Math 2270 § 1. **Second Midterm Exam** Name: Solutions
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1. *For each of the given matrices, find the determinant in as simple a way as possible, preferably without doing the full computation (2 points). Explain (3 points).*

   (a) $\det A = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 0$

   because the matrix has a zero column. For example, expanding the determinant on the second column yields

   $$\det A = \sum_{i=1}^{3} (-1)^{i+2} a_{i2} \det(A_{i2}) = -0 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 0.$$

   (b) $\det B = \begin{vmatrix} 1 & -2 & 3 \\ 4 & 4 & 4 \\ 6 & 0 & 10 \end{vmatrix} = 0$

   because the third row is the sum of twice the first row plus the second row. Hence the RREF form of B has a zero row so B is not invertible so $\det B = 0$.

   (c) $\det C = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 5 & 5 & 5 & 0 \\ 7 & 7 & 7 & 7 \end{vmatrix} = 1 \cdot 3 \cdot 5 \cdot 7 = 105$

   because the determinant of the lower triangular matrix C is the product of its diagonal entries.

   (d) Expanding E by its second column,

   $$\det E = \begin{vmatrix} 2 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 2 & 2 \\ 2 & 0 & 3 & 4 & 4 \\ 2 & 0 & 3 & 3 & 5 \end{vmatrix} = -3 \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 3 & 4 & 4 \\ 2 & 3 & 3 & 5 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \\ 2 & 3 & 4 & 4 \\ 2 & 3 & 3 & 5 \end{vmatrix}$$

   $$= -3 \cdot 0 + 1 \cdot \begin{vmatrix} 2 & 3 & 3 & 3 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 1 \cdot 2 \cdot (-1) \cdot 1 \cdot 2 = -4.$$

   The first four by four determinant vanishes because the second row is twice the first row. The second four by four determinant equals the determinant of the matrix obtained by doing row replacement operations: the second row is replaced by the second minus the first, the third row is replaced by the third minus the first, the fourth row is replaced by the fourth minus the first. The resulting matrix is upper triangular, thus its determinant is the product of its diagonals.

2. *Let* $X \subset \mathbf{R}^4$ *be the subset all* $\mathbf{x}$ *satisfying the equation*

$$X = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + 2x_2 = 3x_3 + 4x_4 \right\}$$

State the definition X *is a* vector subspace *of* $\mathbf{R}^4$*. Explain why* X *is a vector subspace of* $\mathbf{R}^4$*. State the definition* $\mathcal{B}$ *is a* basis *of* X*. Find a basis of the vector subspace of* X*.*

A *vector subspace* $X \subset \mathbf{R}^4$ is a subset that is closed under vector addition and scalar multiplication. It satisfies three properties: (1) $\mathbf{0} \in X$, (2) If $\mathbf{u}, \mathbf{v} \in X$ then $\mathbf{u} + \mathbf{v} \in X$ and (2) If $\mathbf{u} \in X$ and $c \in \mathbf{R}$ then $c\mathbf{u} \in X$.

The given set X satisfies the three conditions of being a vector subspace.

First, $0 + 2 \cdot 0 = 3 \cdot 0 + 4 \cdot 0$ so $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in X$. Second, suppose $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \in X$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in X$. This means $u_1 + 2u_2 = 3u_3 + 4u_4$ and $v_1 + 2v_2 = 3v_3 + 4v_4$. Hence $(u_1 + v_1) + 2(u_2 + v_2) = 3(u_3 + v_3) = 4(u_4 + v_4)$ so $\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ u_4 + v^4 \end{pmatrix} \in X$. Third, suppose $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \in X$ and $c \in \mathbf{R}$. This means $u_1 + 2u_2 = 3u_3 + u_4$. Hence $(cu_1) + 2(cu_2) = 3(cu_3) + 4(cu_4)$ so $c\mathbf{u} = \begin{pmatrix} cu_1 \\ cu_2 \\ cu_3 \\ cu_4 \end{pmatrix} \in X$. The three conditions of a vector subspace holds for this X. In fact, $X = \mathrm{Nul}(A)$ is the nullspace of the 1×4 matrix $A = [1 \quad 2 \quad -3 \quad -4]$ and a nullspace is a vector subspace.

A *basis* of a vector subspace is a set of vectors $\mathcal{B} \subset X$ which spans X and is linearly independent.

Solving the equation

$$\begin{pmatrix} 1 & 2 & -3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

we see that x_2, x_3 and x_4 are free so that $x_1 = -2x_2 + 3x_3 + 4x_4$ and

$$X = \left\{ \begin{pmatrix} -2x_2 + 3x_3 + 4x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_2,\ x_3 \text{ and } x_4 \text{ any reals} \right\} = \operatorname{span} \mathcal{B}$$

where

$$\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

This shows $\mathcal{B}$ is a spanning set for X. Since the equation

$$c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

implies $c_1 = c_2 = c_3 = 0$ from the bottom three equations, $\mathcal{B}$ is also linearly independent, thus a basis.

3. *Answer for the given matrix A. Define* $\mathrm{Col}(A)$*. Find a basis for* $\mathrm{Col}(A)$*. Define:* $\mathrm{Nul}(A)$*. Find a basis for* $\mathrm{Nul}(A)$*. Find* $\mathrm{rank}(A)$*.*

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & 3 & 3 \\ 2 & 0 & 0 & 3 & 0 \\ 1 & 5 & 5 & 1 & 4 \end{pmatrix}$$

$\mathrm{Col}(A)$ consists of all linear combinations of columns of A, namely

$$\mathrm{Col}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbf{R}^5\}.$$

Doing row operations we find

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & 3 & 3 \\ 2 & 0 & 0 & 3 & 0 \\ 1 & 5 & 5 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & -2 & -2 & -1 & -4 \\ 0 & 4 & 4 & -1 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & -3 & -6 \\ 0 & 0 & 0 & 3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

x_1, x_2 and x_4 are the pivot variables, thus a basis for $\mathrm{Col}(A)$ consists of pivot columns of A, namely,

$$\text{basis of } \mathrm{Col}(A) \text{ is } \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

$\mathrm{Nul}(A)$ consits of all vectors annihilated by A, namely,

$$\mathrm{Nul}(A) = \{\mathbf{x} \in \mathbf{R}^5 : A\mathbf{x} = \mathbf{0}\}.$$

Solving the homogeneous equation using the reduced echelon matrix, x_3 and x_5 are free so that $x_4 = -2x_5$, $x_2 = -x_3 + x_4 + x_5 = -x_3 - x_5$ and $x_1 = -x_2 - x_3 - 2x_4 - 2x_5 = -2x_3 + 3x_5$. Thus

$$\mathrm{Nul}(A) = \left\{ \begin{pmatrix} -2x_3 + 3x_5 \\ x_3 - x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{pmatrix} : \text{any real } x_3,\, x_5 \right\} = \mathrm{Span}\, \mathcal{B} \quad \text{where} \quad \mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

The vectors of $\mathcal{B}$ are not multiples of one another so $\mathcal{B}$ is the basis for $operatorname Nul(A)$.

The rank is the dimension of the column space so $\operatorname{rank}(A) = 3$.

4. *Define: the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p\}$ is linearly independent. Determine whether the set $\mathcal{S}$ is linearly independent. Explain.*

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -5 \\ 7 \end{bmatrix} \right\}$$

The vectors $\mathbf{a}_1$, $\mathbf{a}_2$,..., $\mathbf{a}_p$ are *linearly independent* if the zero linear combination

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_p\mathbf{a}_p = \mathbf{0}$$

is possible only for $c_1 = c_2 = \cdots = c_p = 0$.

Inserting the vectors as columns of A and solving for $\mathbf{c}$ in $A\mathbf{c} = \mathbf{0}$ by doing row operations

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \\ 1 & 0 & -5 \\ 1 & 3 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 4 \\ 0 & -1 & -4 \\ 0 & 2 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since c_3 is free, it can take any value, so the solution of $A\mathbf{c} = \mathbf{0}$ need not be dead zero, therefore $\mathcal{S}$ is linearly dependent.

5. (a) *Recall that $[\mathbf{x}]_{\mathcal{B}}$ denotes the coordinates of the vector $\mathbf{x}$ in the basis $\mathcal{B}$. Let the basis be $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$. Given $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, what is $[\mathbf{x}]_{\mathcal{B}}$? Given $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, what is $\mathbf{y}$?*

The coordinates $[\mathbf{x}]_{\mathcal{B}} = \mathbf{c}$ solve $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \mathbf{x}$. Doing row operations on the augmented matrix,

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{pmatrix}$$

which says $c_2 = -1$ and $c_1 = 3 - 2c_2 = 3 - 2(-1) = 5$. Thus

$$\left[\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \qquad \text{Check:}\quad 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

On the other hand, $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ implies $\mathbf{y} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$.

(b) *Given the $A = LU$ decomposition, solve $A\mathbf{x} = \mathbf{b}$ where*

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 5 & 6 & 7 \\ 0 & 8 & -9 \\ 0 & 0 & 10 \end{pmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The solution is done in two steps. The first equation $L\mathbf{y} = \mathbf{b}$ has the obvious solution

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{implies} \qquad \mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The second equation $U\mathbf{x} = \mathbf{y}$ is done by back substitution

$$\begin{pmatrix} 5 & 6 & 7 \\ 0 & 8 & -9 \\ 0 & 0 & 10 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{implies} \qquad \mathbf{x} = \begin{pmatrix} -\frac{11}{8} \\ \frac{9}{80} \\ \frac{1}{10} \end{pmatrix}$$

(c) *Suppose that the vectors* $\mathbf{a}_1$, $\mathbf{a}_2$ *and* $\mathbf{a}_3$ *span* $\mathbf{R}^3$. *Explain why the set* $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ *is also linearly independent, where*

$$\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}.$$

Inserting the vectors as columns into matrix A, the statement that the columns span is the same as saying $A\mathbf{x} = \mathbf{b}$ can be solved for every $\mathbf{b}$. But that means that the RREF form of A must have a pivot in every row. But since the matrix is square, that means that there is also a pivot in every column. But that means that there are no free variables so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. But that is the condition that the columns of A are linearly independent.

All of these statements are equivalent to the invertibility of A. So another argument may be: the columns of A spanning $\mathbf{R}^3$ is equivalent to the condition that the columns are linearly independent, by the main Theorem of Invertible Matrices.