Math 2270 § 1. **Third Midterm Exam** Name: Solutions
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1. *Diagonalize A. Check your answer. [Hint: the eigenvalues are* $\lambda = 1, 2, 2$.*]*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{pmatrix}$$

We find an eigenvector for $\lambda_1 = 1$ by inspection.

$$\mathbf{0} = (A - \lambda_1 I)\mathbf{v}_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

We find eigenvectors for $\lambda_2 = 2$ by inspection.

$$[\mathbf{0}, \mathbf{0}] = (A - \lambda_2 I)[\mathbf{v}_2, \mathbf{v}_3] = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The diagonalizing matrix P such that $P^{-1}AP = D$ is

$$P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

We check by seeing if $AP = PD$. Indeed,

$$AP = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$PD = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

which are equal.

2. *Find the rank of A. Let* $H = \operatorname{Col} A$. *Show that* $\mathcal{S}$ *is a linearly independent subset of* H. *Show also that* $\mathcal{S}$ *is not a basis for* H. *Show that* $\mathcal{S}$ *can be extended to a basis of* $H = \operatorname{Col} A$ *by finding vectors to add to* $\mathcal{S}$ *to make a basis.*

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \qquad \mathcal{S} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

Row reducing A we find

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are three pivots in the echelon matrix, thus $\boxed{\operatorname{rank} A = 3}$. Moreover, a basis for $H = \operatorname{Col} A$ is given by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, the pivot columns of A. Observe that $\dim H = 3$.

Second, observe that

$$\mathbf{s}_1 = \begin{pmatrix}2\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\0\\1\\1\end{pmatrix} = \mathbf{a}_1 + \mathbf{a}_2, \qquad \mathbf{s}_2 = \begin{pmatrix}2\\1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\\1\end{pmatrix} + \begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \mathbf{a}_2 + \mathbf{a}_3.$$

Thus both $\mathbf{s}_1$ and $\mathbf{s}_2$ are linear combinations of columns of A, thus are in $H = \operatorname{Col} A$. Also $\mathbf{s}_1$ and $\mathbf{s}_2$ are not multiples of each other, so are independent. $\mathcal{S}$ has two vectors which is too few to be a basis of H which is three dimensional.

A basis for H may be achieved by adding the vector $\mathbf{s}_3 = \mathbf{a}_2$. The set $\mathcal{B} = \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ is a basis. To see it, we observe that

$$\mathbf{a}_1 = \mathbf{s}_1 - \mathbf{s}_3, \qquad \mathbf{a}_2 = \mathbf{s}_3, \qquad \mathbf{a}_3 = \mathbf{s}_2 - \mathbf{s}_3.$$

Thus vectors in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, the basis for H, are linear combinations of vectors of $\mathcal{B}$. Thus three vectors in $\mathcal{B}$ span the three dimensional space H. By the Basis Theorem, $\mathcal{B}$ is a basis for H.

3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) *Let λ be an eigenvalue of the matrix A with multiplicity m. Then A has m independent eigenvectors corresponding to λ.*

   False. Not every matrix has as as many independent eigenvectors as the algebraic multiplicity. For example, the characteristic polynomial of

   $$A = \begin{pmatrix}3 & 1\\0 & 3\end{pmatrix}, \qquad A - 3I = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$$

   is $\det(A - \lambda I) = (3-\lambda)^2$ so $\lambda = 3$ is an eigenvalue with multiplicity $m = 2$ but $A - 3I$ is a rank one matrix and has only a one dimensional nullspace, the space of $\lambda = 3$ eigenvectors of A.

   (b) it If λ is an eigenvalue of the invertible matrix A then $\dfrac{1}{\lambda}$ is an eigenvalue of A^{-1}.

   True. Since A is invertible,

   $$0 \ne \det(A) = \det(A - 0I)$$

   so that zero is not an eigenvalue of A. Let $\mathbf{v}$ be a λ-eigenvector. Thus

   $$A\mathbf{v} = \lambda\mathbf{v}.$$

   Multiplying by A^{-1} and dividing by $\lambda \ne 0$ we find

   $$\frac{1}{\lambda}\mathbf{v} = A^{-1}\mathbf{v}$$

   so $\mathbf{v}$ is a $\frac{1}{\lambda}$ eigenvector of A^{-1}.

   (c) *Eigenvectors corresponding to distinct eigenvalues of the matrix A are orthogonal.*

   False. The eigenvalues of $A = \begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}$ are $\lambda_1 = 1$ and $\lambda_2 = 2$. Eigenvectors are

   $$\mathbf{0} = (A - \lambda_1 I)\mathbf{v}_1 = \begin{pmatrix}0 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}, \qquad \mathbf{0} = (A - \lambda_2 I)\mathbf{v}_2 = \begin{pmatrix}-1 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

   But

   $$\mathbf{v}_1 \bullet \mathbf{v}_2 = 1\cdot 1 + 0 \cdot 1 = 1$$

   which is not zero so that $\mathbf{v}_1$ and $\mathbf{v}_2$ are not orthogonal.

4. *In $\mathbb{P}_2$, the polynomials of degree two or less, let $\mathcal{B} = \{1, 1+t, 1+t+t^2\}$ and $\mathcal{C} = \{1, t, t^2\}$. Show that $\mathcal{B}$ is a basis for $\mathbb{P}_2$. Find the change of coordinates matrix from basis $\mathcal{B}$ to basis $\mathcal{C}$. Let $\mathbf{x} = 3 + 4t + 5t^2$. Find $[\mathbf{x}]_{\mathcal{B}}$.*

The degree two or less polynomials make up the set $\mathbb{P}_2 = \{\alpha + \beta t + \gamma t^2 : \alpha, \beta, \gamma \in \mathbf{R}\}$, which is a vector space under usual addition and scalar multiplication. Observe that

$$1 = \mathbf{b}_1, \qquad t = \mathbf{b}_2 - \mathbf{b}_1, \qquad t^2 = \mathbf{b}_3 - \mathbf{b}_2$$

so that any vector $\mathbf{x} \in \mathbb{P}_2$ may be written

$$\mathbf{x} = \alpha + \beta t + \gamma t^2 = \alpha\mathbf{b}_1 + \beta(\mathbf{b}_2 - \mathbf{b}_1) + \gamma(\mathbf{b}_3 - \mathbf{b}_2) = (\alpha - \beta)\mathbf{b}_1 + (\beta - \gamma)\mathbf{b}_2 + \gamma\mathbf{b}_3. \quad (1)$$

It is a linear combination of $\mathbf{b}_i$'s, thus $\mathbb{P}_2$ is spanned by $\mathcal{B}$. Also the dependency condition

$$0 = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = c_1 \cdot 1 + c_2(1+t) + c_3(1+t+t^2)$$

implies the equations

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 0 &= \phantom{c_1 + {}} c_2 + c_3 \\ 0 &= \phantom{c_1 + c_2 + {}} c_3 \end{aligned}$$

whose only solution is $c_1 = c_2 = c_3 = 0$. Thus $\mathcal{B}$ is also independent, therefore a basis.

Alternatively, we could consider coordinates $[\mathbf{b}_i]_{\mathcal{C}}$ in the $\mathcal{C}$ basis, the columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, and argue that they form a basis in $\mathbf{R}^3$. One way is to check that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is invertible (its determinant is nonzero) so that its columns are a basis of $\mathbf{R}^3$ so the corresponding functions $\mathbf{b}_i$ are a basis for $\mathbb{P}_2$.

The coordinates for the $\mathbf{b}_i$'s in the $\mathcal{C}$ basis are

$$[\mathbf{b}_1]_{\mathcal{C}} = [1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\mathbf{b}_2]_{\mathcal{C}} = [1+t]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{b}_3]_{\mathcal{C}} = [1+t+t^2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \big[[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, [\mathbf{b}_3]_{\mathcal{C}}\big] = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Finally, this matrix relates the coordinates according to

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$$

For $\mathbf{x} = 3 + 4t + 5t^2$ we know $[\mathbf{x}]_{\mathcal{C}}$ and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and must solve for $[\mathbf{x}]_{\mathcal{B}}$. Writing the augmented matrix $\left[\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}, [\mathbf{x}]_{\mathcal{C}}\right]$ and solving

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

we find $c_3 = 5$, $c_2 = 4 - c_3 = -1$ and $c_1 = 3 - c_2 - c_3 = 3 - (-1) - 5 = -1$. Thus

$$[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}.$$

Check:

$$-\mathbf{b}_1 - \mathbf{b}_2 + 5\mathbf{b}_3 = -1 - (1+t) + 5(1+t+t^2) = 3 + 4t + 5t^2.$$

Alternately, we may use (1) to see that

$$3 + 4t + 5t^2 = (3-4)\mathbf{b}_1 + (4-5)\mathbf{b}_2 + 5\mathbf{b}_3 = -\mathbf{b}_1 - \mathbf{b}_2 + 5\mathbf{b}_3.$$

5. (a) *Find the eigenvalues and the corresponding eigenvectors of* $A = \begin{pmatrix} 6 & 2 \\ -1 & 4 \end{pmatrix}$.

Find the eigenvalue by solving

$$0 = \begin{vmatrix} 6-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = (6-\lambda)(4-\lambda) + 2 = \lambda^2 - 10\lambda + 26 = (\lambda-5)^2 + 1.$$

Thus the two eigenvalues are

$$\lambda = 5 - i; \qquad \bar{\lambda} = 5 + i.$$

The λ eigenvector $\mathbf{v}$ is found by inspection

$$(A - \lambda I)\mathbf{v} = \begin{pmatrix} 1+i & 2 \\ -1 & -1+i \end{pmatrix} \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$$

The $\bar{\lambda}$ eigenvector is the complex conjugate

$$\overline{\mathbf{v}} = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}.$$

(b) *Let* $B = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 3 & 3 & 4 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}$. *Find* $(\operatorname{Col} B)^\perp$.

$(\operatorname{Col} B)^\perp$ is the space of all vectors orthogonal to the generators of $\operatorname{Col} B$, in other words solutions of $B^T\mathbf{x} = \mathbf{0}$ which is $\operatorname{Nul} B^T$. Row reducing B^T we find

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 2 \\ 2 & 4 & 2 \\ 2 & 4 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The variable x_3 is free so can be any real. $x_2 = -x_3$ and $x_1 = -3x_2 - 2x_3 = 3x_3 - 2x_3 = x_3$. Thus the solution is

$$(\operatorname{Col} B)^\perp = \operatorname{Nul} B^T = \left\{ \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} : x_3 \in \mathbf{R} \right\} = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$