Math 2270 § 2. **First Midterm** Name: Solutions
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1. *Describe the solution of this system in parametric vector form. Show all steps to receive full credit.*

$$\begin{aligned} x_2 + 2x_3 + 3x_4 + 2x_5 &= 4 \\ x_1 + 2x_2 + x_3 + 3x_4 + 2x_5 &= 4 \\ 2x_1 + 4x_2 + 2x_3 + 3x_4 + 4x_5 &= -1 \end{aligned}$$

Perform row operations on the augmented matrix.

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 2 & 4 \\ 1 & 2 & 1 & 3 & 2 & 4 \\ 2 & 4 & 2 & 3 & 4 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 3 & 2 & 4 \\ 0 & 1 & 2 & 3 & 2 & 4 \\ 2 & 4 & 2 & 3 & 4 & -1 \end{pmatrix} \quad \text{Swap } R_1 \text{ and } R_2$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 3 & 2 & 4 \\ 0 & 1 & 2 & 3 & 2 & 4 \\ 0 & 0 & 0 & -3 & 0 & -9 \end{pmatrix} \quad \text{Replace } R_3 \text{ by } R_3 - 2R_1$$

x_3 and x_5 are free variables and can be set to any real value. Solving we find

$$\begin{aligned} -3x_4 &= -9 \quad \text{so } x_4 = 3 \\ x_2 &= 4 - 2x_3 - 3x_4 - 2x_5 = -5 - 2x_3 - 2x_5 \\ x_1 &= 4 - 2x_2 - x_3 - 3x_4 - 2x_5 = 4 - 2(-5 - 2x_3 - 2x_5) - x_3 - 3(3) - 2x_5 \\ &= 5 + 3x_3 + 2x_5. \end{aligned}$$

Thus the set of solutions is

$$\mathcal{S} = \left\{ \begin{bmatrix} 5 + 3x_3 + 2x_5 \\ -5 - 2x_3 - 2x_5 \\ x_3 \\ 3 \\ x_5 \end{bmatrix} : \text{where } x_3 \text{ and } x_5 \text{ are any real numbers.} \right\}$$

or in parametric vector form

$$\mathcal{S} = \left\{ \begin{bmatrix} 5 \\ -5 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} : \text{where } x_3 \text{ and } x_5 \text{ are any real numbers.} \right\}.$$

2. *Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Is $\mathbf{b}$ in the range of T? Explain why or why not. Describe geometrically which $\mathbf{c}$ are in the range of T.*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Perform row operations on the augmented matrix.

$$\begin{pmatrix} 1 & 2 & 3 & 1 & c_1 \\ 2 & 3 & 4 & 1 & c_2 \\ 3 & 4 & 5 & 1 & c_3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 1 & c_1 \\ 0 & -1 & -2 & -1 & c_2 - 2c_1 \\ 0 & -2 & -4 & -2 & c_3 - 3c_1 \end{pmatrix} \begin{matrix} \\ \text{Replace } R2 \text{ by } R2 - 2R_1 \\ \text{Replace } R3 \text{ by } R3 - 3R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 1 & c_1 \\ 0 & -1 & -2 & -1 & c_2 - 2c_1 \\ 0 & 0 & 0 & 0 & c_3 - 3c_1 - 2(c_2 - 2c_1) \end{pmatrix} \quad \text{Replace } R3 \text{ by } R3 - 2R_2$$

The augmented matrix $(A|\mathbf{b})$ reduces to a matrix that has a zero row on the bottom. Thus there are no zero equals nonzero rows, and the system is consistent. One can solve for $\mathbf{x}$ so that $A\mathbf{x} = \mathbf{b}$. Thus $T(\mathbf{x}) = \mathbf{b}$ for some $\mathbf{x} \in \mathbb{R}^3$, thus $\mathbf{b}$ is in the range of T.

A geometric condition for $\mathbf{c}$ to be in the range of T is that the equations $A\mathbf{x} = \mathbf{c}$ be consistent. This requires that the bottom row of the REF be zero for $\mathbf{c}$, or $c_3 - 3c_1 - 2(c_2 - 2c_1) = c_3 + c_1 - 2c_2 = 0$.

3. *Let $P : \mathbf{R}^2 \to \mathbf{R}^2$ be a projection to the line $x + y = 0$. It is given by*

$$P\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} .5\,x_1 - .5\,x_2 \\ -.5\,x_1 + .5\,x_2 \end{bmatrix}.$$

Define P is a linear transformation. *Is P a linear transformation? Explain why or why not. Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be a linear transformation such that*

$$T(\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad T(\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad T(\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}.$$

Find $T(\mathbf{v})$ and explain. Give an example of a nonlinear mapping $N : \mathbf{R}^2 \to \mathbf{R}^2$ whose range is contained in the line $x + y = 0$. Explain why your N is nonlinear.

A transformation $P : \mathbf{R}^2 \to \mathbf{R}^2$ is *linear* if (1) $P(\mathbf{x}) + P(\mathbf{y})$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^2$; and (2) $P(c\mathbf{x}) = cP(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^2$ and all $c \in \mathbf{R}$.

To see that this particular transformation is linear, we must check (1) and (2). For this purpose, suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ are any vectors in $\mathbf{R}^2$ and c is any number. Then using the definition of P and rules for vector addition and scalar multipliication,

$$P(\mathbf{x}+\mathbf{y}) = P\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = P\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} .5(x_1+y_1) - .5(x_2+y_2) \\ -.5(x_1+y_1) + .5(x_2+y_2) \end{bmatrix}$$
$$= \begin{bmatrix} .5x_1 - .5x_2 \\ -.5x_1 + .5x_2 \end{bmatrix} + \begin{bmatrix} .5y_1 - .5y_2 \\ -.5y1 + .5y_2 \end{bmatrix} = P\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + P\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = P(\mathbf{x}) + P(\mathbf{y});$$
$$P(c\mathbf{x}) = P\left(c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = P\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) = \begin{bmatrix} .5cx_1 - .5cx_2 \\ -.5cx_1 + .5cx_2 \end{bmatrix} = c\begin{bmatrix} .5x_1 - .5x_2 \\ -.5x_1 + .5x_2 \end{bmatrix} = cP(\mathbf{x}).$$

Thus (1) and (2) hold for any vectors and scalars, thus P is linear.

To find $T(\mathbf{v})$, use superposition. Because the vectors are basic vectors of $\mathbf{R}^3$ we may express

$$\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Using linearity and the equations for T we deduce

$$T(\mathbf{v}) = T\left(3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = 3T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) - 2T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -7 \\ -6 \end{bmatrix}.$$

Give an example of a nonlinear mapping $N : \mathbf{R}^2 \to \mathbf{R}^2$ whose range is contained in the line $y_1 + y_2 = 0$. One such mapping is given by

$$N(\mathbf{x}) = N\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + x_1 + x_2 \\ -1 - x_1 - x_2 \end{bmatrix}.$$

We have $y_1 + y_2 = 1 + x_1 + x_2 + (-1 - x_1 - x_2) = 0$ so $N(\mathbf{x})$ has range in the line. Also N is nonlinear because

$$N(\mathbf{0}) = N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq \mathbf{0}.$$

A linear map must have $N(\mathbf{0}) = \mathbf{0}$.

4. *Define: the set of vectors* $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\}$ *is* linearly independent. *Is the set* $\mathcal{S}$ *is linearly independent? Explain. Is* $\mathbf{b}$ *in* $\operatorname{span}\mathcal{S}$*? Explain.*

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 2 \\ 8 \end{bmatrix} \right\}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

A set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\}$ is *linearly independent* if whenever there are constants $c_1, \ldots, c_p$ such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_p\mathbf{a}_p = \mathbf{0}$$

then $c_1 = c_2 = \ldots = c_p = 0$. (Linear independence is not a statement about matrices.)

Let us put the vectors of $\mathcal{S}$ into columns of a matrix A, augmented with $\mathbf{b}$ and try to solve the homogeneous system for $\mathbf{c}$ in $A\mathbf{c} = \mathbf{0}$ and the inhomogeneous system $A\mathbf{x} = \mathbf{b}$. Using elimination we find

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 3 & 1 & 2 & 2 \\ 2 & 9 & 8 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & -5 & -4 & -1 \\ 0 & 5 & 4 & 1 \end{pmatrix} \begin{matrix} \\ \text{Replace } R2 \text{ by } R2 - 3R_1 \\ \text{Replace } R3 \text{ by } R3 - 2R2 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & -5 & -4 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ \text{Replace } R3 \text{ by } R3 + R2 \end{matrix}$$

The third column is free, so we have a solution with the free variable nonzero. Taking $c_3 = 1$ so $c_2 = -\frac{4}{5}$ and $c_1 = -\frac{2}{5}$ we find that there is a nontrivial dependency relation between the vectors so $\mathcal{S}$ is linearly dependent,

$$-\frac{2}{5}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The last row of the reduced matrix of $\mathbf{b}$ is zero as well. This says that the system $A\mathbf{x} = \mathbf{b}$ is consistent so can be solved for $\mathbf{x}$, namely by choosing $x_3 = 0$ we get $x_2 = \frac{1}{5}$ and $x_1 = \frac{3}{5}$. Thus as linear combination of $\mathcal{S}$ makes $\mathbf{b}$ so $\mathbf{b} \in \operatorname{span}\mathcal{S}$:

$$\frac{3}{5}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

5. *Suppose that the row echelon form of the matrix A is R, where*

$$R = \begin{bmatrix} 1 & * & * & * \\ 0 & 2 & * & * \\ 0 & 0 & 3 & * \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and "$*$" is any number. Is the solution of $A\mathbf{x} = 0$ unique? Explain why or why not. Can $Ax = \mathbf{v}$ be solved for the $\mathbf{v}$ above? Expalin why or why not. Is the range of the operator $T(\mathbf{x}) = A\mathbf{x}$ all of $\mathbf{R}^6$? Explain.

The solution of $A\mathbf{x} = 0$ is unique because the row equivalent reduced system $R\mathbf{x} = 0$ has no free variables. Hence $\mathbf{x} = \mathbf{0}$ is the only solution.

$Ax = \mathbf{v}$ can not be be solved for the the given $\mathbf{v}$ in general. After reducing the augmented system $(A|\mathbf{v})$ we get the new system $(R|\mathbf{v}^*)$ where $\mathbf{v}^*$ usually will not have its bottom two rows equal to zero. Usually the system $A\mathbf{x} = \mathbf{v}$ will be inconsistent. Only for special A's can it be solved.

The range of the operator $T(\mathbf{x}) = A\mathbf{x}$ is not all of $\mathbf{R}^6$. This is because only four of the six rows are pivot rows. And there would be some $\mathbf{b} \in \mathbf{R}^6$ such that the system $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$ would be inconsistent. Such $\mathbf{b}$ are not in the range of T.

6. *Balance the chemical equation by formulating a vector equations and solving it for for the numbers of each molecule so that the numbers of each type of atom on the left matches the numbers of corresponding atoms on the right. Iron sulfide combines with oxygen to produce iron oxide and sulfur dioxide.*

$$\mathrm{Fe\ S} + \mathrm{O_2} \longrightarrow \mathrm{Fe_2\ O_3} + \mathrm{S\ O_2}.$$

Write an equation in vectors of the number of atoms of iron, sulfur and oxygen. Letting x_1, x_2, x_3 and x_4 denote the number of iron sulfide, oxygen, iron dioxide and sulfur dioxide molecules, resp., the vector equation is

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

This is a homogeneous equation $A\mathbf{x} = \mathbf{0}$ with columns equal to the vectors. Row reduce as usual.

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 2 & -3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 2 & -3 & -2 \end{pmatrix} \text{Replace } R2 \text{ by } R2 - R_1$$

$$\to \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & -3 & -2 \\ 0 & 0 & 2 & -1 \end{pmatrix} \text{Swap } R2 \text{ and } R3$$

x_4 is free so we set it to a value to be determined $x_4 = t$. Hence $x_3 = \frac{1}{2}t$, $x_2 = \frac{7}{4}t$ and $x_1 = t$. Choosing $t = 4$ gives the smallest possible whole number solution, as is required for chemical balabcing: $x_1 = 4$, $x_2 = 7$, $x_3 = 2$ and $x_4 = 4$. Finally we check that these numbers balance the reaction.

$$4\ \mathrm{Fe\ S} + 7\ \mathrm{O_2} \longrightarrow 2\mathrm{Fe_2\ O_3} + 4\ \mathrm{S\ O_2}.$$