Math 2270 § 1. **Second Midterm Exam** Name: Practice Problems

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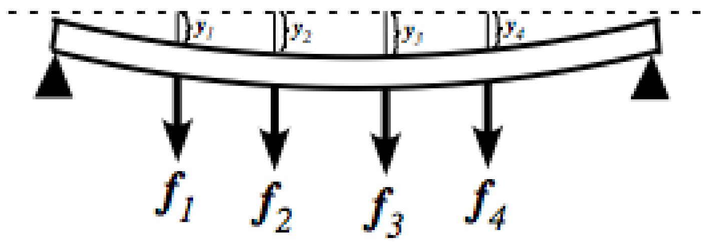
1. *Suppose that a horizontal elastic beam is supported at each end and is subjected to forces at four points. Let the flexibility matrix D be given in centimeters per newton. Suppose forces of* 20, 35, 40 *and* 15 *newtons are applied at the four points. What will be the deflection at the four points? What forces would need to be applied so that the deflections are* 0, 0.1, 0.1 *and* 0 *cm at the corresponding points?*



$$D = \begin{pmatrix} .040 & .021 & .015 & .012 \\ .030 & .042 & .030 & .024 \\ 0.020 & .028 & .045 & .032 \\ .010 & .014 & .022 & .048 \end{pmatrix}.$$

The deflections are given by $\mathbf{y} = D\mathbf{f}$. Inserting given forces we find using the program ©**R**,

$$\mathbf{y} = D\mathbf{f} = \begin{pmatrix} .040 & .021 & .015 & .012 \\ .030 & .042 & .030 & .024 \\ 0.020 & .028 & .045 & .032 \\ .010 & .014 & .022 & .048 \end{pmatrix} \begin{pmatrix} 10 \\ 35 \\ 40 \\ 15 \end{pmatrix} = \begin{pmatrix} 1.915 \\ 3.330 \\ 3.460 \\ 2.190 \end{pmatrix}.$$

The stiffness matrix is D^{-1}. The forces for the given deflections are $\mathbf{f} = D^{-1}\mathbf{y} =$

$$\begin{pmatrix} 40.00000 & -20.0000000 & 0.0000000 & 0.0000000 \\ -28.57143 & 57.1900047 & -26.874120 & -3.536068 \\ 0.00000 & -26.4026403 & 49.504950 & -19.801980 \\ 0.00000 & -0.4125413 & -14.851490 & 30.940590 \end{pmatrix} \begin{pmatrix} 0 \\ 0.1 \\ 0.1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2.000000 \\ 3.031589 \\ 2.310231 \\ -1.526403 \end{pmatrix}$$

2. *Suppose that $AB = AC$ where B and C are $p \times n$ matrices and A is an invertible $n \times n$ matrix. Show that $B = C$. Is it true in general when A is not invertible?*

   Since A is invertible, the inverse matrix A^{-1} exists. Premultiplying the equation by A^{-1} we find

$$B = IB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}(AC) = (A^{-1}A)C = IC = C$$

   as desired. For example, the matrix $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ is not invertible because its determinant is $ad - bc = 2 \cdot 2 - 4 \cdot 1 = 0$ but there are different C and B that yield the same product

$$AB = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 22 \\ 2 & 11 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 7 \\ 1 & 2 \end{pmatrix} = AC$$

3. *Explain why the columns of an $n \times n$ matrix are linearly independent and span $\mathbf{R}^n$ when A is invertible.*

   Recall that the column vectors $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are linearly independent if the only solution of the vector equation

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

   is the trivial solution. But since A is invertible, if we premultiply by the inverse we find

$$\mathbf{x} = I\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{0} = \mathbf{0}.$$

   Thus the solution $\mathbf{x}$ must be zero, showing that the columns are linearly independent.

   Recall that the columns of A span $\mathbf{R}^n$ if every vectror $\mathbf{b} \in \mathbf{R}^2$ can be realized as a linear combination of the columns. This means, for every $\mathbf{b}$ we can solve the equation

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

   But since A is invertible, if we premultiply by the inverse we find

$$\mathbf{x} = I\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}.$$

   Thus there is solution $\mathbf{x}$ giving the weights for a linear combination of the columns that equals $\mathbf{b}$. Since any $b \in \mathbf{R}^n$ is the linear combination of the column vectors, $\mathbf{R}^n$ is spanned by the columns of A.

   Knowing one of these, we could have also deduced the other from the fact that n linearly independent vectors of $\mathbf{R}^n$ automatically span; or from the fact that n spanning vectors of $\mathbf{R}^n$ are automatically linearly independent.

4. *Find the inverse matrix of A. Check your answer.*

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 3 & -1 & -1 \end{pmatrix}$$

Make an augmented matrix $[A; I]$ and reduce to RREF

$$\begin{pmatrix} 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \quad \text{Swap } R_1 \text{ and } R_2$$

$$\to \begin{pmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & -4 & -3 & 1 & -2 & 0 \\ 0 & -7 & -7 & 0 & -3 & 1 \end{pmatrix} \quad \begin{matrix} \text{Replace } R_2 \text{ by } R_2 - 2R_1 \\ \text{Replace } R_3 \text{ by } R_3 - 3R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & -7 & -7 & 0 & -3 & 1 \end{pmatrix} \quad \text{Multiply } R_2 \text{ by } -\tfrac{1}{4}$$

$$\to \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{7}{4} & -\frac{7}{4} & \frac{1}{2} & 1 \end{pmatrix} \quad \begin{matrix} \text{Replace } R_1 \text{ by } R_1 - 2R_2 \\ \text{Replace } R_3 \text{ by } R_3 + 7R_2 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{3}{4} & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & -\frac{2}{7} & -\frac{4}{7} \end{pmatrix} \quad \text{Multiply } R_3 \text{ by } -\tfrac{4}{7}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & 0 & -1 & \frac{5}{7} & \frac{3}{7} \\ 0 & 0 & 1 & 1 & -\frac{2}{7} & -\frac{4}{7} \end{pmatrix} \quad \begin{matrix} \text{Replace } R_1 \text{ by } R_1 - \frac{1}{2}R_3 \\ \text{Replace } R_2 \text{ by } R_2 - \frac{3}{4}R_3 \end{matrix}$$

We check $A^{-1}A =$

$$\begin{pmatrix} 0 & \frac{1}{7} & \frac{2}{7} \\ -1 & \frac{5}{7} & \frac{3}{7} \\ 1 & -\frac{2}{7} & -\frac{4}{7} \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 3 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 3 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{7} & \frac{2}{7} \\ -1 & \frac{5}{7} & \frac{3}{7} \\ 1 & -\frac{2}{7} & -\frac{4}{7} \end{pmatrix} = AA^{-1}$$

5. *Guess the form of A^{-1} and prove $AA^{-1} = A^{-1}A = I$.*

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & \cdots & 0 \\ 1 & 2 & 3 & \cdots & 0 \\ \vdots & & & & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

The matrix A can be written in terms of elementary matrices. If $E(j, c)$ corresponds to multiplying R_j by c and $E(i, j, c)$ corresponds to replacing R_i by $R_i + cR_j$ then

$$\begin{aligned} A = {} & E(n,1,1) \cdots E(3,1,1)E(2,1,1) \\ & \cdot E(n,2,1) \cdots E(4,2,1)E(3,2,1) \\ & \cdot E(n,3,1) \cdots E(5,3,1)E(4,3,1) \\ & \qquad \cdots \\ & \cdot E(n,n-2,1)E(n-1,n-2,1) \\ & \cdot E(n,n-1,1) \\ & \cdot E(n,n) \cdots E(3,3)E(2,2) \end{aligned}$$

To see why,

$$E(n,n) \cdots E(3,3)E(2,2) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & n \end{pmatrix}$$

$$\begin{array}{c} E(n,n-2,1)E(n-1,n-2,1) \\ E(n,n-1,1) \cdot E(n,n) \cdots E(3,3)E(2,2) \end{array} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & n-2 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & n-2 & n-1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & n-2 & n-1 & n \end{pmatrix}$$

And so on to fill up A by going down each column in turn.
Using $E(j,c)^{-1} = E(j, \frac{1}{c})$ and $E(i,j,c)^{-1} = E(i,j,-c)$ the inverse is

$$\begin{aligned}
A^{-1} = E(2, \frac{1}{2})E(3, \frac{1}{3}) \cdots E(n, \frac{1}{n}) \\
\cdot E(n, n-1, -1) \\
\cdot E(n-1, n-2, -1)E(n, n-2, -1) \\
\cdots \\
\cdot E(4,3,-1)E(5,3,-1) \cdots E(n,3,-1) \\
\cdot E(3,2,-1)E(4,2,-1) \cdots E(n,2,-1) \\
\cdot E(2,1,-1)E(3,1,-1) \cdots E(n,1,-1) \\
= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{n-1} & \frac{1}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{n} & \frac{1}{n} \end{pmatrix}
\end{aligned}$$

To see why,

$$E(2,1,-1)E(3,1,-1) \cdots E(n,1,-1) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 1 & 0 \\ -1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} E(3,2,-1)E(4,2,-1) \cdots E(n,2,-1) \\ \cdot E(2,1,-1)E(3,1,-1) \cdots E(n,1,-1) \end{matrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -1 & 0 & \cdots & 1 & 0 \\ 0 & -1 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

And so on wiping out each lower -1 column

$$\begin{array}{c} E(n,n-1,-1) \\ \cdot E(n-1,n-2,-1)E(n,n-2,-1) \\ \cdots \\ \cdot E(4,3,-1)E(5,3,-1)\cdots E(n,3,-1) \\ \cdot E(3,2,-1)E(4,2,-1)\cdots E(n,2,-1) \\ \cdot E(2,1,-1)E(3,1,-1)\cdots E(n,1,-1) \end{array} = \begin{pmatrix} 1 & 0 & 0 & \cdots & & 00 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 1 \end{pmatrix}$$

This is multiplied by diag $\left[1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right]$ to get A^{-1}. One checks that $A^{-1}A = I$ and $AA^{-1} = I$.

6. *Explain why the columns of the $n \times n$ matrix A span $\mathbf{R}^n$ whenever the columns of A are linearly independent.*

   Both conditions happen if and only if A is invertible, by the Invertible Matrix Theorem.

   A less glib explanation goes as follows: Suppose the columns of A span $\mathbf{R}^n$. Then for any $\mathbf{b} \in \mathbf{R}^n$, we find a linear combination of columns that equals $\mathbf{b}$, in other words for every $\mathbf{b}$ you can solve $A\mathbf{x} = \mathbf{b}$. If the augmented matrix $[A, \mathbf{b}]$ is row-reduced, then the corresponding echelon matrix must have a pivot in every row. But since A is $n \times n$, there must be a pivot in every column. This means that the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. But this says that the only linear combination of the columns of A that makes $\mathbf{0}$ is the trivial linear combination. In other words, the columns of A satisfy the condition to be linearly independent.

   Now these steps may be reversed: if the columns are independent then the only solution to $A\mathbf{x} = 0$ is trivial. Hence the echelon form of A must have a pivot in every column. But since A is $n \times n$, this says there is a pivot in every row. This implies that the equation $A\mathbf{x} = \mathbf{b}$ can be solved for every $\mathbf{b}$, in other words every $\mathbf{b}$ is a linear combination of the columns of A, in other words the columns span $\mathbf{R}^n$.

7. *Suppose you receive a stream of vectors $\mathbf{x}_1, \ldots \mathbf{x}_k \in \mathbf{R}^n$, and need to compute $G_k = X_k X_k^T$ where $X_k = [\mathbf{x}_1, \ldots \mathbf{x}_k]$ matrix whose columns are $\mathbf{x}_i$. Using partitioned matrices, show how to update to G_{k+1} using G_k and $\mathbf{x}_{k+1}$ without having to construct X_{k+1} and $X_{k+1}X_{k+1}^T$ from scratch. (Text problem 124[17].)*

   The partitioning is $X_{k+1} = [X_k, \mathbf{x}_{k+1}]$. Thus

$$G_{k+1} = X_{k+1}X_{k+1}^T = \begin{pmatrix} X_k & \mathbf{x}_{k+1} \end{pmatrix} \begin{pmatrix} X_k^T \\ \mathbf{x}_{k+1}^T \end{pmatrix} = X_k X_k^T + \mathbf{x}_{k+1}\mathbf{x}_{k+1}^T = G_k + \mathbf{x}_{k+1}\mathbf{x}_{k+1}^T.$$

8. *Use partitioned matrices to show for $n = 1, 2, \ldots$ that the $n \times n$ matrix A is invertible and that B is its inverse. (Text problem 124[24].)*

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

An induction argument to prove sequence of logical propositions $\mathcal{P}_n$, one proves two statements. The first is the base case, to show that $\mathcal{P}_2$ is true. The second is the induction case, to show for all $n \ge 2$ that if $\mathcal{P}_n$ is true then so is $\mathcal{P}_{n+1}$.

In the present problem $\mathcal{P}_n$ is the statement "the $n \times n$ matrix A is invertible and that B is its inverse, where A and B are as above."

In the bases case, when $n = 2$, we use the formula for the inverse of 2×2 matrices on A:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{1 \cdot 1 - 0 \cdot 1} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = B$$

as desired.

In the induction case we assume $A^{-1} = B$ for $n \times n$ matrices. We need the formula for inverting the block lower triangular matrices. Assume A_{11} is an invertible $p \times p$ matrix, A_{22} is an invertible $q \times q$ matrix and A_{21} is any $q \times p$ matrix. Then we seek a matrix B such that $AB = I$ or

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

This leads to the equations

$$\begin{aligned} A_{11}B_{11} &= I \\ A_{11}B_{12} &= 0 \\ A_{21}B_{11} + A_{22}B_{21} &= 0 \\ A_{21}B_{12} + A_{22}B_{22} &= I \end{aligned}$$

The first and second equations say $B_{11} = A_{11}^{-1}$ and $B_{12} = 0$ since A_{11} is invertible. The fourth says $0 + A_{22}B_{22} = I$ so $B_{22} = A_{22}^{-1}$ since A_{22} is invertible. It follows from the third

$A_{22}B_{21} = -A_{21}A_{11}^{-1}$ which implies $B_{21} = -A_{22}^{-1}A_{21}A_{11}^{-1}$. Summarizing

$$A^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1} \cdot A_{21}A_{11}^{-1} & A_{22}^{-1} \end{pmatrix}.$$

Let's build the $(n+1) \times (n+1)$ matrix $\mathcal{A}$ out of the $n \times n$ matrix A with inverse B, the 1×1 matrix 1 with inverse 1 and the $1 \times n$ matrix U consisting of all ones:

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ U & 1 \end{pmatrix}.$$

Applying the inverse formula we get

$$\mathcal{A}^{-1} = \begin{pmatrix} B & 0 \\ -1 \cdot UB & 1 \end{pmatrix}$$

Computing

$$-1 \cdot UB = (-1)\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

Thus $\mathcal{A}^{-1} = \mathcal{B}$, the $(n+1) \times (n+1)$ matrix in the same pattern as B, as desired. The induction proof is complete.

We could do problem (5.) by the same method.

9. *Solve the equation* $A\mathbf{x} = \mathbf{b}$ *using LU factorization, where*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ -4 & 0 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$$

Start by row reducing A to echelon form.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ -4 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 8 & 7 \end{pmatrix} \qquad \begin{matrix} \\ \text{Replace } R_2 \text{ by } R_2 - 2R_1 \\ \text{Replace } R_3 \text{ by } R_3 + 4R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{matrix} \\ \\ \text{Replace } R_3 \text{ by } R_3 + 8R_2 \end{matrix}$$

The echelon matrix is U upper triangular. The lower triangular matrix L records the row operations column by column.

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -8 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ -4 & 0 & 3 \end{pmatrix} = A$$

The equation $LU\mathbf{x} = A\mathbf{x} = \mathbf{b}$ is solved in two steps, solving for $\mathbf{y}$ then $\mathbf{x}$ in

$$L\mathbf{y} = \mathbf{b}$$
$$U\mathbf{x} = \mathbf{y}.$$

First back-substitution solves for $\mathbf{y}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -8 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}.$$

Thus

$$\begin{aligned} y_1 &= 4 \\ y_2 &= 5 - 2y_1 = 5 - 2 \cdot 4 = -3 \\ y_3 &= 7 + 4y_1 + 8y_2 = 7 + 4 \cdot 4 + 8 \cdot (-3) = -1 \end{aligned}$$

Second back-substitution solves for $\mathbf{x}$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix}.$$

Hence

$$\begin{aligned}
x_3 &= 1 \\
x_2 &= -(-3 + x_3) = -(-3 + 1) = 2 \\
x_1 &= 4 - 2x_2 - x_3 = 4 - 2 \cdot 2 - 1 = -1.
\end{aligned}$$

Finally we check that this is correct:

$$A\mathbf{x} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ -4 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = \mathbf{b}.$$

10. *Suppose that $A = QR$ where Q and R are $n \times n$ matrices, R is invertible and upper triangular, and $Q^T Q = I$. Show that for each $\mathbf{b} \in \mathbf{R}^n$ the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. What computation with Q and R will produce that solution? (Text problem 132[24].)*

    Note that since $Q^T Q = I$ then Q is invertible and $Q^{-1} = Q^T$. The equation

    $$QR\mathbf{x} = A\mathbf{x} = \mathbf{b}$$

    can be solved in two steps: solve for $\mathbf{y}$ then $\mathbf{x}$ in

    $$\begin{aligned} Q\mathbf{y} &= \mathbf{b} \\ R\mathbf{x} &= \mathbf{y}. \end{aligned}$$

    Because Q and R are invertible, these equations imply

    $$\begin{aligned} \mathbf{y} &= Q^{-1}\mathbf{b} = Q^T\mathbf{b} \\ \mathbf{x} &= R^{-1}\mathbf{y} = R^{-1}Q^T\mathbf{b} \end{aligned}$$

    This formula says both that the solution exists (one is given by the formula) and that it is unique (every solution has the same expression).

11. *Find bases for* Nul(A) *and* Col(A)*, the rank and the nullity of A where*

    $$A = \begin{pmatrix} 1 & 2 & 3 & 5 & 7 \\ 2 & 4 & 4 & 1 & 2 \\ 3 & 6 & 4 & 1 & 2 \\ 4 & 8 & 1 & 1 & 1 \end{pmatrix}.$$

Row reducing

$$\begin{pmatrix} 1 & 2 & 3 & 5 & 7 \\ 2 & 4 & 4 & 1 & 2 \\ 3 & 6 & 4 & 1 & 2 \\ 4 & 8 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 5 & 7 \\ 0 & 0 & -2 & -9 & -12 \\ 0 & 0 & -5 & -14 & -19 \\ 0 & 0 & -11 & -19 & -27 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 5 & 7 \\ 0 & 0 & -2 & -9 & -12 \\ 0 & 0 & 0 & \frac{17}{2} & 11 \\ 0 & 0 & 0 & \frac{61}{2} & 39 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 5 & 7 \\ 0 & 0 & -2 & -9 & -12 \\ 0 & 0 & 0 & \frac{17}{2} & 11 \\ 0 & 0 & 0 & 0 & -\frac{8}{17} \end{pmatrix}$$

Thus columns 1,3,4 and 5 are pivot columns. Thus a basis for Col(A) is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ 2 \\ 1 \end{pmatrix} \right\}$$

There are four vectors in the basis, so the rank $\mathrm{r}(A) = \dim(\mathrm{Col}(A)) = 4$. x_2 is a free variable, and can take any real value. Solving $A\mathbf{x} = \mathbf{0}$ we see that

$$\begin{aligned} x_5 &= 0 \\ x_4 &= 0 \\ x_3 &= 0 \\ x_1 &= -2x_2 \end{aligned}$$

Hence

$$\mathrm{Nul}(A) = \left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} : x_2 \in \mathbf{R} \right\} \qquad \text{which has the basis} \qquad \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

The nullity, $\mathrm{n}(A) = \dim(\mathrm{Nul}(A)) = 1$.

12. *Suppose that the columns of the matrix $A = [\mathbf{a}_1 \ldots, \mathbf{a}_p]$ are linearly independent. Explain why $\{\mathbf{a}_1 \ldots, \mathbf{a}_p\}$ is a basis for* $\mathrm{Col}(A)$. *(Text problem 154[30].)*

    $\mathrm{Col}(A)$ is by definition $\mathrm{span}(\{\mathbf{a}_1 \ldots, \mathbf{a}_p\})$ hence the vectors $\{\mathbf{a}_1 \ldots, \mathbf{a}_p\}$ span $\mathrm{Col}(A)$. But a linearly independent spanning set is a basis.

13. *Prove if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_q\}$ is a basis for the linear subspace $H \subset \mathbf{R}^n$, then any finite subset $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subset H$ with $p > q$ is linearly dependent. Conclude that every basis of H has the same dimension. (See problem2 161[27,28].)*

    Let $B = [\mathbf{b}_1, \ldots, \mathbf{b}_q]$ be the $n \times q$ matrix whose columns are the $\mathbf{b}_i$'s and $V = [\mathbf{v}_1, \ldots, \mathbf{v}_p]$ be the $n \times p$ matrix whose columns are the $\mathbf{v}_i$'s. For $1 \le i \le p$, let $\mathbf{c}_i = [\mathbf{v}_i]_{\mathcal{B}}$ be the coordinates of the $\mathbf{v}_i$'s relative to $\mathcal{B}$, thus $\mathbf{v}_i = B\mathbf{c}_i$. Let $C = [\mathbf{c}_1, \ldots, \mathbf{c}_q]$ be the $q \times p$ matrix whose columns are the $\mathbf{c}_i$'s. Then $V = BC$.

    Now let us prove that the $\mathbf{v}_i$'s are dependent. Because there are more columns than rows, the columns of C are not independent so there are weights $\mathbf{u} \in \mathbf{R}^p$ such $\mathbf{u} \neq \mathbf{0}$ and that a the combination of columns of C is zero, $C\mathbf{u} = \mathbf{0}$. Multiplying by B we find

    $$V\mathbf{u} = BC\mathbf{u} = B\mathbf{0} = \mathbf{0}.$$

    It follows that there are nonzero weights $\mathbf{u}$ such that the linear combination of the $\mathbf{v}_i$'s equals zero. But this is the definition that the $\mathbf{v}_i$'s are linearly dependent.

    Now suppose that $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subset H$ is another basis besides $\mathcal{B}$. We must have $p \le q$, because, as we showed above, $p > q$ would mean that the $\mathbf{v}_i$'s are linearly dependent. Reversing the roles of $\mathcal{B}$ and $\mathcal{V}$, we must have $q \le p$ also, because if $q > p$, then there are more $\mathbf{b}_i$'s than in the basis $\mathcal{V}$, and as above, the $\mathbf{b}_i$'s would be dependent. Together these inequalities say that any two bases of H must have the same dimension.

14. *Let $\mathcal{B}$ be a basis for the linear subspace $H = \mathrm{span}(\mathcal{B}) \subset \mathbf{R}^4$. Let $\mathbf{u}, \mathbf{v} \in H$. Show that $\mathcal{B}$ is a basis. Find $[\mathbf{u}]_{\mathcal{B}}$. If $[\mathbf{v}]_{\mathcal{B}} = \mathbf{c}$, what is $\mathbf{v}$, where*

    $$\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}, \qquad \mathbf{u} = \begin{pmatrix} 2 \\ 6 \\ 5 \\ 6 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}?$$

The set $\mathcal{B}$ spans, by definition of H. It is a basis if it is also linearly independent. Inserting as column vectors, augmenting with $\mathbf{u}$ we row reduce.

$$[B, \mathbf{u}] = \begin{pmatrix} -2 & 1 & 0 & 2 \\ 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 2 & 6 \end{pmatrix} \to \begin{pmatrix} -2 & 1 & 0 & 2 \\ 0 & \frac{3}{2} & 1 & 7 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 2 & 6 \end{pmatrix} \to \begin{pmatrix} -2 & 1 & 0 & 2 \\ 0 & \frac{3}{2} & 1 & 7 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{4}{3} & \frac{4}{3} \end{pmatrix} \to \begin{pmatrix} -2 & 1 & 0 & 2 \\ 0 & \frac{3}{2} & 1 & 7 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There is a pivot in every column so no free variables. The only solution of $B\mathbf{x} = \mathbf{0}$ is trivial, hence the $\mathbf{b}_i$'s are independent, so $\mathcal{B}$ is a basis. Solving $B\mathbf{y} = \mathbf{u}$ we get

$$\begin{aligned} y_3 &= 1 \\ y_2 &= \frac{2}{3}(7 - y_3) = \frac{2}{3}(7 - 1) = 4 \\ y_1 &= -\frac{1}{2}(2 - y_2) = -\frac{1}{2}(2 - 4) = 1 \end{aligned}$$

Hence we have the coefficients of $\mathbf{u}$ relative to $\mathcal{B}$,

$$[\mathbf{u}]_{\mathcal{B}} = \mathbf{y} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \qquad \text{because } B\mathbf{y} = \mathbf{u}, \text{ or} \qquad 1 \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 5 \\ 6 \end{pmatrix}.$$

Finally $[\mathbf{v}]_{\mathcal{B}} = \mathbf{c}$ means

$$\mathbf{v} = B\mathbf{c} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 7 \\ 11 \end{pmatrix}$$

15. *Construct a nonzero* 3×3 *matrix* A *and a nonzero vectors* $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$ *such that* $\mathbf{b}$ *is in* $\mathrm{Col}(A)$ *but is not the same as any of the columns of* A, $\mathbf{c}$ *is not in* $\mathrm{Col}(A)$ *and* $\mathbf{d}$ *is in* $\mathrm{Nul}(A)$. *(Text problems 154[27, 28, 29].)*

Just about any nonivertible matrix will do. Try

$$A = \begin{pmatrix} 3 & 4 & 5 \\ 2 & 3 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -5 \\ -2 \\ -3 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix}, \qquad \mathbf{d} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

We see that $\mathbf{b} \in \mathrm{Col}(A)$ and $\mathbf{d} \in \mathrm{Nul}(A)$ because

$$A \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = \mathbf{b}, \qquad \text{and} \qquad A\mathbf{d} = \mathbf{0}$$

We see that $A\mathbf{y} = \mathbf{c}$ is inconsistent so $\mathbf{c} \notin \mathrm{Col}(A)$ by row reduction

$$[A, \mathbf{c}] = \begin{pmatrix} 3 & 4 & 5 & 7 \\ 2 & 3 & 3 & 5 \\ 1 & 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & 3 & 5 \\ 3 & 4 & 5 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & -1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is an inconsitent system.

16. *Find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?*

$$\begin{pmatrix} 6 \\ 5 \\ -12 \\ 15 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -6 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 2 \\ -3 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -3 \\ 3 \end{pmatrix}$$

We put the vectors in as columns, starting with the last because its simpler, row reduce and pick the vectors corresponding to pivot columns for the basis

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 6 \\ 1 & 1 & 2 & 1 & 5 \\ -3 & 0 & -3 & -6 & -12 \\ 3 & 3 & 6 & 3 & 15 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 0 & 6 \\ 0 & -1 & -1 & 1 & -1 \\ 0 & 6 & 6 & -6 & 6 \\ 0 & -3 & -3 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 0 & 6 \\ 0 & -1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The first two columns are pivots, thus the basis is the corresponding columns of the original matrix and the dimension of H is two.

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -3 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix} \right\}$$

17. *Compute the determinant by cofactor expansion.*

$$\Delta = \begin{vmatrix} 2 & 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{vmatrix}$$

The second column has the most zeros so use it to expand.

$$\Delta = +1 \cdot \begin{vmatrix} 2 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 0 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 3 & 2 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

Now expand by the last rows.

$$\Delta = \left\{ 1 \cdot \begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 3 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix} \right\} - 2 \left\{ -1 \cdot \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & 3 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 3 & 2 \end{vmatrix} \right\}$$

Finally, expand the second rows (or invoke the 3×3 determinant formulæ).

$$\Delta = \left\{ -1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} \right\} + \left\{ -1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} \right\}$$

$$+ 2 \left\{ -1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \right\} - 2 \left\{ -1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} \right\}$$

$$= \{-(1 \cdot 2 - 2 \cdot 3) - 3(2 \cdot 3 - 1 \cdot 0)\} + \{-(1 \cdot 3 - 1 \cdot 1) + (2 \cdot 3 - 1 \cdot 0)\}$$

$$+ 2\{-(1 \cdot 2 - 2 \cdot 3) - 3(1 \cdot 3 - 1 \cdot 1)\} - 2\{-(1 \cdot 2 - 2 \cdot 3) - 3(2 \cdot 3 - 1 \cdot 0)\}$$

$$= \{4 - 18\} + \{-2 + 6\} + 2\{4 - 6\} - 2\{4 - 18\} = -14 + 4 - 4 + 28 = 14.$$

18. *Find the determinant from Problem 17 using row reduction.*

$$\Delta = \begin{vmatrix} 2 & 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{vmatrix} = (-1)^3 \begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 1 \end{vmatrix}$$

Swap R_3 and R_4

Then swap R_2 and R_3.

Then swap R_1 and R_2.

$$= - \begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 2 \\ 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 2 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 2 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix}$$

$$\Delta = -\begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 2 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 2 \\ 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 2 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 2 \\ 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 & \frac{7}{2} \end{vmatrix}$$

$$= -\left(1 \cdot (-2) \cdot \left(\tfrac{1}{2}\right) \cdot 4 \cdot \tfrac{7}{2}\right) = 14.$$

19. *Find a formula for* $\det(rA)$ *where* A *is an* $n \times n$ *matrix. Prove your formula. (Text problem 178[36].)*

Claim: $\det(rA) = r^n \det(A)$. Proof by induction (see problem 8 about induction arguments.)

Base case $n = 1$, where $A = (a)$. Then $\det(rA) = ra = r\det(A)$.

Induction case. Assume the statement is true for $n \times n$ matrices to show it for $(n+1) \times (n+1)$ matrices. Let A be an $(n+1) \times (n+1)$ matrix. By definition, the determinant is given inductively by first row expansion

$$\det(rA) = \sum_{j=1}^{n+1} (-1)^{1+j} (ra_{1j}) \det(rA_{1j})$$

where A_{1j} is obtained by deleting the first row and jth column from A. Similarly rA_{1j} is obtained by deleting the first row and jth column from rA. rA_{1j} is an $n \times n$ matrix, so the induction hypothesis may be applied to this matrix

$$\det(rA_{1j}) = r^n \det(A_{1j}).$$

Inserting this into the formula,

$$\det(rA) = \sum_{j=1}^{n+1} (-1)^{1+j} (ra_{1j})\, r^n \det(A_{1j}) = r^{n+1} \sum_{j=1}^{n+1} (-1)^{1+j} a_{1j} \det(A_{1j}) = r^n \det(A)$$

completes the induction step.

20. *Show that if two rows of square matrix A are equal then* $\det(A) = 0$. *The same is true for columns. Why? (Text problem 178[30].)*

Suppose rows i and j are equal, $i \neq j$. Then the elementary row operation of replacing row R_j by $R_j - R_i$ results in a zero row j

$$\Delta = \begin{vmatrix} & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & R_j & - \\ & \vdots & \end{vmatrix} = \begin{vmatrix} & \vdots & \\ - & R_i & - \\ & \vdots & \\ - & \mathbf{0} & - \\ & \vdots & \end{vmatrix}$$

Now using row expansion on the jth row

$$\Delta = \sum_{k=1}^{n} (-1)^{j+k}\, 0 \cdot \det(A_{jk}) = 0.$$

If columns C_i and C_j are equal, then they become equal rows in the transposed matrix A^T. The row statement says for such matrices $\det(A^T) = 0$. But the determinat of a matrix equals the determinant of its transpose

$$\det A = \det(A^T) = 0.$$

Or we could have argued as above by using column operations instead of row operations.

21. *Solve using Cramer's Rule.*

$$\begin{pmatrix} 2 & 3 & 3 \\ 1 & 2 & 4 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

Put

$$\Delta = \begin{vmatrix} 2 & 3 & 3 \\ 1 & 2 & 4 \\ 3 & 1 & 1 \end{vmatrix} = 2\cdot 2\cdot 1 + 3\cdot 4\cdot 3 + 3\cdot 1\cdot 1 - 2\cdot 4\cdot 1 - 3\cdot 1\cdot 1 - 3\cdot 2\cdot 3 = 14$$

The solutions are ratios of determinants

$$x_1 = \frac{1}{\Delta}\begin{vmatrix} 5 & 3 & 3 \\ 6 & 2 & 4 \\ 7 & 1 & 1 \end{vmatrix} = \frac{5\cdot 2\cdot 1 + 3\cdot 4\cdot 7 + 3\cdot 6\cdot 1 - 5\cdot 4\cdot 1 - 3\cdot 6\cdot 1 - 3\cdot 2\cdot 7}{14} = \frac{16}{7}$$

$$x_2 = \frac{1}{\Delta}\begin{vmatrix} 2 & 5 & 3 \\ 1 & 6 & 4 \\ 3 & 7 & 1 \end{vmatrix} = \frac{2\cdot 6\cdot 1 + 5\cdot 4\cdot 3 + 3\cdot 1\cdot 7 - 2\cdot 4\cdot 7 - 5\cdot 1\cdot 1 - 3\cdot 6\cdot 3}{14} = -\frac{11}{7}$$

$$x_3 = \frac{1}{\Delta}\begin{vmatrix} 2 & 3 & 5 \\ 1 & 2 & 6 \\ 3 & 1 & 7 \end{vmatrix} = \frac{2\cdot 2\cdot 7 + 3\cdot 6\cdot 3 + 5\cdot 1\cdot 1 - 2\cdot 6\cdot 1 - 3\cdot 1\cdot 7 - 5\cdot 2\cdot 3}{14} = \frac{12}{7}$$

22. *Find the inverse using Cramer's Inverse Formula.*

$$A = \begin{pmatrix} 2 & 3 & 3 \\ 1 & 2 & 5 \\ 3 & 3 & 2 \end{pmatrix}$$

Put

$$\Delta = \begin{vmatrix} 2 & 3 & 3 \\ 1 & 2 & 5 \\ 3 & 3 & 2 \end{vmatrix} = 2\cdot 2\cdot 2 + 3\cdot 5\cdot 3 + 3\cdot 1\cdot 3 - 2\cdot 5\cdot 3 - 3\cdot 1\cdot 2 - 3\cdot 2\cdot 3 = 8$$

Then, Cramer's rule expresses A^{-1} in terms of cofactors $C_{ij} = (-1)^{i+j}\det(A_{ij})$

$$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{8}\begin{pmatrix} +\left|\begin{smallmatrix}2&5\\3&2\end{smallmatrix}\right| & -\left|\begin{smallmatrix}3&3\\3&2\end{smallmatrix}\right| & +\left|\begin{smallmatrix}3&3\\2&5\end{smallmatrix}\right| \\ -\left|\begin{smallmatrix}1&5\\3&2\end{smallmatrix}\right| & +\left|\begin{smallmatrix}2&3\\3&2\end{smallmatrix}\right| & -\left|\begin{smallmatrix}2&3\\1&5\end{smallmatrix}\right| \\ +\left|\begin{smallmatrix}1&2\\3&3\end{smallmatrix}\right| & -\left|\begin{smallmatrix}2&3\\3&3\end{smallmatrix}\right| & +\left|\begin{smallmatrix}2&3\\1&2\end{smallmatrix}\right| \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -11 & 3 & 9 \\ 13 & -5 & -7 \\ -3 & 3 & 1 \end{pmatrix}$$

23. *Let $T \subset \mathbf{R}^2$ be a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) Show that its area is given by A. (Text problems 187[29,30].)*

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The area of a triangle whose vertices are 0, $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbf{R}^2$ is half the area of a parallelogram generated by $\mathbf{v}_1$ and $\mathbf{v}_2$, thus is the determinant of a matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$

$$A = \frac{1}{2}\det([\mathbf{v}_1, \mathbf{v}_2])$$

Translating the triangle by $(-x_1, -y_1)$ to the origin, the sides are

$$\mathbf{v}_1 = \begin{pmatrix} x_2 - x_1 \\ \\ y_2 - y_1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} x_3 - x_1 \\ \\ y_3 - y_1 \end{pmatrix}.$$

View the vectors in the x-y plane of $\mathbf{R}^3$ by augmenting by zero,

$$\tilde{\mathbf{v}}_1 = \begin{pmatrix} \mathbf{v}_1 \\ \\ 0 \end{pmatrix}, \qquad \tilde{\mathbf{v}}_2 = \begin{pmatrix} \mathbf{v}_2 \\ \\ 0 \end{pmatrix}$$

Now the right cylinder with base given by the parallelogram spanned by $\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{v}}_2$ and height 1 is the parallelopiped generated by $\tilde{\mathbf{v}}_1$, $\tilde{\mathbf{v}}_2$ and $\mathbf{e}_3$. Its volume is equal to the area of its base times height, or in other words

$$V = \det([\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \mathbf{e}_3]) = 2A \cdot 1$$

By row operations and then column operations, then transpose and row swaps

$$\begin{aligned} 2A &= \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & 0 \\ \\ y_2 - y_1 & y_3 - y_1 & 0 \\ \\ 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & x_1 \\ \\ y_2 - y_1 & y_3 - y_1 & y_1 \\ \\ 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} x_2 & x_3 & x_1 \\ \\ y_2 & y_3 & y_1 \\ \\ 1 & 1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} x_2 & y_2 & 1 \\ \\ x_3 & y_3 & 1 \\ \\ x_1 & y_1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} x_1 & y_1 & 1 \\ \\ x_2 & y_2 & 1 \\ \\ x_3 & y_3 & 1 \end{vmatrix}. \end{aligned}$$

24. *Consider the subset H of the vector space of continuous functions on the interval $[a,b]$. Is H a subspace of $\mathcal{C}([a,b])$? (H is the subset satisfying periodic boundary conditions.)*

$$H = \{f \in \mathcal{C}([a,b]) : f(a) = f(b)\}$$

To be a subspace, we have to show that H is closed under vector addition and scalar multiplication. So we check if the sum satisfies the condition to belong to H. Choose any two continuous function $f, g \in H$ and any number $c \in \mathbf{R}$. Then $f(a) = f(b)$ and $g(a) = g(b)$. Thus the vector sum and scalar multiple satisfy

$$(f+g)(a) = f(a) + g(a) = f(b) + g(b) = (f+g)(b),$$
$$(cf)(a) = c(f(a)) = c(f(b)) = (cf)(b)$$

Thus $f + g \in H$ and $cf \in H$ so H is closed under addition and scalar multiplication. It is a vector subspace of $\mathcal{C}([a,b])$.

25. *Let $\mathcal{T}(V,W) = \{T : V \to W : T \text{ is a linear transformation}\}$ be the set of linear transformations mapping the vector space V to the vector space W. The addition of two transformations $S, T \in \mathcal{T}(V,W)$ and scalar multiplication by $c \in \mathbf{R}$ is defined by*

$$(S+T)(x) := S(x) + T(x), \qquad (cT)(x) := cT(x).$$

Show that $\mathcal{T}(V,W)$ is a real vector space.

The proof that as set with operations is a vector space amounts to checking that the ten axioms for a vector space are satisfied. It turns out that the fact that W is a vector space is used for every step. Following the order on p. 192, suppose that $S, T, U \in \mathcal{T}(V,W)$ and that $c, d \in \mathbf{R}$.

(1) The sum $S+T$ is a linear transformation because $(S+T)(x+y) = S(x+y)+T(x+y) = [S(x)+S(y)]+[T(x)+T(y)] = [S(x)+T(x)]+[S(y)+T(y)] = (S+T)(x)+(S+T)(y)$ for every $x, y \in V$ because of the definition of addition on transformations, the linearity of S and T, and again, the definition of addition on transformations. Similarly, $(S+T)(cx) = S(cx) + T(cx) = cS(x) + cT(x) = c(S(x) + T(x)) = c(S+T)(x)$ for all $x \in V$ and all $c \in \mathbf{R}$ because of the definition of addition on transformations, the linearity of S, and the definition of the addition of transformations.

(2) For every $x \in V$, $(S+T)(x) = S(x) + T(x) = T(x) + S(x) + (T+S)(x)$ because of the commutativity of vector addition in W.

(3) For every $x \in V$, $[(S+T)+U](x) = (S+T)(x) + U(x) = [S(x) + T(x)] + U(x) = S(x) + [T(x) + U(x)] = S(x) + (T+U)(x) = [S + (T+U)](x)$ because of the associativity of vector addition in W.

(4) Let $Z(x) = 0$ be the transformation that sends all vectors to zero. It is the additive identity because, for all $x \in V$, $(T+Z)(x) = T(x) + Z(x) = T(x) + 0 = T(x)$, by the additive identity property of $0 \in W$.

(5) Let $(-T)(x) = -T(x)$ be the transformation that takes x to the additive inverse of $T(x)$ in W. One checks that it is a linear transformation. It serves as the additive inverse in the space of transformations because, for all $x \in V$, $(T + [-T])(x) = T(x) + [-T](x) = T(x) + [-T(x)] = 0 = Z(x)$, using the fact that $-T(x)$ is the additive inverse of $T(x)$ in W.

(6) The scalar multiple cT is a linear transformation because $(cT)(x+y) = cT(x+y) = c[T(x) + T(y)] = [cT(x)] + [cT(y)] = (cT)(x) + (cT)(y)$ for every $x, y \in V$ because of the definition of scalar multiplication on transformations, the linearity of T, and again, the definition of addition on transformations. Similarly, $(cT)(kx) = cT(kx) = c[kT(x)] =$

$k[cT(x)] = k(cT)(x)$ for all $x \in V$ and all $k \in \mathbf{R}$ because of the definition of scalar multiplication on transformations, the linearity of S, and the definition of the scalar multiplication of transformations.

(7) For all $x \in V$, $[c(S+T)](x) = c[(S+T)(x)] = c[S(x)+T(x)] = [cS(x)] + [cT(x)] = (cS)(x) + (cT)(x) = [(cS)+(cT)](x)$ using the distributivity with respect to vector addition in W.

(8) For all $x \in V$, $[(c+d)T](x) = (c+d)T(x) = [cT(x)] + [dT(x)] = [(cT)(x)] + [(dT)(x)] = [(cT)+(dT)](x)$ using the distributivity with respect to scalar addition in W.

(9) For all $x \in V$, $[c(dT)](x) = c[(dT)(x)] = c[dT(x)] = (cd)T(x) = [(cd)T](x)$ using the associativity of scalar multiplication in W.

(10) For all $x \in V$, $(1T)(x) = 1[T(x)] = T(x)$, using $1y = y$ property of W.

The steps require careful interpretation but it is all a little tedious.

26. *Let H and K be vector subspaces of the vector space $\mathbb{V}$. Show that the intersection $H \cap K$ is also a vector subspace. Show by example, however, that the union of two subspaves is not necessarily a vector subspace. (Text problem 199[32].)*

Let us check the three properties of a vector subspace. First, sicee zero is in both $\mathbf{0} \in H$ and $\mathbf{0} \in K$, it is in the intersection $\mathbf{0} \in H \cap K$. Second let $\mathbf{u}, \mathbf{v} \in H \cap K$ be two vectors in the intersection, we must show that their sum is in the intersection as well. But since $\mathbf{u}$ and $\mathbf{v}$ are in the intersection, both must belong to both subspaces. For instance $\mathbf{u}, \mathbf{v} \in H$. But because H is a subspace, it contains sums $\mathbf{u} + \mathbf{v} \in H$. Similarly since $\mathbf{u}, \mathbf{v} \in K$, because K is a subspace, it contains sums $\mathbf{u} + \mathbf{v} \in K$. Now that the sum is in both subspaces, it is in their intersection too $\mathbf{u} + \mathbf{v} \in H \cap K$. Third, let $\mathbf{u} \in H \cap K$ be a vector in the intersection and $c \in \mathbf{R}$ be any number. We must show that their scalar product is in the intersection as well. But since $\mathbf{u}$ is in the intersection, it must belong to both subspaces. For instance, since $\mathbf{u} \in H$, because H is a subspace, it contains scalar multiples $c\mathbf{u} \in H$. Similarly since $\mathbf{u} \in K$, because K is a subspace, it contains scalar products $c\mathbf{u} \in K$. Now that $c\mathbf{u}$ is in both subspaces, it is in their intersection too $c\mathbf{u} \in H \cap K$.

Let $\mathbf{e}_i$ be unit coordinate vectors in $\mathbf{R}^3$. Let's give an example that shows unions are not necessarily subspaces. If $H = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}$ then it is a subspace because it is a span. Indeed it is the x-y coordinate plane. Let $K = \operatorname{span}\{\mathbf{e}_2, \mathbf{e}_3\}$ which is a subspace because it is a span which is the y-z coordinate plane. The union $H \cup K$ is the union of these planes which does not contain all points. Both $\mathbf{e}_1$ and $\mathbf{e}_3$ are vectors in one of the subspaces so also in their union. But the sum $\mathbf{v} = \mathbf{e}_1 + \mathbf{e}_3$ is in neither plane, and so not in the union. $\mathbf{v}$ is not spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ so not in H and is not spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$ so not in K either, thus $\mathbf{v} \notin H \cup K$. Thus $H \cup K$ is not closed under addition, so fails to be a vector subspace. Another way to see it is $H \cup K$ contains the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbf{R}^3$. If it were a subspace, it would contain all linear combinations, which is all of $\mathbf{R}^3$, but it does not.

27. *Let $\mathcal{C}[0,1]$ denote the vector space of continuous functions on the interval $[0,1]$. Define a mapping $T : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ by Tf is the antiderivative of f with $(Tf)(0) = 0$. Show that T is a linear transformation and describe the kernel of T. (Text problem 209[34].)*

The operator may be given as the definite integral. For $f \in \mathcal{C}[0,1]$,

$$(Tf)(x) = \int_0^x f(t)\,dt.$$

Since the integral of a continuous function is continuous in x, it maps to $\mathcal{C}[0,1]$. It is an antiderivative since $(Tf)'(x) = f(x)$ and $(Tf)(0) = 0$. It remains to show that it is closed under addition and scalar multiplication. To check closure under addition, choose $f, g \in \mathcal{C}[0,1]$. We have by linearity of the integral,

$$(T[f+g])(x) = \int_0^x [f+g](t)\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt = (Tf)(x) + (Tg)(x).$$

To check closure under scalar multiplication, choose $f \in \mathcal{C}[0,1]$ and $c \in \mathbf{R}$. We have by linearity of the integral,

$$(T[cf])(x) = \int_0^x [cf](t)\,dt = \int_0^x cf(t)\,dt = c\int_0^x f(t)\,dt = c[(Tf)(x)].$$

To find the kernel we seek $f \in \mathcal{C}[0,1]$ such that $(Tf)(x) = z(x)$, where $z(x) = 0$ is the additive zero vector (function) in $\mathcal{C}[0,1]$ (since for any f, $f + z = f$). In other words, we seek f so that

$$(Tf)(x) = \int_0^x f(t)\,dt = z(x) = 0, \qquad \text{for all } x \in [0,1].$$

Differentiating, using the Fundamental Theorem of Calculus,

$$f(x) = (Tf)'(x) = z'(x) = 0, \qquad \text{for all } x \in [0,1].$$

Thus $\ker(T) = \{z\}$ is just the zero vector (the dead zero function.)