Math 2270 § 1. **Third Midterm Exam** Name: Practice Problems
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1. *Find a basis for the subspace spanned by the following vectors.*

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 5 \\ 8 \\ 6 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 14 \\ 23 \\ 18 \end{bmatrix}$$

We put the vectors in as columns. Then row reduce and choose the pivot columns.

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 5 & 14 \\ 3 & 1 & 8 & 23 \\ 2 & 0 & 6 & 18 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -3 & 3 & 12 \\ 0 & -5 & 5 & 20 \\ 0 & -4 & 4 & 16 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The first and second are pivot columns. Thus a basis for the spanned subspace is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

2. *Suppose $n \geq 1$ and that the vectors $\mathcal{H} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ span the n-dimensional vector space $\mathbb{V}$. Show that $\mathcal{H}$ is a basis for $\mathbb{V}$.*

We already know that $\mathcal{H}$ spans $\mathbb{V}$. It remains to argue that the vectors in $\mathcal{H}$ are linearly independent, making $\mathcal{H}$ a basis. We apply the *Spanning Set Theorem*, which says that some subset of $\mathcal{H}$ is a basis for the nonzero space $\mathbb{V}$. Here is the argument: suppose that the set $\mathcal{H}$ is not independent. Then one of the vectors, say $\mathbf{v}_k \in \mathcal{H}$, is a linear combination of the other vectors and may be removed from $\mathcal{H}$ leaving a smaller set that still spans $\mathbb{V}$. So long as there are more than two vectors remaining, we may continue to remove vectors until the reduced spanning set $\mathcal{H}'$ is independent or there remains only one vector. By assumption, the space spanned must be nonzero and so if the spanning set $\mathcal{H}'$ consists of a single vector, it must be the basis for the space.

Now every basis has to have the the same number of vectors which is the dimension $n = \dim(\mathbb{V})$. Thus the independent spanning set $\mathcal{H}'$ must have n vectors, thus $\mathcal{H}' = \mathcal{H}$ (no vectors were removed). This is part of the *Basis Theorem* in the text.

3. *Find the determinant by row reduction. (Text problem 177[8].)*

$$D = \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{vmatrix}$$

The row operation of subtracting a multiple of one row from another does not change the determinant. Swapping rows in the last equality multiplies the determinant by -1. The determinant of an upper triangular matrix is the product of the diagonal entries. $D =$

$$\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -15 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & 10 \end{vmatrix} = -\big(1 \cdot 1 \cdot 1 \cdot (10)\big) = -10.$$

4. *Let A and B be square matrices. Show that even though AB and BA may not be equal, it is always true that* $\det(AB) = \det(BA)$. *(Problem 177[33] of the text.)*

   This is a simple consequence of the Theorem that the determinant of a product of square matrices is the product of determinants. So

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA).$$

5. *Using just Theorem 3 of Section 3.2 that gives the effect of elementary row operations on the determinant of a matrix, show that if two rows of a square matrix A are equal, then* $\det(A) = 0$. *The same is true for two columns. Why? (Text problem 177[30].*

   Assume that rows R_i and R_j of A are equal, where $i \neq j$. Let E be the elementary row operation matrix that swaps R_i and R_j. Then because the rows are equal, $A = EA$. Taking determinants we find

$$\det(A) = \det(EA) = -\det(A)$$

   because this row operation flips the sign of the determinant. It follows that $\det(A) = 0$ as desired.

   Now suppose two columns $\mathbf{a}_i$ and $\mathbf{a}_j$ are equal, where $i \neq j$. If we are allowed to transpose the matrix, then A^T has the same determinant as A but now A^T has two equal rows $\mathbf{a}_i^T$ and $\mathbf{a}_j^T$, thus has zero determinant by what we showed already. However, we can argue just from Theorem 3. First observe that if A had a zero row, say R_i, then $\det(A) = 0$. To see this, let E be the elementary row operation that multiplies R_i by 2. By Theorem 3, this operation doubles the determinant. However $A = EA$ because $2R_i$ is still a zero row. Thus

$$\det(A) = \det(EA) = 2\det(A)$$

   which also implies $\det(A) = 0$. Now, if A has two equal columns, let $\mathbf{c}$ be a column vector whose entries are zero except the ith which is 1 and the jth which is -1. Thus $A\mathbf{c} = \mathbf{0}$ so there is a nontrivial vector in $\mathbf{c} \in \mathrm{Nul}(A)$. It follows that A may be row-reduced to an echelon U matrix with a free column, and therefore, a zero row. Let E_1, E_2,...,E_p be matrices that reduce A to echelon form. Hence

$$E_p \cdots E_1 \cdot A = U$$

Take determinant of this equation. Now, by Theorem 3, each of the row operations E_i multiplies the determinant by a nonzero constant k_i. It follows that

$$k_p \cdots k_1 \cdot \det(A) = \det(U) = 0$$

because U has a zero row. But since the k_i's are nonzero, it follows that $\det(A) = 0$ too.

6. *Find basis for the column space, the row space and the null space of the matrix. (Text problem 238[3].)*

$$A = \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{pmatrix}$$

Row reducing, we find

$$A = \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = U$$

The first third and fourth columns are pivot columns, so a basis for the column space is

$$\mathcal{B}_{\text{col}} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\}$$

The nonzero rows of the echelon matrix U are a basis for the row space.

$$\mathcal{B}_{\text{row}} = \left\{ \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 3 \end{bmatrix} \right\}$$

The basis for the null space are solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$, or what is the same, $U\mathbf{x} = \mathbf{0}$. x_2 and x_5 are free variables. The null space is thus $x_4 = -3x_5$, $x_3 = \frac{1}{3}x_4 - \frac{1}{3}x_5 = -\frac{4}{3}x_5$ and $x_1 = \frac{3}{2}x_2 - 3x_3 - x_4 - \frac{5}{2}x_5 = \frac{3}{2}x_2 + \frac{9}{2}x_5$ thus

$$\text{Nul}(A) = \left\{ \begin{bmatrix} \frac{3}{2}x_2 + \frac{9}{2}x_5 \\ x_2 \\ -\frac{4}{3}x_5 \\ -3x_5 \\ x_5 \end{bmatrix} \ : \ x_2, x_5 \in \mathbf{R} \right\} \quad \text{so} \quad \mathcal{B}_{\text{Nul}} = \left\{ \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{9}{2} \\ 0 \\ -\frac{4}{3} \\ -3 \\ 1 \end{bmatrix} \right\}$$

7. *Let A be an $m \times n$ matrix. Show that $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbf{R}^m$ if and only if the equations $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution. (Text problem 238[29].)*

   Assume that $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution. This means that the dimension of the null space of the transpose is $\dim(\text{Nul}(A^T)) = 0$. This is the same as the number of free columns of A^T which means that every column of A^T is independent, or $\dim(\text{Col}(A^T)) = m$, the number of columns of A^T. This says that $m = \dim(\text{Row}(A))$. In other words, every row of the echelon form of A must be a pivot row, which form the basis of $\text{Row}(A)$. But this means that every row of the row reduction of the augmented matrix $[A, \mathbf{b}]$ to echelon form has every row a pivot row. But this means we can solve $A\mathbf{x} = \mathbf{b}$ for every $\mathbf{b} \in \mathbf{R}^m$.

   Now assume that we can solve $A\mathbf{x} = \mathbf{b}$ for every $\mathbf{b} \in \mathbf{R}^m$. This means that the row reduction of the augmented matrix $[A, \mathbf{b}]$ to echelon form has a pivot in every row (otherwise there is a zero row that says zero equals a linear combination of the b_j's so not all $\mathbf{b}$ make a consistent system). But this says that there are m basis elements in $\text{Row}(A)$ so $\dim(\text{Row}(A)) = m$. Thus every one of the m columns of A^T is linearly independent. It follows that the only dependence relation $A^T\mathbf{c} = \mathbf{0}$ is the trivial one $\mathbf{c} = \mathbf{0}$ thus $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

8. *In $\mathbb{P}_2$, find the change of coordinates matrix from the basis $\mathcal{B} = \{1-3t^2, 2+t-5t^2, 1+2t\}$ to the standard basis. Then write t^2 as a linear combination of the polynomials in $\mathcal{B}$. (Problem 244[14] of the text.)*

   The standard basis is $\mathcal{C} = \{1, t, t^2\}$. By the theorem, the change of basis matrix is given by finding coordinates of the basic vectors in one basis in terms of the other basis.

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \left[[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}, [\mathbf{b}_3]_{\mathcal{C}}\right] = \left[[1-3t^2]_{\mathcal{C}}, [2+t-5t^2]_{\mathcal{C}}, [1+2t]_{\mathcal{C}}\right] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$$

   To find the coefficients of $[t^2]_{\mathcal{B}}$ we solve

$$P_{\mathcal{C}\leftarrow\mathcal{B}}[t^2]_{\mathcal{B}} = [t^2]_{\mathcal{C}}.$$

   Row reduce the augmented matrix $\left[P_{\mathcal{C}\leftarrow\mathcal{B}}, [t^2]_{\mathcal{C}}\right]$.

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

   Hence $x_3 = 1$, $x_2 = -2x_3 = -2$ and $x_1 = -2x_2 - x_3 = -2(-2) - 1 = 3$. Then

$$[t^2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

   so

$$t^2 = 3\mathbf{b}_2 - 2\mathbf{b}_1 + \mathbf{b}_3 = 3(1-3t^2) - 2(2+t-5t^2) + (1+2t).$$

9. *Let $\mathcal{B} = \{\mathbf{x}_0, \ldots, \mathbf{x}_6\}$ and $\mathcal{C} = \{\mathbf{y}_0, \ldots, \mathbf{y}_6\}$ where $\mathbf{x}_k = \cos^k t$ and $\mathbf{y}_k = \cos kt$. Let $H = \operatorname{span}(\mathcal{B})$. Show that $\mathcal{C}$ is another basis for H. Set $P = \big[[\mathbf{y}_0]_{\mathcal{B}}, \ldots, [\mathbf{y}_6]_{\mathcal{B}}\big]$ and calculate P^{-1}. Explain why the columns of P^{-1} are the $\mathcal{C}$-coordinate vectors of $\mathbf{x}_0, \ldots, \mathbf{x}_6$. Then use these coordinate vectors to write trigonometric identities that express powers of $\cos t$ in terms of the functions in $\mathcal{C}$. (Text problems 230[34] and 244[17].)*

   We use the Pythagorean identity $\sin^2 t + \cos^2 t = 1$ and the addition formulæ

$$\cos(A+B) = \cos A \cos B - \sin A \sin B; \qquad \sin(A+B) = \sin A \cos B + |cos A \sin B$$

   This thus we get the equations expressing the $\mathcal{C}$ in terms of $\mathcal{B}$.

$$\begin{aligned}
\cos 2t &= \cos^2 t - \sin^2 t = \cos^2 t - (1 - \cos^2 t) = 2\cos^2 t - 1 \\
\cos 3t &= \cos 2t \cos t - \sin 2t \sin t = (2\cos^2 t - 1)\cos t - 2\sin^2 t \cos t \\
&= 2\cos^3 t - \cos t - 2(1 - \cos^2 t)\cos t = 4\cos^3 t - 3\cos t \\
\sin 3t &= \sin 2t \cos t + \cos 2t \sin t = 2 \sin t \cos^2 t + (2\cos^2 t - 1)\sin t \\
&= (4\cos^2 t - 1)\sin t \\
\cos 4t &= 2\cos^2 2t - 1 = 2(2\cos^2 t - 1)^2 - 1 = 8\cos^4 t - 8\cos^2 t + 1 \\
\cos 5t &= \cos 2t \cos 3t - \sin 2t \sin 3t \\
&= (2\cos^2 t - 1)(4\cos^3 t - 3\cos t) - (2\cos t \sin t)(4\cos^2 t - 1)\sin t \\
&= (2\cos^2 t - 1)(4\cos^3 t - 3\cos t) - (8\cos^3 t - 2\cos t)(1 - \cos^2 t) \\
&= 8\cos^5 t - 10\cos^3 t + 3\cos t + 8\cos^5 t - 10\cos^3 t + 2\cos t \\
&= 16\cos^5 t - 20\cos^3 t + 5\cos t \\
\cos 6t &= 2\cos^2 3t - 1 = 2(4\cos^3 t - 3\cos t)^2 - 1 = 32\cos^6 t - 48\cos^4 t + 18\cos^2 t - 1
\end{aligned}$$

   These formulas say that each function in $\mathcal{C}$ is a linear combination of the vectors in $\mathcal{B}$. Thus $\operatorname{span}(\mathcal{C}) \subset H$. However, the functions in $\mathcal{C}$ are linearly independent, thus $\mathcal{C}$ is a basis for $\operatorname{span}(\mathcal{C})$ making it seven dimensional, thus must equal H.

   Reading off the coefficients starting from $\cos(0 \cdot t) = 1$ and $\cos(1 \cdot t) = \cos t$ we find

$$P = P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & -8 & 0 & 18 \\ 0 & 0 & 0 & 4 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & -48 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 32 \end{pmatrix}$$

We invert using the usual augmented matrix $[P, I]$ and row reduce.

$$\left[\begin{array}{ccccccccccccccc} 1 & 0 & -1 & 0 & 1 & 0 & -1 & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 & & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -8 & 0 & 18 & & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & -20 & 0 & & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & -48 & & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 & & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 32 & & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccccccccccccc} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 & 9 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -5 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -6 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} \end{array}\right]$$

$$\to \left[\begin{array}{cccccccccccccc} 1 & 0 & 0 & 0 & -3 & 0 & 8 & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -10 & 0 & 0 & 1 & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 & 9 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -5 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -6 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} \end{array}\right]$$

$$\to \left[\begin{array}{ccccccc|ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & -10 & 1 & 0 & \frac{1}{2} & 0 & \frac{3}{8} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{3}{4} & 0 & \frac{5}{8} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -15 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{5}{16} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -6 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} \end{array}\right]$$

$$\to \left[\begin{array}{ccccccc|ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{3}{8} & 0 & \frac{5}{16} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{3}{4} & 0 & \frac{5}{8} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{15}{32} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{5}{16} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & \frac{3}{16} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} \end{array}\right]$$

Thus

$$P^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{3}{8} & 0 & \frac{5}{16} \\ 0 & 1 & 0 & \frac{3}{4} & 0 & \frac{5}{8} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{15}{32} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{5}{16} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & \frac{3}{16} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} \end{pmatrix}$$

Because

$$P = P_{\mathcal{B}\leftarrow\mathcal{C}} = \big[[\mathbf{y}_0]_{\mathcal{B}}, \ldots, [\mathbf{y}_6]_{\mathcal{B}}\big]$$

for any $\mathbf{x} \in H$ we have

$$P[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{B}}.$$

Premultiplying by the inverse we find

$$P^{-1}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$$

thus, P^{-1} is the unique change of basis matrix, whose columns are basic vectors in coordinates

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = P^{-1} = P = \big[[\mathbf{x}_0]_{\mathcal{C}}, \dots, [\mathbf{x}_6]_{\mathcal{C}}\big].$$

In other words, the columns of P^{-1} are $\mathcal{C}$-coordinate vectors of the $\mathbf{x}_j$'s. In terms of expressing the $\mathcal{C}$ functions, the powers of $\cos t$, in terms of the $\mathcal{B}$ functions we get

$$\begin{aligned}
\cos^2 t &= \frac{1}{2} + \frac{1}{2}\cos 2t \\
\cos^3 t &= \frac{3}{4}\cos t + \frac{1}{4}\cos 3t \\
\cos^4 t &= \frac{3}{8} + \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t \\
\cos^5 t &= \frac{5}{8}\cos t + \frac{5}{16}\cos 3t + \frac{1}{32}\cos 5t \\
\cos^6 t &= \frac{5}{16} + \frac{15}{32}\cos 2t + \frac{3}{16}\cos 4t + \frac{1}{32}\cos 6t
\end{aligned}$$

10. *Determine whether the matrix can be diagonalized. If it can, do so. (text problem 288[33].)*

$$A = \begin{pmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{pmatrix}$$

Expand according to the third column to find the characteristic polynomial.

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} -6-\lambda & 4 & 0 & 9 \\ -3 & -\lambda & 1 & 6 \\ -1 & -2 & 1-\lambda & 0 \\ -4 & 4 & 0 & 7-\lambda \end{vmatrix} \\
&= -\begin{vmatrix} -6-\lambda & 4 & 9 \\ -1 & -2 & 0 \\ -4 & 4 & 7-\lambda \end{vmatrix} + (1-\lambda)\begin{vmatrix} -6-\lambda & 4 & 9 \\ -3 & -\lambda & 6 \\ -4 & 4 & 7-\lambda \end{vmatrix} \\
&= -[2(6+\lambda)(7-\lambda) - 36 + 4(7-\lambda) - 72] \\
&\quad + (1-\lambda)[\lambda(6+\lambda)(7-\lambda) - 96 - 72 + 24(6+\lambda) + 12(7-\lambda) - 36\lambda] \\
&= -[-2\lambda^2 - 2\lambda + 4] + (1-\lambda)[-\lambda^3 + \lambda^2 + 18\lambda + 24] \\
&= \lambda^4 - 2\lambda^3 + -15\lambda^2 - 4\lambda + 20 \\
&= (\lambda - 1)(\lambda - 5)(\lambda + 2)^2.
\end{aligned}$$

We find the eigenvectors by solving homogeneous systems. For $\lambda_1 = 1$, we row reduce $A - \lambda_1 I$

$$\begin{pmatrix} -7 & 4 & 0 & 9 \\ -3 & -1 & 1 & 6 \\ -1 & -2 & 0 & 0 \\ -4 & 4 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 \\ -3 & -1 & 1 & 6 \\ -7 & 4 & 0 & 9 \\ -4 & 4 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 5 & 1 & 6 \\ 0 & 18 & 0 & 9 \\ 0 & 12 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & \frac{7}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So if $x_4 = 2$, $x_3 = -7$, $x_2 = -1$ and $x_1 = 2$. For $\lambda_2 = 5$, we row reduce $A - \lambda_2 I$

$$\begin{pmatrix} -11 & 4 & 0 & 9 \\ -3 & -5 & 1 & 6 \\ -1 & -2 & -4 & 0 \\ -4 & 4 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & 0 \\ -3 & -5 & 1 & 6 \\ -11 & 4 & 0 & 9 \\ -4 & 4 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 13 & 6 \\ 0 & 26 & 44 & 9 \\ 0 & 12 & 16 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 13 & 6 \\ 0 & 0 & -294 & -147 \\ 0 & 0 & -140 & -70 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 13 & 6 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So if $x_4 = 2$, $x_3 = -1$, $x_2 = -13x_3 - 6x_4 = 1$ and $x_1 = -2x_2 - 4x_3 = 2$. For $\lambda_2 = -2$, we row reduce $A - \lambda_2 I$

$$\begin{pmatrix} -4 & 4 & 0 & 9 \\ -3 & 2 & 1 & 6 \\ -1 & -2 & 3 & 0 \\ -4 & 4 & 0 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -3 & 0 \\ -3 & 2 & 1 & 6 \\ -4 & 4 & 0 & 9 \\ -4 & 4 & 0 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 8 & -8 & 6 \\ 0 & 12 & -12 & 9 \\ 0 & 12 & -12 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 4 & -4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So if $x_3 = 1$ and $x_4 = 0$ then $x_2 = 1$ and $x_1 = -2x_2 + 3x_3 = 1$. Also if $x_3 = 0$ and $x_4 = 4$ then $x_2 = -3$ and $x_1 = -2x_2 + 3x_3 = 6$. We build our diagonalizing matrix

$P = [\mathbf{v}_1, \mathbf{v}_2\mathbf{v}_3, \mathbf{v}_4]$ and check $AP = PD$.

$$AP = \begin{pmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{pmatrix} \begin{pmatrix} 2 & -2 & 1 & 6 \\ -1 & 1 & 1 & -3 \\ -7 & -1 & 1 & 0 \\ 2 & 2 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 10 & -2 & -12 \\ -1 & 5 & -2 & 6 \\ -7 & -5 & -2 & 0 \\ 2 & 10 & 0 & -8 \end{pmatrix}$$

$$PD = \begin{pmatrix} 2 & -2 & 1 & 6 \\ -1 & 1 & 1 & -3 \\ -7 & -1 & 1 & 0 \\ 2 & 2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 10 & -2 & -12 \\ -1 & 5 & -2 & 6 \\ -7 & -5 & -2 & 0 \\ 2 & 10 & 0 & -8 \end{pmatrix}$$

11. *Show that if the $n \times n$ matrix A has n linearly independent eigenvectors, then so does A^T. (Use the Diagonalization Theorem.) (Text problem 288[28].)*

    Recall the Diagonalization Theorem which says that the $n \times n$ matrix A is diagonalizable if and only if A has n independent eigenvectors.

    Now since A is diagonalizable, the Diagonalization Theorem implies that A has n independent eigenvectors. Put these vectors in as columns $P = [\mathbf{p}_1, \ldots, \mathbf{p}_n]$. Then

$$AP = PD$$

    where D is the diagonal matrix of eigenvalues corresponding to the $\mathbf{p}_i$'s. From this equation we can see that A^T is diagonalizable as well. Namely, taking the transpose of the equation we find

$$(AP)^T = P^T A^T = (PD)^T = D^T P^T = DP^T$$

    because for diagonal matrices, $D^T = D$. Rearranging,

$$Q^{-1}AQ = P^T A^T (P^T)^{-1} = D$$

    which says A^T is diagonalizable by the matrix $Q = (P^T)^{-1}$. Again using the Diagonalization Theorem, the fact that A^T is diagonalizable implies that A^T has n linearly independent eigenvectors as desired. In fact, the eigenvectors are the columns of $Q = (P^T)^{-1}$ and the eigenvalues are the same as for A, namely the diagonals of D.

12. *Suppose A is an invertible $n \times n$ matrix which is similar to B. Show that then B is invertible and A^{-1} is similar to B^{-1}. (Text problem 295[19].)*

    First we show that B is invertible. Being similar to A implies that there is an invertible matrix P such that

$$P^{-1}AP = B.$$

    But the matrices on the left side are invertible so we consider

$$C = P^{-1}A^{-1}P$$

    We claim that C is the inverse of B, therefore B is invertible. Indeed

$$CB = P^{-1}A^{-1}P)(P^{-1}AP) = P^{-1}A^{-1}PP^{-1}AP = P^{-1}A^{-1}AP = P^{-1}P = I,$$

thus C is a left inverse of B. But a left inverse is an inverse by the Invertible Matrix Theorem.

Now taking the inverse of the first equation yields

$$(P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P = B^{-1}$$

which says that B^{-1} is similar to A^{-1} via the same matrix P, as to be shown.

13. *For the matrix A, show that it is similar to a composition of a rotation and a scaling. (Text problem 302[15].)*

$$A = \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$$

First find the complex eigenvalue and eigenvector.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 5 \\ -2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 10 = \lambda^2 - 4\lambda + 13$$

The eigenvalues are thus given by the quadratic formula

$$\lambda, \bar{\lambda} = \frac{4 \pm \sqrt{16 - 4 \cdot 13}}{2} = 2 \pm 3i.$$

Now $\lambda = 2 - 3i$ so $a = 2$ and $b = 3$. Let's find the complex eigenvector for $\lambda = 2 - 3i$ by inspection

$$0 = (A - \lambda I)\mathbf{v} = \begin{pmatrix} -1+3i & 5 \\ -2 & 1+3i \end{pmatrix} \begin{pmatrix} 1+3i \\ 2 \end{pmatrix}$$

Consider the matrix

$$P = [\Re e\, \mathbf{v}, \Im m\, \mathbf{v}] = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$$

We claim that this matrix establishes the similarity to rotation composed with dilation. Indeed

$$AP = \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 11 & 3 \\ 4 & -6 \end{pmatrix}$$

$$P \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 11 & 3 \\ 4 & -6 \end{pmatrix}$$

are the same, as claimed. Let $r = \sqrt{a^2 + b^2} = \sqrt{2^2 + 3^2} = \sqrt{13}$. Then

$$P^{-1}AP = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} = \sqrt{13} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

where the scaling is dilation by a factor $r = \sqrt{13}$ and the rotation is by an angle ϕ where

$$\cos\phi = \frac{2}{\sqrt{13}}, \qquad \sin\phi = \frac{3}{\sqrt{13}}.$$

This says $\phi = 0.9827937 + 2\pi k$ radians for k an integer.

14. *Let A be an $m \times n$ matrix. Show that*

$$(\operatorname{Row} A^T)^\perp = \operatorname{Nul} A^T; \qquad (\operatorname{Col} A^T)^\perp = \operatorname{Nul} A$$

The equality of sets is shown by establishing both "$\subset$" and "$\supset$." Technically, vectors in $\operatorname{Nul} A^T$ are $m \times 1$ column vectors whereas vectors in $(\operatorname{Row} A^T)^\perp$ are $1 \times m$ row vectors, but we identify these via transpose. To show $(\operatorname{Row} A^T)^\perp \subset \operatorname{Nul} A^T$ choose an arbitrary $\mathbf{v} \in (\operatorname{Row} A^T)^\perp$. This means $\mathbf{v}$ is orthogonal to all vectors in $\operatorname{Row} A^T$. However, this space is spanned by the rows of A^T which are the transposes of the columns of A, thus $\operatorname{Row} A^T = \operatorname{span}\{\mathbf{a}_1^T, \ldots, \mathbf{a}_n^T\}$. In particular $\mathbf{a}_i^T \bullet \mathbf{v} = 0$ for $i = 1, \ldots, n$. Viewing $\mathbf{v}$ as a column vector in $\mathbf{R}^m$, we can write this as matrix multiplication of a $1 \times m$ matrix times an $m \times 1$ matrix $\mathbf{a}_i^T \mathbf{v}^T = 0$. But this occurs in matrix multiplication

$$A^T \mathbf{v}^T = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \mathbf{v}^T = \begin{bmatrix} \mathbf{a}_1^T \mathbf{v}^T \\ \vdots \\ \mathbf{a}_n^T \mathbf{v}^T \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

showing that $\mathbf{v}^T \in \operatorname{Nul}(A^T)$, so $(\operatorname{Row} A^T)^\perp \subset \operatorname{Nul} A^T$ as to be shown.

Conversely, if $\mathbf{v}^T \in \operatorname{Nul}(A^T)$ then $A^T \mathbf{v}^T = \mathbf{0}$ which says by the row-column rule for multiplication that $\mathbf{a}_i^T \mathbf{v}^T = 0$ for every row of A^T. But this says the $\mathbf{v}$ is orthogonal to the vectors $\mathbf{a}_1^T, \ldots \mathbf{a}_n^T$. However, these vectors span the row space of A^T so that $\mathbf{v} \in (\operatorname{Row} A^T)^\perp$ so $\operatorname{Nul} A^T \subset (\operatorname{Row} A^T)^\perp$ as to be shown.

To show $(\operatorname{Col} A^T)^\perp \subset \operatorname{Nul} A$, choose an arbitrary vector $\mathbf{w} \in (\operatorname{Col} A^T)^\perp$. The column space of A^T is spanned by the transposes of the rows R_j of A, namely $\operatorname{Col} A^T = \operatorname{span}\{R_1^T, \ldots, R_m^T\}$. Hence $\mathbf{w}$ is orthogonal to these $R_j^T \bullet \mathbf{w} = 0$ for all $j = 1, \ldots, m$. Viewing as multiplication of $1 \times n$ matrix by a $n \times 1$ matrices, $R_j \mathbf{w} = 0$ for all $j = 1, \ldots, m$. By the row-column rule for multiplication, this says

$$A\mathbf{w} = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} \mathbf{w} = \begin{bmatrix} R_1 \mathbf{w} \\ \vdots \\ R_m \mathbf{w} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

Hence $\mathbf{w} \in \operatorname{Nul} A$ as to be shown so $(\operatorname{Col} A^T)^\perp \subset \operatorname{Nul} A$.

Conversely, if $\mathbf{w} \in \operatorname{Nul} A$, by the row-column rule for multiplication, this says $R_j \mathbf{w} = 0$ for all $j = 1, \ldots, m$. Thus $\mathbf{w}$ is orthogonal to the generating set of $\operatorname{Col} A^T = \operatorname{span}(R_1^T, \ldots, R_m^T)$. It follows that $\mathbf{w} \in (\operatorname{Col} A^T)^\perp$ as to be shown so $\operatorname{Nul} A \subset (\operatorname{Col} A^T)^\perp$.

15. *For the matrix A construct a matrix N whose column form a basis for* $\operatorname{Nul} A$ *and construct another matrix R whose rows form a basis for* $\operatorname{Row} A$. *Perform a matrix computation that illustrates a fact from Theorem 3. (Text problem 338[34].)*

$$A = \begin{pmatrix} -6 & 3 & -27 & -33 & -13 \\ 6 & -5 & 25 & 28 & 14 \\ 8 & -6 & 34 & 38 & 18 \\ 12 & -10 & 50 & 41 & 23 \\ 14 & -21 & 49 & 29 & 33 \end{pmatrix}$$

$$\begin{pmatrix} -6 & 3 & -27 & -33 & -13 \\ 6 & -5 & 25 & 28 & 14 \\ 8 & -6 & 34 & 38 & 18 \\ 12 & -10 & 50 & 41 & 23 \\ 14 & -21 & 49 & 29 & 33 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{6} \\ 6 & -5 & 25 & 28 & 14 \\ 8 & -6 & 34 & 38 & 18 \\ 12 & -10 & 50 & 41 & 23 \\ 14 & -21 & 49 & 29 & 33 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -\frac{1}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{6} \\ 0 & -2 & -2 & -5 & 1 \\ 0 & -2 & -2 & -6 & \frac{2}{3} \\ 0 & -4 & -4 & -25 & -3 \\ 0 & -14 & -14 & -48 & \frac{8}{3} \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{6} \\ 0 & -2 & -2 & -5 & 1 \\ 0 & 0 & 0 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & -15 & -5 \\ 0 & 0 & 0 & -13 & -\frac{13}{3} \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -\frac{1}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{6} \\ 0 & -2 & -2 & -5 & 1 \\ 0 & 0 & 0 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus x_3 and x_5 are free. It follows that $x_4 = -\frac{1}{3}x_5$, $x_2 = -x_3 - \frac{5}{2}x_4 + \frac{1}{2}x_5 = -x_3 + \frac{4}{3}x_5$ and $x_1 = \frac{1}{2}x_2 - \frac{9}{2}x_3 - \frac{11}{2}x_4 - \frac{13}{6}x_5 = -5x_3 + \frac{1}{3}x_5$. The null space and N whose columns are a basis of the null space is thus

$$\operatorname{Nul} A = \left\{ \begin{pmatrix} -5x_3 - \frac{1}{3}x_5 \\ -x_3 + \frac{4}{3}x_5 \\ x_3 \\ -\frac{1}{3}x_5 \\ x_5 \end{pmatrix} : x_3, x_5 \in \mathbf{R} \right\}; \qquad N = \begin{pmatrix} -5 & \frac{1}{3} \\ -1 & \frac{4}{3} \\ 1 & 0 \\ 0 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix}.$$

A matrix R whose rows are a basis of the row space is

$$R = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{6} \\ 0 & -2 & -2 & -5 & 1 \\ 0 & 0 & 0 & -1 & -\frac{1}{3} \end{pmatrix}$$

Theorem 3 says that $(\operatorname{Row} A)^\perp = \operatorname{Nul} A$. This is equivalent to all the basis vectors of $\operatorname{Row} A$, namely the pivotal row vectors of the echelon U_i, $i = 1, 2, 3$, are orthogonal to all basis vectors of $\operatorname{Nul} A$, namely the columns $\mathbf{n}_j$, $j = 1, 2$. Using the matrix multiplication version, $U_i \mathbf{n}_j = 0$ for all $i = 1, 2, 3$ and $j = 1, 2$. We can check by matrix multiplication

$$RN = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{6} \\ 0 & -2 & -2 & -5 & 1 \\ 0 & 0 & 0 & -1 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} -5 & \frac{1}{3} \\ -1 & \frac{4}{3} \\ 1 & 0 \\ 0 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

16. *Verify the parallelogram law for vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbf{R}^n$. *(Text problem 338[24].)*

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \bullet (\mathbf{u} + \mathbf{v}) + (\mathbf{u} - \mathbf{v}) \bullet (\mathbf{u} - \mathbf{v}) \\ &= (\mathbf{u} \bullet \mathbf{u} + \mathbf{u} \bullet \mathbf{v} + \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v}) \\ &\quad + (\mathbf{u} \bullet \mathbf{u} - \mathbf{u} \bullet \mathbf{v} - \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v}) \\ &= 2\mathbf{u} \bullet \mathbf{u} + 2\mathbf{v} \bullet \mathbf{v} \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2. \end{aligned}$$