Math 2270 § 1. **Second Midterm Exam** Name: Solutions
Treibergs $\sigma\tau$ October 21, 2015

1. (a) *Solve for x_2 using Cramer's Rule.*

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$x_2 = \frac{\begin{vmatrix} 0 & 3 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 0 & 2 \end{vmatrix}} = \frac{0+3+0-0-12-2}{0+1+0-0-4-6} = \frac{-11}{-9} = \frac{11}{9}.$$

(b) *Using determinants, find the area of the triangle whose vertices in the plane are*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$$

The area of the triangle is half the area of the parallelogram whose sides are $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ thus

$$\begin{aligned} A &= \frac{1}{2}|\det([\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1]) = \frac{1}{2}\left| \det\begin{pmatrix} 2-1 & 4-1 \\ 5-1 & 9-1 \end{pmatrix} \right| \\ &= \frac{1}{2}\left| \det\begin{pmatrix} 1 & 3 \\ 4 & 8 \end{pmatrix} \right| = \frac{1}{2}|1\cdot 8 - 4\cdot 3| = \frac{1}{2}|-4| = 2. \end{aligned}$$

2. *Find the LU factorization of A and check your answer.*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -4 & 0 & -4 \end{pmatrix}$$

Do elementary row operations and record which ones are used.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -4 & 0 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 8 & 8 \end{pmatrix} \qquad \begin{matrix} \text{Replace } R_2 \text{ by } R_2 - 2R_1 \\ \text{Replace } R_3 \text{ by } R_3 + 4R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -8 \end{pmatrix} \qquad \text{Replace } R_3 \text{ by } R_3 + 8R_2$$

The matrices L and U are recovered from this. We check

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -8 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -4 & 0 & -4 \end{pmatrix}.$$

3. *Suppose that row reduction of A yields the following echelon matrix. For each of the problems, answer the question with justification or explain why the problem can't be done from the information given. Find* $\mathrm{Col}(A)$*. Find* $\det(A)$*. Find* $\mathrm{Nul}(A)$*. Find* $\mathrm{rank}(A)$*.*

$$U = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 0 & 7 & 0 \\ 0 & 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{Col}(A)$ spanned by the columns of A corresponding to pivot columns of U. However, the matrix A is not given so we cannot say what $\mathrm{Col}(A)$ is.

The row reduced matrix U has a zero row, thus A is not invertible by the Invertible Matrix Theorem (RREF is not the identity matrix). Therefore $\det(A) = 0$. Alternately, there are elementary row operation invertible matrices E_i such that repeated application yields the echelon matrix.

$$E_p \cdot E_{p-1} \cdots E_1 \cdot A = U$$

so

$$\begin{aligned} \det(A) &= \det(E_1^{-1} \cdots E_p^{-1} \cdot U) \\ &= \det(E_1^{-1}) \cdots \det(E_p^{-1}) \cdot \det(U) \\ &= \det(E_1^{-1}) \cdots \det(E_p^{-1}) \cdot 0 = 0 \end{aligned}$$

because $\det(U)$ is the product of entries down the diagonal of U which is zero.

Since U is row equivalent to A, $A\mathbf{x} = \mathbf{0}$ has the same solution set as $U\mathbf{x} = \mathbf{0}$, namely $\mathrm{Nul}(A)$. x_3 and x_5 are the only free variables, so they can be set to any real values. Solving the equations we find

$$x_4 = -\frac{9}{8}x_5; \quad x_2 = -\frac{7}{6}x_4 = \frac{21}{16}x_5;$$
$$x_1 = -2x_2 - 3x_3 - 4x_4 - 5x_5 = -\frac{21}{8}x_5 - 3x_3 + \frac{9}{2}x_5 - 5x_5 = -3x_3 - \frac{25}{8}x_5$$

so the solution is

$$\mathrm{Nul}(A) = \left\{ \begin{pmatrix} -3x_3 - \frac{25}{8}x_5 \\ \frac{21}{16}x_5 \\ x_3 \\ -\frac{9}{8}x_5 \\ x_5 \end{pmatrix} : x_3, x_5 \in \mathbf{R} \right\} = \mathrm{span}\left\{ \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{25}{8} \\ \frac{21}{16} \\ 0 \\ -\frac{9}{8} \\ 1 \end{pmatrix} \right\}.$$

The rank of a matrix is the dimension of its column space, which is the same as the number of pivot variables which correspond to independent columns. Thus $\mathrm{rank}(A) = 3$.

4. *Let $H = \mathrm{Col}(A)$ be the column space of the matrix A. Define: $\mathcal{B}$ is a basis for the subspace $H \subset \mathbf{R}^4$. Is $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ a basis for H? Prove your answer.*

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A finite subset of a vector space $\mathcal{B} \subset \mathbf{R}^4$ is a *basis for the subspace* H if it is linearly independent and it spans H.

$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for H.

First we check linear independence. We see if there are nonzero solutions of the dependence relation

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The fourth equation says $c_3 = 0$. The third $c_2 = 0$ and the second $c_1 = 0$. Since only the trivial combination makes zero, the vectors $\mathcal{B}$ are linearly independent.

To see that $\mathcal{B}$ spans H, we have to show that each column of A is a linear combination of the $\mathbf{b}_i$'s. Then any vector in $H = \text{Col}(A)$ which is a linear combination of the columns of A is in turn a linear combnation of the $\mathbf{b}_i$'s. Inspection shows that

$$\mathbf{b}_1 + \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{a}_1, \qquad \mathbf{b}_2 + \mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{a}_2,$$

$$\mathbf{b}_1 + \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{a}_3, \qquad \mathbf{b}_3 - \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{a}_4.$$

Thus the columns of A are in the span of $\mathcal{B}$, hence Col(A) is also in the span of $\mathcal{B}$.

Alternately, another check that the $\mathbf{a}_i$'s are all in span$(\mathcal{B})$ is showing we can solve for the 3×4 matrix c_{ij} such that

$$c_{1j}\mathbf{b}_1 + c_{2j}\mathbf{b}_1 + c_{3j}\mathbf{b}_1 = \mathbf{a}_j$$

for all $j = 1, \ldots, 4$. In other words such that $BC = A$. But by row reducing the augmented matrix $[B, A]$ we see that

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Back substituting we find that we have the solution

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Thus all the columns of A are linear combinations of the $\mathbf{b}_i$'s.

5. *Let* $T : \mathbb{V} \to \mathbb{W}$ *be a mapping between the vector spaces* $\mathbb{V}$ *and* $\mathbb{W}$*. Define:* T *is a linear transformation from* $\mathbb{V}$ *to* $\mathbb{W}$*. Define:* $\mathrm{Nul}(T)$*, the null space of the transformation. Assuming that* T *is a linear transformation, show that* $\mathrm{Nul}(T)$ *is a vector subspace.*

The mapping $T : \mathbb{V} \to \mathbb{W}$ is a *linear transformation* if for every $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ there holds $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u})+T(\mathbf{v})$ and for every $\mathbf{u} \in \mathbb{V}$ and for every $c \in \mathbf{R}$ there holds $T(c\mathbf{u}) = cT(\mathbf{u})$.

The null space of T is the set of vectors in $\mathbb{V}$ that are annihilated by T, namely,

$$\mathrm{Nul}(T) = \{\mathbf{v} \in \mathbb{V} : T(\mathbf{v}) = \mathbf{0}\}.$$

To show that $\mathrm{Nul}(T)$ is a vector subspace of $\mathbb{V}$ for linear T we have to show that it contains zero vector and that it is closed under vector addition and scalar multiplication. To see zero is in the subspace, observe that $T(\mathbf{0}) = \mathbf{0}$ for any linear map, so $\mathbf{0} \in \mathrm{Nul}(T)$. For example, this is true because for any vector $\mathbf{v} \in \mathbb{V}$ we have by linearity of T,

$$T(\mathbf{0}) = T(0 \cdot \mathbf{v}) = 0 \cdot T(\mathbf{v}) = \mathbf{0}.$$

For the second condition, choose any $\mathbf{u}, \mathbf{v} \in \mathrm{Nul}(T)$. These vectors satisfy the condition to be in $\mathrm{Nul}(T)$, namely $T(\mathbf{u}) = 0$ and $T(\mathbf{v}) = 0$. Then, by linearity of T, their sum

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus the sum $\mathbf{u} + \mathbf{v}$ satisfies the condition to belong to $\mathrm{Nul}(T)$. For the third condition, choose any $\mathbf{u} \in \mathrm{Nul}(T)$ and any $c \in \mathbf{R}$. This vector satisfies the condition to be in $\mathrm{Nul}(T)$, namely $T(\mathbf{u}) = 0$. Then, by linearity of T, the scalar product

$$T(c\mathbf{u}) = cT(\mathbf{u}) = c \cdot \mathbf{0} = \mathbf{0}.$$

Thus the scalar product $c\mathbf{u}$ satisfies the condition to belong to $\mathrm{Nul}(T)$. This completes the argument that $\mathrm{Nul}(T)$ contains zero and is closed under both vector addition and scalar multiplication, and so is a vector subspace of $\mathbb{V}$.