production $\mathbf{x}$ is production = intermediate demand + demand or

$$\mathbf{x} = C\mathbf{c} + \mathbf{d}.$$

The production is thus

$$\mathbf{x} = (I-C)^{-1}\mathbf{d} = \frac{1}{3}\begin{bmatrix} 10 & 6 & 4 \\ 2 & 6 & 2 \\ 4 & 6 & 10 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 20 \\ 8 \\ 26 \end{bmatrix}.$$

The approximation for $\mathbf{x}$ may be described economically as follows. You must produce at least $\mathbf{x}_1 = d$. However to produce $\mathbf{x}_1$ you need to produce an additional intermendiate demand of $C\mathbf{x}_1$ which means you have produce at least $\mathbf{x}_2 = \mathbf{d} + C\mathbf{x}_1$. Continuing this way, you have to produce at least $\mathbf{x}_k$. But to produce this much you also need to produce the additional intermediate demand, $C\mathbf{x}_k$ which means you have to produce at least $\mathbf{x}_{k+1} = \mathbf{d} + C\mathbf{x}_k$. This leads to the recursion

$$\mathbf{x}_1 = \mathbf{d}, \qquad \mathbf{x}_{k+1} = \mathbf{d} + C\mathbf{x}_k.$$

A few iterates are

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 5.7 \\ 2.2 \\ 10.2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 8.1 \\ 3.25 \\ 13.35 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 10.17 \\ 4.12 \\ 15.795 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} 11.9175 \\ 4.8325 \\ 17.73 \end{bmatrix},$$

The sequence is getting closer to $\mathbf{x}$, but slowly if at all.

4. (a) *Let* $T : \mathbf{R}^3 \to \mathbf{R}^4$ *be a linear transformation. Can* T *ever be one-to-one? Can* T *ever be onto? Why or why not?*

   Being linear, it is given by the 4×3 matrix $T(\mathbf{x}) = A\mathbf{x}$. Doing row operations may yield at most three pivots, one in each column. Then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and T is one-to-one. On the other hand there cannot be a pivot in every row since there are four rows. It follows, that T is never onto.

(b) *For this A, let $T(\mathbf{x}) = A\mathbf{x}$. Is T is one-to-one? Is T onto? Explain.*

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 4 & 2 \\ 9 & -6 & -3 \\ -2 & 10 & 5 \end{bmatrix}$$

Doing row operations, one finds

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 4 & 2 \\ 9 & -6 & -3 \\ -2 & 10 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 \\ 2 & 2 & 1 \\ 9 & -6 & -3 \\ -2 & 10 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 \\ 0 & -6 & -3 \\ 0 & -42 & -21 \\ 0 & 18 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 \\ 0 & -6 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There is a free variable so $T(\mathbf{x})$ is not one-to-one. There is a zero row in REF so $T(\mathbf{x})$ is not onto.

5. *Find an LU factorization of the matrix. Check your answer.*

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

We perform only the row replacement operations. We build up L according to the columns in the intermediate steps.

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix} \to B = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix} \to C = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & \frac{7}{3} \end{bmatrix}$$

From the first column of A we put a_{21}/a_{11} and a_{31}/a_{11} in the first column

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & * & 1 \end{bmatrix}$$

From the second column of B we put b_{32}/b_{22} in the second column

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix}$$

The REF matrix becomes $U = C$. Now check by multiplying:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & \frac{7}{3} \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix} = A.$$

6. *Determine whether the following statements are true or false. If true, give a short explanation. If false, find matrices for which the statement fails.*

(a) STATEMENT. *For 2×2 matrices A, B and C, if $AC = BC$ then $A = B$.*
FALSE. Let $A = \left(\begin{smallmatrix} 0 & 2 \\ 1 & 1 \end{smallmatrix}\right) \neq B = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $C = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $AB = \left(\begin{smallmatrix} 0 & 2 \\ 0 & 1 \end{smallmatrix}\right) = BC$. There is no right cancellation rule for matrix multiplication.

(b) Statement. *If the 2×2 matrix A is invertible then elementary row operations that reduce A to the identitiy I also reduce A^{-1} to I.*
False. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $E = A^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Then $EA = I$ but $EA^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \neq I$. What is true is the same row operations reduce I to A^{-1}.

(c) Statement. *For invertible 2×2 matrices A and B we have $(AB)^{-1} = A^{-1}B^{-1}$.*
False. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $AB = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. Then $(AB)^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$ which is not $A^{-1}B^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$. What is true is that $(AB)^{-1} = B^{-1}A^{-1}$.