Math 2270 § 1. **First Midterm Exam** Name:____________________
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1. *Find the general solution to the linear system*

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 3x_1 + 4x_2 - 2x_3 &= 5 \\ x_1 - 7x_2 - 9x_3 &= -10 \end{aligned}$$

Use elementary row operations to reduce the augmented matrix.

$$\begin{pmatrix} 1 & 3 & 1 & 4 \\ 3 & 4 & -2 & 5 \\ 1 & -7 & -9 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & -5 & -5 & -7 \\ 0 & -10 & -10 & -14 \end{pmatrix} \qquad \begin{array}{c} \text{Replace } R_2 \text{ by } R_2 - 3R_1 \\ \text{Replace } R_3 \text{ by } R_3 - R_1 \end{array}$$

$$\to \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & -5 & -5 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{Replace } R_3 \text{ by } R_3 - 2R_2$$

The equation is consistent since there is no zero equals nonzero row. x_3 is a free variable so it can take any real value. Solving for the other variables yields $-5x_2 = 5x_3 - 7$ or $x_2 = \frac{7}{5} - x_3$. Also $x_1 = 4 - 3x_2 - x_3 = 4 - 3(\frac{7}{5} - x_3) - x_3 = -\frac{1}{5} + 2x_3$. Thus the set of solutions is

$$\left\{ \begin{bmatrix} -\frac{1}{5} + 2x_3 \\ \\ \frac{7}{5} - x_3 \\ \\ x_3 \end{bmatrix} : x_3 \in \mathbf{R} \right\}.$$

2. *Let*

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Determine whether $\mathbf{b}$ *is a linear combination of* $\mathbf{a}_1$*,* $\mathbf{a}_2$ *and* $\mathbf{a}_3$*. Do the vectors* $\mathbf{a}_1$*,* $\mathbf{a}_2$ *and* $\mathbf{a}_3$ *span* $\mathbf{R}^3$*?*

We wish to know wether there are constants x_1, x_2 and x_3 such that

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b},$$

where we inserted $\mathbf{a}_i$ as columns of A. Let us augment the matrix with both the given $\mathbf{b}$ and a general $\mathbf{b}$ and reduce.

$$\begin{pmatrix} 1 & 2 & -1 & -1 & b_1 \\ 2 & 3 & 1 & 2 & b_2 \\ 3 & 5 & 0 & 1 & b_3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & -1 & b_1 \\ 0 & -1 & 3 & 4 & b_2 - 2b_1 \\ 0 & -1 & 3 & 4 & b_3 - 3b_1 \end{pmatrix} \qquad \begin{array}{c} \text{Replace } R_2 \text{ by } R_2 - 2R_1 \\ \text{Replace } R_3 \text{ by } R_3 - 3R_1 \end{array}$$

$$\to \begin{pmatrix} 1 & 2 & -1 & -1 & b_1 \\ 0 & -1 & 3 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & -b_1 - b_2 + b_3 \end{pmatrix} \qquad \text{Replace } R_3 \text{ by } R_3 - R_2$$

Since the given $\mathbf{b}$ satisfies $-b_1 - b_2 + b_3 = 1 - 2 + 1 = 0$, the system is consistent and x_3 is a free variable. It means that we may solve for $\mathbf{x}$ and $\mathbf{b}$ is a linear combination of $\mathbf{a}_i$ in infinitely many ways. For one particular solution, put $x_3 = 0$ so $x_2 = -4$ and $x_1 = 7$ so

$$7\mathbf{a}_1 - 4\mathbf{a}_2 + 0\mathbf{a}_3 = 7\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \mathbf{b}.$$

To determine whether $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ span $\mathbf{R}^3$, we need to know whether the equation

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

can be solved for any $\mathbf{b}$. But, this is not possible because, for the last row to be consistent, namely zero equals zero, the vector $\mathbf{b}$ has to satisfy $-b_1 - b_2 + b_3 = 0$. But not all $\mathbf{b}$ satisfy this equation, thus the system cannot be solved for all $\mathbf{b}$ so the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ don't span $\mathbf{R}^3$. We could also see this by noticing that the not every row of the row-echelon matrix of A has a pivot.

3. *Define the map* $T : \mathbf{R}^3 \to \mathbf{R}^2$ *by*

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Define: $T(\mathbf{x})$ *is a* linear transformation. *Is* $T(\mathbf{x})$ *a linear transformation? Explain why or why not. Define: The map* $T(\mathbf{x})$ *is* onto. *Is* $T(\mathbf{x})$ *onto? Explain why or why not. Define: The map* $T(\mathbf{x})$ *is* one-to-one. *Is* $T(\mathbf{x})$ *one-to-one? Explain why or why not.*

A map $T : \mathbf{R}^3 \to \mathbf{R}^2$ is a linear transformation if (1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$ and (2) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every $c \in \mathbf{R}$ and every $\mathbf{u} \in \mathbf{R}^3$. These properties hold for $T(\mathbf{x}) = A\mathbf{x}$ as a result of properties of matrix multiplication. Indeed, (1) $T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u}+A\mathbf{v} = T(\mathbf{u})+T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$ and (2) $T(c\mathbf{u}) = A(c\mathbf{u}) = cA\mathbf{u} = cT(\mathbf{u})$ for every $c \in \mathbf{R}$ and every $\mathbf{u} \in \mathbf{R}^3$.

A map $T : \mathbf{R}^3 \to \mathbf{R}^2$ is onto if for every $\mathbf{b} \in \mathbf{R}^2$ there is a $\mathbf{x} \in \mathbf{R}^3$ such that $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$. Writing the equation as an augmented matrix,

$$\begin{pmatrix} 3 & 2 & 1 & b_1 \\ 1 & 2 & 3 & b_2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & b_2 \\ 3 & 2 & 1 & b_1 \end{pmatrix} \qquad \text{Swap } R_1 \text{ and } R_2$$

$$\to \begin{pmatrix} 1 & 2 & 3 & b_2 \\ 0 & -4 & -8 & b_1 - 3b_2 \end{pmatrix} \qquad \text{Replace } R_2 \text{ and } R_2 - 3R_1$$

There is a pivot entry in every row, so that no matter what $\mathbf{b}$ is, it can be solved for $\mathbf{x}$. Since x_3 is free it can be taken to be any number. For example, if $x_3 = 0$ we have $x_2 = -\frac{1}{4}b_1 + \frac{3}{4}b_2$ and $x_1 = b_2 - 2x_2 = b_2 - 2(-\frac{1}{4}b_1 + \frac{3}{4}b_2) = \frac{1}{2}b_1 - \frac{1}{2}b_2$. Thus

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2}b_1 - \frac{1}{2}b_2 \\ -\frac{1}{4}b_1 + \frac{3}{4}b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

A map $T : \mathbf{R}^3 \to \mathbf{R}^2$ is one-to-one if for every $\mathbf{b} \in \mathbf{R}^2$ there is at most one $\mathbf{x} \in \mathbf{R}^3$ such that $T(\mathbf{x}) = \mathbf{b}$. The given map is not one-to-one because we can find at least two $\mathbf{x}$'s that map to any $\mathbf{b}$. Indeed, if $\mathbf{b} = \mathbf{0}$ then by choosing x_3 to be any number, $x_2 = -2x_3$ and $x_1 = -3x_3 - 2x_2 = x_3$ we have

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus infinitely many $\mathbf{x}$'s map to $\mathbf{b} = \mathbf{0}$. Hence the map is not one-to-one. Equivalently, we could say that $T(\mathbf{x})$ is not one-to-one because the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

4. *Define: the set of vectors* $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\}$ *is linearly independent. Determine whether the set* $\mathcal{S}$ *is linearly independent, where*

$$\mathcal{S} = \left\{ \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -2 \\ 1 \\ 9 \end{bmatrix} \right\}$$

The set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p\}$ is linearly independent if the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p = \mathbf{0}.$$

has only the all-zero solution $(x_1 \ldots, x_p) = (0, \ldots, 0)$.

Write the dependency condition as $A\mathbf{x} = \mathbf{0}$ where the $\mathbf{a}_i$ are the columns of A. Elementary row operations on the matrix A yield

$$\begin{pmatrix} 3 & 2 & 0 \\ 4 & 2 & -2 \\ 1 & 1 & 1 \\ 0 & 3 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & -2 \\ 3 & 2 & 0 \\ 0 & 3 & 9 \end{pmatrix} \qquad \text{Swap } R_1 \text{ and } R_3$$

$$\to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & -1 & -3 \\ 0 & 3 & 9 \end{pmatrix} \qquad \begin{array}{l} \text{Replace } R_2 \text{ by } R_2 - 4R_1 \\ \text{Replace } R_3 \text{ by } R_3 - 3R_1 \end{array}$$

$$\to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \text{Replace } R_3 \text{ by } R_3 - \frac{1}{2}R_2 \\ \text{Replace } R_4 \text{ by } R_4 + \frac{3}{2}R_2 \end{array}$$

We see that x_3 is a free variable. Thus there are infinitely many solutions so zero is not the only solution. Thus the vectors are not linearly independent. Alternately, we see that the third vector is a linear combination of the first two so the vectors are not linearly independent.

$$-2\begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 9 \end{bmatrix}$$

5. *Let* $T : \mathbf{R}^2 \to \mathbf{R}^4$ *be a linear transformation such that*

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

Find $T\left(\begin{bmatrix} 5 \\ 10 \end{bmatrix}\right)$. *Find the standard matrix of* $T(\mathbf{x})$ *For this transformation* T*, is there a vector* $\mathbf{b} \in \mathbf{R}^4$ *such that the system* $T(\mathbf{x}) = \mathbf{b}$ *is inconsistent? If there is such* $\mathbf{b}$*, find one and check. If there in no such* $\mathbf{b}$*, explain why not.*

Using linearity of T, we see that

$$T\left(\begin{bmatrix} 5 \\ 10 \end{bmatrix}\right) = T\left(5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 10\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 5T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 10T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 5\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 10\begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \\ 20 \\ 5 \end{bmatrix}$$

The standard matrix has columns

$$\mathbf{a}_1 = T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_2 = T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

Thus

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 2 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{so} \quad T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 0 & 2 \\ 1 & 2 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The system $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$ is inconsistent if there is no solution. Notice that both $\mathbf{a}_1$ and $\mathbf{a}_2$ have equal first and third components so their linear combinations will do too. Thus we can show that $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ cannot be a linear combination and $A\mathbf{x} = \mathbf{b}$ has no solution for this $\mathbf{b}$. To see it, do row operations on the augmented matrix.

$$\begin{aligned}
\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} &\to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} && \text{Swap } R_1 \text{ and } R_2 \\
&\to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{pmatrix} && \text{Replace } R_4 \text{ by } R_4 - R_1 \\
&\to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{pmatrix} && \begin{array}{l}\text{Replace } R_3 \text{ by } R_3 - 2R_2 \\ \text{Replace } R_4 \text{ by } R_4 + 2R_2\end{array} \\
&\to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} && \text{Replace } R_4 \text{ by } R_4 + R_3
\end{aligned}$$

The system is inconsistent with this $\mathbf{b}$. The third row says zero equals -2, which cannot hold.