Math 2270 § 3. **Second Midterm Exam** Name: Sample
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The following questions are typical of what you might expect on the midterm exam.

$$A = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 10 & 8 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} -4 \\ 1 \\ 5 \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} 3 \\ 25 \\ 19 \end{pmatrix}, \ \mathbf{z} = \begin{pmatrix} 6 \\ 8 \\ -1 \end{pmatrix}.$$

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{w}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

1. Let $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Find the angle between $\mathbf{u}$ and $\mathbf{v}$. Find the area of the triangle whose vertices are $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. Find a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Find the volume of a parallelotope spanned by the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.
2. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by the formula $f(x,y) = (y,x)$. Show that f is a linear transformation.
3. Show that if the Euclidean vectors $\mathbf{p}, \mathbf{q} \in \mathbf{R}^n$ satisfy the Pythagorean equation $\|\mathbf{p}+\mathbf{q}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{q}\|^2$ then $\mathbf{p}$ and $\mathbf{q}$ are orthogonal.
4. Let $\rho : \mathbf{R}^2 \to \mathbf{R}^2$ be a rotation in the plane by an angle θ. Write a formula for ρ. Show that if $\mathbf{p}, \mathbf{q} \in \mathbf{R}^2$ then $\mathbf{p}$ and $\rho(\mathbf{p})$ have the same length and $\angle(\mathbf{p}, \mathbf{q})$ and $\angle(\rho(\mathbf{p}), \rho(\mathbf{q}))$ are the same angle.
5. Find the equation of the plane through $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{z}$.
6. Find a point on the line $\dfrac{z-7}{4} = \dfrac{y-6}{3} = \dfrac{z+5}{2}$ which is closest to the point $(8, 9, 10)$.
7. Find the distance of the plane in $\mathbf{R}^3$ satisfying $x + 2y + 3z = 4$ to the origin.
8. Do the curves $\mathbf{z} + s\mathbf{u}$ and $\mathbf{y} + t\mathbf{v}$ intersect?
9. Find the equation of the plane which contains the line $\mathbf{u} + t\mathbf{v}$ and the point $\mathbf{z}$.
10. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear transformation. Suppose $T(\mathbf{w}_1) = \mathbf{u}$, $T(\mathbf{w}_2) = \mathbf{v}$, $T(\mathbf{w}_3) = \mathbf{y}$. Find $T(\mathbf{z})$. Find a matrix M so that $T(\mathbf{x}) = M\mathbf{x}$.
11. Which of the following sets V with operations $\oplus$ and $\odot$ are vector spaces? If $(V, \oplus, \odot)$ is a vector space, give a proof. If not, give a counterexample.
    (a) $V = \{f : [0,1] \to \mathbf{R}\}$ is the space of all real valued functions on the interval and, $(f \oplus g)(t) = f(t) + g(t)$, $(c \odot f)(t) = cf(t)$ are pointwise addition and scalar multiplication.
    (b) $V = \{(x,y) \in \mathbf{R}^2 : x \ge 0, y \ge 0\}$, $(x,y) \oplus (p,q) = (x+p, y+q)$, $c \odot (x,y) = (|c|x, |c|y)$.
    (c) $V := \mathbf{R}^2$, $(x,y) \oplus (p,q) = (x+p+1, y+q+1)$, $c \odot (x,y) = (cx, cy)$.
12. Which of the following subsets $W \subset \mathbf{R}^3$ of the usual real vector space $(\mathbf{R}^3, +, \cdot)$ are vector subspaces $(W, +, \cdot)$? Why?
    (a) $W = \{(0,0,0)\}$.
    (b) $W = \{(x,y,z) : x + y + z = 1\}$.
    (c) $W = \{(x,y,z) : x + y + z = 0\}$.

(d) $W = \{(x, y, z) : xyz = 0\}$.

13. Find conditions on $\mathbf{b} = (a, b, c)$ so that $A\mathbf{x} = \mathbf{b}$ is solvable.

14. Find a basis for the range of A.

15. Find a basis for the row space of A. Find a basis for the row space of A consisting of rows of A.

16. Do $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{z}$ span $\mathbf{R}^3$?

17. Are $\{t^2 + t + 1, t^2 - 3, t^2 + 2t\}$ independent in P_2, the space of real polynomials of degree ≤ 2?

18. Let $M_1 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find M_3 and M_4 so that $\{M_1, M_2, M_3, M_4\}$ is a basis for $\mathcal{M}_{2,2}$, the space of all real 2×2 matrices.

19. Find a basis for the null space of A.

20. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space. Show that S is linearly dependent if and only if one of the vectors of S is a linear combination of the all the other vectors of S.

21. Let V be a real vector spave with operations $\oplus$ and $\odot$. Show that the additive identitiy is unique. Then show that the additive inverse is unique. Finally, show that $(-1) \odot \mathbf{x}$ equals the additive inverse of $\mathbf{x}$.