1. *Solve the system using elementary row operations on the augmented matrix.*

$$2x_1 + 3x_2 + x_3 = 4$$
$$x_1 - 2x_2 + 4x_3 = 1$$
$$5x_1 + 18x_2 - 8x_3 = 13$$

$$\begin{pmatrix} 2 & 3 & 1 & 4 \\ 1 & -2 & 4 & 1 \\ 5 & 18 & -8 & 13 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 1 & 4 \\ 5 & 18 & -8 & 13 \end{pmatrix} \qquad \text{swap R1 and R2}$$

$$\to \begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 7 & -7 & 2 \\ 0 & 28 & -28 & 8 \end{pmatrix} \qquad \begin{array}{l} \text{Replace } R_2 \text{ by } R_2 - 4R_2 \\ \text{Replace } R_3 \text{ by } R_3 - 5R_1 \end{array}$$

$$\to \begin{pmatrix} 1 & -2 & 4 & 1 \\ 0 & 7 & -7 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{Replace } R_3 \text{ by } R_3 - 4R_2$$

x_3 is free, so it can be any real number. Then $x_2 = \frac{2}{7} + x_3$ and $x_1 = 1 - 4x_3 + 2x_2 = 1 - 4x_3 + 2\left(\frac{2}{7} + x_3\right) = \frac{11}{7} - 2x_3$. Thus the set of solutions is

$$\mathcal{S} = \left\{ \begin{pmatrix} \frac{11}{7} - 2x_3 \\ \frac{2}{7} + x_3 \\ x_3 \end{pmatrix} : x_3 \in \mathbf{R} \right\}$$

2. *Determine whether the linear system is consistent. Do not completely solve the system.*

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 + 2x_4 &= 4 \\ x_1 - 2x_2 + 3x_3 + &= 1 \\ 2x_1 + 6x_2 + 3x_4 &= 3 \\ x_1 - x_2 + 3x_3 &= 2 \end{aligned}$$

Eliminating from the augmented matrix yields

$$\begin{pmatrix} 2 & 3 & 2 & 2 & 4 \\ 1 & -2 & 3 & 0 & 1 \\ 2 & 6 & 0 & 3 & 3 \\ 1 & -1 & 3 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 0 & 1 \\ 2 & 3 & 2 & 2 & 4 \\ 2 & 6 & 0 & 3 & 3 \\ 1 & -1 & 3 & 0 & 2 \end{pmatrix} \qquad \text{Swap } R_1 \text{ and } R_2$$

$$\to \begin{pmatrix} 1 & -2 & 3 & 0 & 1 \\ 0 & 7 & -4 & 2 & 2 \\ 0 & 10 & -6 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{l} \text{Replace } R_2 \text{ by } R_3 - 2R_1 \\ \text{Replace } R_3 \text{ by } R_3 - 2R_1 \\ \text{Replace } R_4 \text{ by } R_4 - R_1 \end{array}$$

$$\to \begin{pmatrix} 1 & -2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 7 & -4 & 2 & 2 \\ 0 & 10 & -6 & 3 & 1 \end{pmatrix} \qquad \text{Permute } R_2 \to R_3 \to R_4 \to R_2$$

$$\to \begin{pmatrix} 1 & -2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -4 & 2 & -5 \\ 0 & 0 & -6 & 3 & -9 \end{pmatrix} \qquad \begin{array}{l} \text{Replace } R_3 \text{ by } R_3 - 7R_2 \\ \text{Replace } R_4 \text{ by } R_4 - 10R_2 \end{array}$$

$$\to \begin{pmatrix} 1 & -2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -4 & 2 & -5 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \qquad \text{Replace } R_4 \text{ by } R_4 - \tfrac{3}{2}R_3$$

The last equation equates zero to nonzero, thus the system is inconsitent.

3. *Find an equation between g, h and k to make the system corresponding to this augmented matrix consitent.*

$$\begin{pmatrix} 1 & 2 & 3 & 2 & g \\ 2 & 2 & 9 & 3 & h \\ 3 & 8 & 6 & 7 & k \end{pmatrix}$$

Do elementary row operations and carry the rightmost column.

$$\begin{pmatrix} 1 & 2 & 3 & 2 & g \\ 2 & 2 & 9 & 3 & h \\ 3 & 8 & 6 & 7 & k \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 2 & g \\ 0 & -2 & 3 & -1 & h-2g \\ 0 & 2 & -3 & 1 & k-3g \end{pmatrix} \quad \begin{array}{l} \\ \text{Replace } R_2 \text{ by } R_2 - 2R_1 \\ \text{Replace } R_3 \text{ by } R_2 - 3R_1 \end{array}$$

$$\to \begin{pmatrix} 1 & 2 & 3 & 2 & g \\ 0 & -2 & 3 & -1 & h-2g \\ 0 & 0 & 0 & 0 & k-5g+h \end{pmatrix} \quad \begin{array}{l} \\ \\ \text{Replace } R_3 \text{ by } R_3 + R_2 \end{array}$$

The last equation is consistent exactly when

$$k - 5g + h = 0.$$

Then there are two free variables and we can solve for the rest of the pivot variables.

4. *Suppose that a system of linear equations has fewer equations than unknowns, called an* underdetermined system. *Does such a system have to be consistent? If it is, why must there be an infinite number of solutions?*

Such a system need not be consistent. An example of an inconsistent system with two unknowns and one equation is

$$0x_1 + 0x_2 = 1.$$

Another example of an inconsistent system with two equations and three unknowns is

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ x_1 + 2x_2 + 3x_3 &= 2. \end{aligned}$$

If the system were consistent, then the echelon form of the augmented matrix does not have a zero equals nonzero row. Its reduced row-echelon matrix looks something like this:

$$\begin{pmatrix} \blacksquare & * & * & 0 & * & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If there are m equations and n unknowns with $m < n$ then in the augmented $m \times (n+1)$ matrix, there is at most one pivot per row, thus the number of pivots is at most $m < n$. Since the system is consistent, the pivots must occur among the first n columns (and not in the last column of the augmented matrix.) Thus there must be a free variable because there are at least $n - m > 0$ non-pivotal columns among the first n, which correspond to the free variables. In solving, each free variable may be set to any real number, and the remaining variables may be determined from these. Because there are free variables, they each contribute an infinity of solutions.

5. *Determine whether the vector* $\mathbf{b}$ *is a linear combination of the vectors* $\mathbf{a}_1$, $\mathbf{a}_2$ *and* $\mathbf{a}_3$

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 5 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 5 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 7 \\ 6 \\ 8 \\ 11 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 4 \end{pmatrix}.$$

We try to solve

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

Writing the augmented matrix of this system and doing elementary row operations, we find

$$\begin{pmatrix} 1 & 5 & 7 & 1 \\ 2 & 2 & 6 & 3 \\ 4 & 0 & 8 & 1 \\ 5 & 1 & 11 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 7 & 1 \\ 0 & -8 & -8 & 1 \\ 0 & -20 & -20 & -3 \\ 0 & -24 & -24 & -1 \end{pmatrix} \quad \begin{matrix} \\ \text{Replace } R_2 \text{ by } R_2 - 2R_1 \\ \text{Replace } R_3 \text{ by } R_3 - 4R_1 \\ \text{Replace } R_4 \text{ by } R_4 - 5R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 5 & 7 & 1 \\ 0 & -8 & -8 & 1 \\ 0 & 0 & 0 & -\frac{11}{2} \\ 0 & 0 & 0 & -4 \end{pmatrix} \quad \begin{matrix} \\ \\ \text{Replace } R_3 \text{ by } R_3 - \frac{5}{2}R_2 \\ \text{Replace } R_4 \text{ by } R_4 - 3R_2 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 5 & 7 & 1 \\ 0 & -8 & -8 & 1 \\ 0 & 0 & 0 & -\frac{11}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \\ \\ \\ \text{Replace } R_4 \text{ by } R_4 - \frac{8}{11}R_3 \end{matrix}$$

The resulting system is inconsistent, thus it is impossible to express $\mathbf{b}$ as a linear combination of the $\mathbf{a}_i$'s.

6. *For what value(s) of h is the vector* $\mathbf{b}$ *in* $\mathbf{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, *where*

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} h \\ 2 \\ 1 \end{pmatrix}.$$

We try to solve

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

Writing the augmented matrix of this system and doing elementary row operations, we find

$$\begin{pmatrix} 1 & 5 & 1 & h \\ 0 & 2 & 6 & 2 \\ 2 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 1 & h \\ 0 & 2 & 6 & 2 \\ 0 & -10 & -1 & 1-2h \end{pmatrix} \qquad \text{Replace } R_3 \text{ by } R_3 - 2R_1$$

$$\to \begin{pmatrix} 1 & 5 & 1 & h \\ 0 & 2 & 6 & 2 \\ 0 & 0 & 29 & 11-2h \end{pmatrix} \qquad \text{Replace } R_3 \text{ by } R_3 + 5R_2$$

The system is consistent for any choice of h. Thus for every h is $\mathbf{b}$ in the span of the $\mathbf{a}_i$'s.

7. *Does the equation* $A\mathbf{x} = \mathbf{b}$ *have a solution for every* $\mathbf{b} \in \mathbf{R}^3$, *where*

$$A = \begin{pmatrix} 4 & 5 & 1 \\ 2 & 3 & 1 \\ 2 & 5 & 3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}?$$

Do elementary row operations on A.

$$\begin{pmatrix} 4 & 5 & 1 \\ 2 & 3 & 1 \\ 2 & 5 & 3 \end{pmatrix} \to \begin{pmatrix} 4 & 5 & 1 \\ 0 & 0.5 & 0.5 \\ 0 & 2.5 & 2.5 \end{pmatrix} \qquad \begin{matrix} \\ \text{Replace } R_2 \text{ by } R_2 - 0.5R_1 \\ \text{Replace } R_3 \text{ by } R_3 - 0.5R_1 \end{matrix}$$

$$\to \begin{pmatrix} 4 & 5 & 1 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{Replace } R_3 \text{ by } R_3 - 5R_2$$

The last row is zero, thus in the equivalent echelon matrix, every linear combination of the columns yields third component zero. Thus not all vectors are linear combinations of the columns, namely, only those vectors with zero third component are linear combinations of the columns. Undoing the row operations says that not all $\mathbf{b}$ are linear combination of the columns of A.

8. *Describe the solutions of the following system in parametric vector form.*

$$\begin{aligned} 2x_1 + x_2 + 4x_3 + 3x_4 &= 4 \\ x_1 + 3x_2 + 7x_3 + 4x_4 &= 7 \\ 4x_1 + 3x_2 + 10x_3 + 7x_4 &= 10 \\ 5x_1 + 5x_3 + 5x_4 &= 5 \end{aligned}$$

Let's do elementary row operations.

$$\begin{pmatrix} 2 & 1 & 4 & 3 & 4 \\ 1 & 3 & 7 & 4 & 7 \\ 4 & 3 & 10 & 7 & 10 \\ 5 & 0 & 5 & 5 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 7 & 4 & 7 \\ 2 & 1 & 4 & 3 & 4 \\ 4 & 3 & 10 & 7 & 10 \\ 5 & 0 & 5 & 5 & 5 \end{pmatrix} \qquad \text{Swap } R1 \text{ and } R2.$$

$$\to \begin{pmatrix} 1 & 3 & 7 & 4 & 7 \\ 0 & -5 & -10 & -5 & -10 \\ 0 & -9 & -18 & -9 & -18 \\ 0 & -15 & -30 & -15 & -30 \end{pmatrix} \qquad \begin{array}{l} \\ \text{Replace } R_2 \text{ by } R_2 - 2R_1 \\ \text{Replace } R_3 \text{ by } R_2 - 4R_1 \\ \text{Replace } R_2 \text{ by } R_2 - 5R_1 \end{array}$$

$$\to \begin{pmatrix} 1 & 3 & 7 & 4 & 7 \\ 0 & -5 & -10 & -5 & -10 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \\ \\ \text{Replace } R_3 \text{ by } R_3 - \frac{9}{5}R_2 \\ \text{Replace } R_3 \text{ by } R_3 - 3R_2 \end{array}$$

We see that x_3 and x_4 are free variables, which take on any real value. Then $x_2 = 2-2x_3-x_4$ and

$$x_1 = 7 - 3x_2 - 7x_3 - 4x_4 = 7 - 3(2 - 2x_3 - x_4) - 7x_3 - 4x_4 = 1 - x_3 - x_4.$$

Thus the solution sat may be written in parametric and in vector form.

$$\mathcal{S} = \left\{ \begin{pmatrix} 1 - x_3 - x_4 \\ 2 - 2x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} : x_3, x_4 \in \mathbf{R} \right\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} : x_3, x_4 \in \mathbf{R} \right\}.$$

The solution set is an affine subspace that is the shifted two-plane, the span of $(-1,-2,1,0)$ and $(-1,-1,0,1)$ through the origin which is translated by the particular solution $(1,2,0,0)$.

9. *Alka Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is dissolved in water, the following reaction produces citrate, water, and carbon dioxoide (gas):*

$$NaHCO_3 + H_3C_6H_5O_7 \longrightarrow Na_3C_6H_5O_7 + H_2O + CO_2$$

*For each compound construct a vector that lists the numbers of atoms. Balance the equation by finding whole number solution such that the total number of atoms match on the left and right sides.*The atoms involved are carbon, hydrogen, oxygen, sodium which corresponds to four equations. Let x_i denote the weights of species involved in the reaction

$$(x_1)NaHCO_3 + (x_2)H_3C_6H_5O_7 \longrightarrow (x_3)Na_3C_6H_5O_7 + (x_4)H_2O + (x_5)CO_2$$

The species correspond to vectors

$$\begin{array}{cccccc} NaHC0_3: & H_3C_6H_5O_7: & Na_3C_6H_5O_7: & H_2O: & CO_2: & \\ \begin{pmatrix}1\\1\\3\\1\end{pmatrix}, & \begin{pmatrix}6\\8\\7\\0\end{pmatrix}, & \begin{pmatrix}6\\5\\7\\3\end{pmatrix}, & \begin{pmatrix}0\\2\\1\\0\end{pmatrix}, & \begin{pmatrix}1\\0\\2\\0\end{pmatrix}, & \begin{array}{l}\leftarrow \text{carbon}\\ \leftarrow \text{hydrogen}\\ \leftarrow \text{oxygen}\\ \leftarrow \text{sodium}\end{array} \end{array}$$

We solve

$$x_1\begin{pmatrix}1\\1\\3\\1\end{pmatrix} + x_2\begin{pmatrix}6\\8\\7\\0\end{pmatrix} = x_3\begin{pmatrix}6\\5\\7\\3\end{pmatrix} + x_4\begin{pmatrix}0\\2\\1\\0\end{pmatrix} + x_5\begin{pmatrix}1\\0\\2\\0\end{pmatrix}$$

Move terms to the left, write the matrix without the $\mathbf{b}=0$ column, and do elementary row operations.

$$\begin{pmatrix} 1 & 6 & -6 & 0 & -1 \\ 1 & 8 & -5 & -2 & 0 \\ 3 & 7 & -7 & -1 & -2 \\ 1 & 0 & -3 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 6 & -6 & 0 & -1 \\ 0 & 2 & 1 & -2 & 1 \\ 0 & -11 & 11 & -1 & 1 \\ 0 & -6 & 3 & 0 & 1 \end{pmatrix} \quad \begin{matrix} \\ \text{Replace } R_2 \text{ by } R_2 - R_1 \\ \text{Replace } R_3 \text{ by } R_2 - 3R_1 \\ \text{Replace } R_4 \text{ by } R_4 - R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & -9 & 6 & -4 \\ 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & \frac{33}{2} & -12 & \frac{13}{2} \\ 0 & 0 & 6 & -6 & 4 \end{pmatrix} \quad \begin{matrix} \text{Replace } R_1 \text{ by } R_1 - 3R_2 \\ \text{Multiply } R_2 \text{ by } \frac{1}{2} \\ \text{Replace } R_3 \text{ by } R_3 + \frac{11}{2}R_2 \\ \text{Replace } R_4 \text{ by } R_4 + 3R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & -9 & 6 & -4 \\ 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 6 & -6 & 4 \\ 0 & 0 & \frac{33}{2} & -12 & \frac{13}{2} \end{pmatrix} \quad \begin{matrix} \\ \\ \text{Swap } R_3 \text{ and } R_4 \\ \\ \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & -9 & 6 & -4 \\ 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & \frac{33}{2} & -12 & \frac{13}{2} \end{pmatrix} \quad \begin{matrix} \\ \\ \text{Multiply } R_3 \text{ by } \frac{1}{6} \\ \\ \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & 0 & \frac{9}{2} & -\frac{9}{2} \end{pmatrix} \quad \begin{matrix} \text{Replace } R_1 \text{ by } R_1 + 9R_3 \\ \text{Replace } R_2 \text{ by } R_2 - \frac{1}{2}R_3 \\ \text{Replace } R_3 \text{ by } R_3 + \frac{11}{2}R_2 \\ \text{Replace } R_4 \text{ by } R_4 - \frac{33}{2}R_3 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad \begin{matrix} \\ \\ \\ \text{Multiply } R_4 \text{ by } \frac{2}{9} \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \qquad \begin{matrix} \text{Replace } R_1 \text{ by } R_1 + 3R_4 \\ \text{Replace } R_2 \text{ by } R_2 + \frac{1}{2}R_4 \\ \text{Replace } R_3 \text{ by } R_3 + R_4 \\ \end{matrix}$$

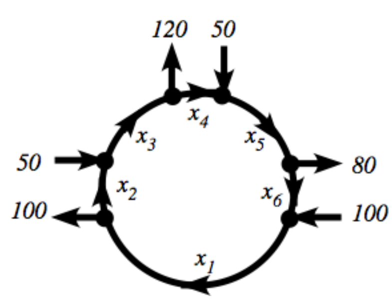
Thus x_5 is a free variable and the solution of the homogeneous system is

$$\mathcal{S} = \left\{ x_5 \begin{pmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{3} \\ 1 \\ 1 \end{pmatrix} : x_5 \in \mathbf{R} \right\}.$$

By taking $x_5 = 3$ we get a simple whole number solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 3 \\ 3 \end{pmatrix}.$$

10. *Consider an English roundabout network such as the one shown in the figure. Find the general solution of the network. Assuming that traffic can only go in the indicated directions, find the largest and smallest value of x_6.*



Let x_i be the traffic flow in vehicles per hour in the direction indicated. There is an equation at each node stating that incoming traffic equals outgoing traffic. Going around the network we get equations

$$
\begin{aligned}
x_1 &= 100 + x_2 \\
50 + x_2 &= x_3 \\
x_3 &= 120 + x_4 \\
150 + x_4 &= x_5 \\
x_5 &= 80 + x_6 \\
100 + x_6 &= x_1.
\end{aligned}
$$

Put the equations into an augmented matrix and do row operations.

$$
\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 100 \\
0 & 1 & -1 & 0 & 0 & 0 & -50 \\
0 & 0 & 1 & -1 & 0 & 0 & 120 \\
0 & 0 & 0 & 1 & -1 & 0 & -150 \\
0 & 0 & 0 & 0 & 1 & -1 & 80 \\
-1 & 0 & 0 & 0 & 0 & 1 & -100
\end{pmatrix}
\to
\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 100 \\
0 & 1 & -1 & 0 & 0 & 0 & -50 \\
0 & 0 & 1 & -1 & 0 & 0 & 120 \\
0 & 0 & 0 & 1 & -1 & 0 & -150 \\
0 & 0 & 0 & 0 & 1 & -1 & 80 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad
\begin{array}{c}
\text{Replace } R_6 \text{ by} \\
R_1 + R_2 + \cdots + R_6
\end{array}
$$

(Why is this the combination of elementary row operations?) Thus x_6 is the free variable which can take any real value. Thus all solutions are of the form

$$
\begin{aligned}
x_1 &= 100 + x_2 = 100 + x_6 \\
x_2 &= -50 + x_3 = x_6 \\
x_3 &= 120 + x_4 = 50 + x_6 \\
x_4 &= -150 + x_5 = -70 + x_6 \\
x_5 &= 80 + x_6.
\end{aligned}
$$

Since all variables must be nonnegative we have $x_4 \geq 0$ so $x_6 \geq 70$ which implies x_5, x_3, x_2 and x_1 are all nonnegative as well.

11. *Determine if the vectors are linearly dependent.*

$$
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 3 \end{pmatrix}, \quad
\mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \quad
\mathbf{v}_3 = \begin{pmatrix} 2 \\ -8 \\ -3 \\ -1 \\ -6 \end{pmatrix}, \quad
\mathbf{v}_4 = \begin{pmatrix} 2 \\ 8 \\ 9 \\ 3 \\ 10 \end{pmatrix}
$$

The vectors are linearly dependent if there is a nonzero choice of weights for a linear combination that adds up to the zero vector. So, let us make the matrix without the zero b column and do elementary row operations.

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 0 & -8 & 8 \\ 3 & 3 & -3 & 9 \\ 1 & 1 & -1 & 3 \\ 3 & 2 & -6 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & -4 & -12 & 4 \\ 0 & -3 & -9 & 3 \\ 0 & -1 & -3 & 1 \\ 0 & -4 & -12 & 4 \end{pmatrix} \quad \begin{matrix} \\ \text{Replace } R_2 \text{ by } R_2 - 2R_1 \\ \text{Replace } R_3 \text{ by } R_3 - 3R_1 \\ \text{Replace } R_4 \text{ by } R_4 - R_1 \\ \text{Replace } R_2 \text{ by } R_5 - 3R_1 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & -3 & -9 & 3 \\ 0 & -1 & -3 & 1 \\ 0 & -4 & -12 & 4 \end{pmatrix} \quad \begin{matrix} \\ \text{Multiply } R_2 \text{ by } -\frac{1}{4} \\ \\ \\ \\ \end{matrix}$$

$$\to \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \\ \\ \text{Replace } R_3 \text{ by } R_3 + 3R_2 \\ \text{Replace } R_4 \text{ by } R_4 + R_2 \\ \text{Replace } R_5 \text{ by } R_5 + 4R_2 \end{matrix}$$

Thus both x_3 and x_4 are free variables. They can be set to any real numbers and the system may be solved for x_1 and x_2. In particular if we put $x_3 = 0$ and $x_4 = 1$ (any nonzero pair will do) then $x_2 = 1$ and $x_1 = -4$. This is a nonzero vector of weights such that the dependency condition holds

$$-4\mathbf{v}_1 + \mathbf{v}_2 + 0\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}.$$

Thus the vectors are linearly dependent.

12. *Suppose that A is an $n \times m$ matrix such that for all $\mathbf{b} \in \mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. Show that then the columns of A are linearly independent.*

We argue by contrapositive ($P \implies Q$ is equivalent to $\sim Q \implies \sim P$). Assume that the columns of A are linearly dependent. By definition of linear dependence, there are coefficients $(x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$ such that the dependency condition holds

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0},$$

where $\mathbf{a}_i$ is the ith column of A. In fact, any constant multiple of the weights $(kx_1, kx_2, \ldots, kx_n)$ where k is a real number is also a solution of the dependency condition. But that says for $\mathbf{b} = \mathbf{0}$, the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. Thus there are $\mathbf{b} \in \mathbf{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ does not have at most one solution.

13. *Use as many columns as possible to construct a matrix B with the property that $B\mathbf{x} = \mathbf{0}$ has only the zero solution. Show that the columns not used are linear combinations of the columns of B. (Problem 63[42,43])*

$$\begin{pmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{pmatrix}$$

Here is the echelon matrix as produced by ©**R**.

$$\begin{pmatrix} 12 & 10.0 & -6.000000e+00 & -3.000000 & 7.000000 & 10.000000 \\ 0 & 1.5 & -4.500000e+00 & -2.750000 & -0.250000 & -8.500000 \\ 0 & 0 & -3.108624e-15 & 4.944444 & -4.944444 & 9.888889 \\ 0 & 0 & 0 & 4.198413 & -4.198413 & 11.396825 \\ 0 & 0 & 0 & 0 & 0 & 2.546314 \end{pmatrix}$$

The fifth column is free, and should be dropped from the matrix. If we do this, then the desired submatrix made from pivot columns is

$$B = \begin{pmatrix} 12 & 10 & -6 & -3 & 10 \\ -7 & -6 & 4 & 7 & 5 \\ 9 & 9 & -9 & -5 & -1 \\ -4 & -3 & 1 & 6 & 9 \\ 8 & 7 & -5 & -9 & -8 \end{pmatrix}$$

Its echelon matrix is the same as the pivot columns of the echelon matrix of A, thus has a pivot in every column. Thus $B\mathbf{x} = \mathbf{0}$ has only the zero solution. The removed column $\mathbf{a}_5$ from A is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_4$ since there is a pivot in those columns for every nonzero entry of fifth column of the echelon matrix of A. Note that the leading entry in the third row is tiny $-3.108624e-15$. It means that the matrix is close to singular and numerical computations are unstable. The package refused to solve the equation

$$B\mathbf{x} = \mathbf{a}_5$$

declaring that "the system is computationally singular." By reducing the tolerance for linear machine dependencies the solution became

$$0.2857143\mathbf{a}_1 + 0.5714286\mathbf{a}_2 + 0.8571429\mathbf{a}_3 - 1.0000000\mathbf{a}_4 + 0.0000000\mathbf{a}_6 = \mathbf{a}_5$$

14. *Determine whether the following transformations from* $\mathbf{R}^3$ *to* $\mathbf{R}^3$ *are linear. If they are, give the proof. If they are not, give a counterexample*

$$R\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 \\ x_3 \\ x_3 - x_1 \end{pmatrix}; \qquad S\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^3 \\ x_3 \\ x_1 x_3 \end{pmatrix}; \qquad T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \sin(x_1) \\ x_3 \\ \cos(x_2) \end{pmatrix};$$

R is linear. To be linear, it has to satisfy the two conditions of linearity:

(1) $R(\mathbf{u}+\mathbf{v}) = R(\mathbf{u}) + R(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$;

(2) $R(c\mathbf{u}) = cR(\mathbf{v})$ for every $\mathbf{u} \in \mathbf{R}^3$ and every $c \in \mathbf{R}$.

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be any vectors. Then by the formula for R, and rules for vector manipulation in $\mathbf{R}^3$,

$$\begin{aligned} R(\mathbf{u}+\mathbf{v}) = R\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix} &= \begin{pmatrix} 2(u_1+v_1) + 3(u_2+v_2) \\ u_3+v_3 \\ (u_3+v_3) - (u_1+v_1) \end{pmatrix} \\ &= \begin{pmatrix} 2u_1 + 3u_2 \\ u_3 \\ u_3 - u_1 \end{pmatrix} + \begin{pmatrix} 2v_1 + 3v_2 \\ v_3 \\ v_3 - v_1 \end{pmatrix} = R(\mathbf{u}) + R(\mathbf{v}) \end{aligned}$$

Hence part (1) holds in the definition of linearity. Let $\mathbf{u} \in \mathbf{R}^3$ be any vector and $c \in \mathbf{R}$ be any number, then

$$R(c\mathbf{u}) = R\begin{pmatrix} cu_1 \\ cu_2 \\ cu_3 \end{pmatrix} = \begin{pmatrix} 2(cu_1) + 3(cu_2) \\ cu_3 \\ (cu_3) - (cu_1) \end{pmatrix} = c\begin{pmatrix} 2u_1 + 3u_2 \\ u_3 \\ u_3 - u_1 \end{pmatrix} = cR(\mathbf{u}).$$

Hence part (2) holds in the definition of linearity as well. Thus R is linear.

To see that S is not linear, consider

$$\mathbf{u} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

then

$$S(\mathbf{u}+\mathbf{v}) = S\begin{pmatrix} 2+3 \\ 0 \\ 0 \end{pmatrix} = S\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 13 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix} = S(\mathbf{u}) + S(\mathbf{v})$$

so condition (1) fails for this $\mathbf{u}$ and $\mathbf{v}$. Thus S is not linear. In fact almost every pair of vectors will yield a counterexample.

To see that T is not linear, we apply it to the zero vector

$$T(\mathbf{0}) = T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq \mathbf{0}.$$

Hence T fails to be linear because it does not map the zero vector to the zero vector.

15. *Suppose that $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation. Suppose that the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent. Then the set of transformed vectors $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is also linearly dependent.*

The assumption of linear dependence says that there are numbers $c_1, c_2, \ldots, c_p$, not all zero such that

$$\mathbf{0} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

Because of linearity, T applied to zero is zero and a linear combination is taken to a linear combination so

$$\mathbf{0} = T(\mathbf{0}) = T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$$

which says that the vectors $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)$ also satisfy the dependency condition with coefficients that are not all zero. Thus the set $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is also linearly dependent.

16. *Find the standard matrix of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that rotates points by $\dfrac{3\pi}{4}$ radians and then reflects across the $x_1 = x_2$ line in the plane.*

Consider the action on the unit vectors in the coordinate plane. The vector $\mathbf{e}_1 = (1, 0)$ is rotated $\dfrac{3\pi}{4}$ radians to the vector $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$, and then reflected across the $x_1 = x_2$ line to the vector $\left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$. The vector $\mathbf{e}_2 = (0, 1)$ is rotated $\dfrac{3\pi}{4}$ radians to the vector $\left(-\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$, and then reflected across the $x_1 = x_2$ line to itself. The columns of the standard matrix A are the $T(\mathbf{e}_i)$ so

$$T(\mathbf{x}) = A\mathbf{x} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

17. *Determine whether the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ is (a) one-to-one and (b) onto. Justify, where*

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & -8 \\ 3 & 2 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

By doing row operations, we reduce the matrix to echelon form.

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & -8 \\ 3 & 2 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 \\ 0 & -4 & -12 \\ 0 & -4 & -12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 \\ 0 & -4 & -12 \\ 0 & 0 & 0 \end{pmatrix}$$

The third column in the echelon form corresponds to a free variable. Thus the homogenoeous equation $A\mathbf{x} = \mathbf{0}$ has more than one solution, in fact infinitely many solutions. This is equivalent to the linear map not being one-to-one. Also there is a zero row, which means that to solve $A\mathbf{x} = \mathbf{b}$, there must be a condition on the components of $\mathbf{b}$ corresponding to the third row of the echelon matrix. Thus not all $\mathbf{b}$ are in the range of T, so T is not onto either.

18. *Suppose that $T : \mathbf{R}^n \to \mathbf{R}^n$ is a linear map between spaces of the same dimension. Show that that T is one-to-one then it is onto.*

Write $T(\mathbf{x}) = A\mathbf{x}$ where A is the standard $n \times n$ matrix. Let's exploit the fact that the matrix of the transformation has the same number of columns and rows. Being one-to-one means that the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the zero solution. This corresponds to A having no free variables and n pivotal variables, one for each column of A. But since each row can have at most one pivot, and there are n pivots, there is a pivotal entry in each of the n rows. But this means that we can solve $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$. This says that every $\mathbf{b}$ is the image of some $\mathbf{x}$, or in other words, T is onto. (It is also true that onto implies one-to-one for square matrices, basically because the argument just given can be run backwards.)

19. *A large apartment building is to be constructed using modular construction techniques. The arrangement of apartments on any particular floor is to be chosen from one of three basic floor plans. Plan A has 18 apartments on one floor, including 3 three-bedroom units, 7 two-bedroom units and 8 one-bedroom units. Each floor of plan B has 4 three-bedroom units, 4 two-bedroom units and 8 one-bedroom units. Each floor of plan C has 5 three-bedroom units, 3 two-bedroom units and 9 one-bedroom units. Suppose that the building contains a total of x_1 floors of plan A, x_2 floors of plan B and x_3 floors of plan C. Is it possible to design the building with exactly 66 three-bedroom units, 74 two-bedroom units and 136 one-bedroom units? If so, is there more than one way to do it? (Problem 91[25] from the text.)*

The vector $\begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}$ gives the number of three-bedroom, two-bedroom and one-bedroom apartments for a single type A floor. Similars column vectors give the count for type B and type C floors. The sum

$$x_1 \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 4 \\ 8 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ 3 \\ 9 \end{pmatrix}$$

is the vector of one, two and three bedroom apartments in the building with x_1 type A, x_2 type B and x_3 type C floors. We equate this with the desired number of apartments vector and try to solve for $\mathbf{x}$.

$$\begin{pmatrix} 3 & 4 & 5 \\ 7 & 4 & 3 \\ 8 & 8 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 66 \\ 74 \\ 136 \end{pmatrix}$$

Row reduction yields

$$\begin{pmatrix} 3 & 4 & 5 & 66 \\ 7 & 4 & 3 & 74 \\ 8 & 8 & 9 & 136 \end{pmatrix} \to \begin{pmatrix} 8 & 8 & 9 & 136 \\ 7 & 4 & 3 & 74 \\ 3 & 4 & 5 & 66 \end{pmatrix}$$

$$\to \begin{pmatrix} 8 & 8 & 9 & 136 \\ 0 & -3 & -\frac{39}{8} & -45 \\ 0 & 1 & \frac{13}{8} & 15 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & \frac{9}{8} & 17 \\ 0 & 1 & \frac{13}{8} & 15 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -\frac{1}{2} & 2 \\ 0 & 1 & \frac{13}{8} & 15 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus there are infinitely many real valued solutions given by the set

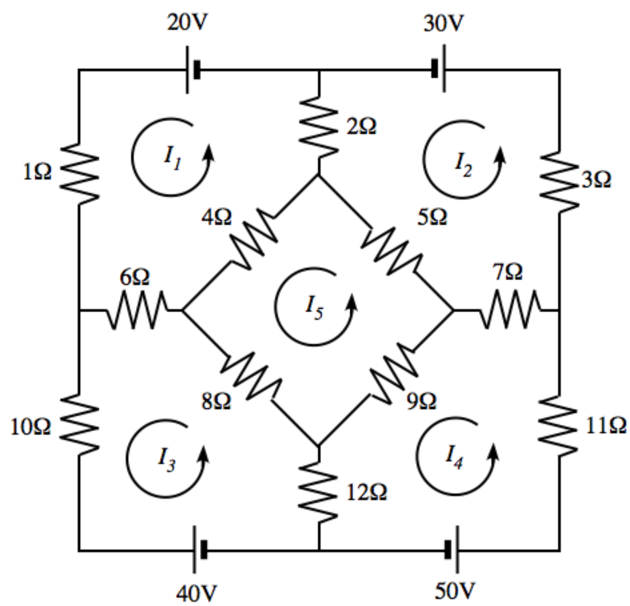
$$\left\{ t \begin{pmatrix} \frac{1}{2} \\ -\frac{13}{8} \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 15 \\ 0 \end{pmatrix} ; t \in \mathbf{R} \right\}$$

Only some of these correspond to positive integer solutons. First the number of floors has to be positive so $0 \le x_3 = t$, so $x_1 = 2 + \frac{t}{2} \ge 0$ automatically, but $0 \le x_2 = 15 - \frac{13}{8}t$ implies $t \le \frac{120}{13} = 9.231$. Second, the number of floors have to be integers, so that the equation for x_2 tells us that t has to be a multiple of 8. The only possibilities are $t = 0$ or $t = 8$, which correspond to building configurations

$$\mathbf{x} = \begin{pmatrix} 2 \\ 15 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 6 \\ 2 \\ 8 \end{pmatrix}.$$

Thus there are two ways to do it.

20. *Write a matrix equation that determines the loop currents. Solve it. (See 88[8] of text.)*



Let the directed loop currents I_i be in the direction given by the diagram. Note that the current in the common boundary is the sum of the currents in the two neigboring loops, thus the current through the 2Ω resistor in the I_1 direction is $I_1 - I_2$. According to Kirchhoff's Law, the total voltage drop around a closed loop is zero. The voltage drop across a resistor is IR and the voltage drop across a battery is $-V$. Note that the batteries are oriented being positive on the long end and negative on the short. Hence the equations corresponding to the five loops are

$$\begin{aligned}
1I_1 + 6(I_1 - I_3) + 4(I_1 - I_5) + 2(I_1 - I_2) - 20 &= 0 \\
2(I_2 - I_1) + 5(I_2 - I_5) + 7(I_2 - I_4) + 3I_2 + 30 &= 0 \\
12(I_3 - I_4) + 8(I_3 - I_5) + 6(I_3 - I_1) + 10I_3 + 40 &= 0 \\
11I_4 + 7(I_4 - I_2) + 9(I_4 - I_5) + 12(I_4 - I_3) - 50 &= 0 \\
4(I_5 - I_1) + 8(I_5 - I_3) + 9(I_5 - I_4) + 5(I_5 - I_2) &= 0
\end{aligned}$$

The corresponding matrix equation is

$$\begin{pmatrix} 13 & -2 & -6 & 0 & -4 \\ -2 & 17 & 0 & -7 & -5 \\ -6 & 0 & 36 & -12 & -8 \\ 0 & -7 & -12 & 39 & -9 \\ -4 & -5 & -8 & -4 & 26 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{pmatrix} = \begin{pmatrix} 20 \\ -30 \\ -40 \\ 50 \\ 0 \end{pmatrix}$$

This problem can be sloved numerically using ©**R**. The echelon form of the augmented matrix is

$$\begin{pmatrix} 13 & -2 & -6 & 0 & -4 & 20 \\ 0 & 16.69231 & -0.9230769 & -7.0000 & -5.615385 & -26.923077 \\ 0 & 0 & 33.1797235 & -12.3871 & -10.156682 & -32.258065 \\ 0 & 0 & 0 & 31.4400 & -15.146667 & 26.666667 \\ 0 & 0 & 0 & 0 & 24.882810 & -4.171614 \end{pmatrix}$$

Back-substitution yields the current vector solution

$$I = \begin{pmatrix} 0.9331273 \\ -1.3882434 \\ -0.7370427 \\ 0.7674084 \\ -0.1676505 \end{pmatrix}$$

21. *Budget©Rent-A-Car in Wichita has a fleet of 500 cars at three locations. A car rented at one location may be returned to any of the three locations. The various fractions of cars returned to any of the locations are shown in the matrix below. Suppose that Monday, there are 295 cars at the airport (or rented from there), 55 cars at the east side office, and 150 cars at the west side office. What will be te approximate distribution of cars Wednesday. (Problem 88[12] from the text.)*

Rented From:			
Airport	East	West	Returned to:
.97	.05	.10	Airport
.00	.90	.05	East
.03	.05	.85	West

Let the vector $\mathbf{x}_k$ denote the number of cars in the three locations on the kth day. Then if the migration matrix stays the same day to day, then we may estimate the distribution of cars on the next day by adding together the the amount of cars that go to a particular location from each of the offices. Thus we have the recursion

$$\mathbf{x}_{k+1} = M\mathbf{x}_k = \begin{pmatrix} .97 & .05 & .10 \\ .00 & .90 & .05 \\ .03 & .05 & .85 \end{pmatrix} \mathbf{x}_k$$

If Monday is the first day, then we are given

$$\mathbf{x}_1 = \begin{pmatrix} 295 \\ 55 \\ 150 \end{pmatrix}$$

Using ©**R** we find the Tuesday and Wednesday distributions to be

$$\mathbf{x}_2 = M\mathbf{x}_1 = \begin{pmatrix} 303.9 \\ 57.0 \\ 139.1 \end{pmatrix}, \qquad \mathbf{x}_3 = M\mathbf{x}_2 = \begin{pmatrix} 311.543 \\ 58.255 \\ 130.202 \end{pmatrix}.$$