Math 2200 § 2. **Second Midterm Exam** Name:________________________
Treibergs March 17, 2011

1. *(a) List the first six terms of the sequence $\{a_n\}$ such that $a_0 = 5$, $a_{n+1} = a_n + 3$ for $n \ge 0$. Give a formula for a_n in terms of n.*

   (b) Compute the sum $\displaystyle\sum_{k=11}^{20} \frac{7}{2^k}$.

   (c) Compute the sum $\displaystyle\sum_{i=1}^{3}\sum_{j=1}^{10}(i+2j)$.

   The recursion tells us to add three each term so

$$a_0 = 5,\ a_1 = 8,\ a_2 = 11,\ a_3 = 14,\ a_4 = 17,\ a_5 = 20, \ldots$$

   Thus $a_n = 5 + 3n$ for $n = 0, 1, 2, 3, \ldots$.

   The sum of a geometric progression with $r \ne 1$ is $\displaystyle\sum_{k=0}^{n} cr^k = \frac{c(1-r^{n+1})}{1-r}$. In our case, $r = \frac{1}{2}$ and $c = 7$. Hence

$$\begin{aligned}\sum_{k=11}^{20}\frac{7}{2^k} &= \sum_{k=0}^{20} 7\left(\frac{1}{2}\right)^k - \sum_{k=0}^{10} 7\left(\frac{1}{2}\right)^k\\ &= \frac{7\left[1-\left(\frac{1}{2}\right)^{21}\right]}{1-\frac{1}{2}} - \frac{7\left[1-\left(\frac{1}{2}\right)^{11}\right]}{1-\frac{1}{2}}\\ &= 14\left[\left(\frac{1}{2}\right)^{11} - \left(\frac{1}{2}\right)^{21}\right] = .00683.\end{aligned}$$

   The sum of the arithmetic progression $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Using this we find

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{10}(i+2j) &= \left(\sum_{i=1}^{3}\sum_{j=1}^{10} i\right) + 2\left(\sum_{i=1}^{3}\sum_{j=1}^{10} j\right)\\ &= 10\left(\sum_{i=1}^{3} i\right) + 6\left(\sum_{j=1}^{10} j\right)\\ &= \frac{10\cdot 3\cdot 4}{2} + \frac{6\cdot 10\cdot 11}{2} = 390.\end{aligned}$$

2. *Let the finite set $A = \{1, 2, 3, 4\}$, and the infinite set $B = \{1, 2, 3, \ldots\}$. Determine whether $S = A \times B$ is countable or uncountable. If S is countable, exhibit a one-to-one correspondence between the set of natural numbers and S. If S is uncountable, show that such a one-to-one correspondence is not possible.*

   The set S is countable. $S = \{(i, j) | i \in A,\ j \in B\}$ can be listed as follows. First we do the four pairs with $j = 1$, then the four pairs $j = 2$ and so on. That is, define a function $f : \{1, 2, 3, \ldots\} \to S$ by

$$\begin{aligned}&f(1) = (1, 1),\ f(2) = (2, 1),\ f(3) = (3, 1),\ f(4) = (4, 1),\\ &f(5) = (1, 2),\ f(6) = (2, 2),\ f(7) = (3, 2),\ f(8) = (4, 2),\\ &f(9) = (1, 3),\ f(10) = (2, 3),\ f(11) = (3, 3),\ f(12) = (4, 3), \ldots\end{aligned}$$

thus $k = i + 4j - 4$. Writing $k - 1 = 4(j-1) + (i-1)$ shows that $i - 1 = (k-1)\,\mathbf{mod}\ 4$ and $j - 1 = (k-1)\,\mathbf{div}\ 4$ so (i,j) is a function of k, inverse to $(i,j) \mapsto i + 4j - 4$. Thus for any pair $(i,j) \in S$ we have $f(k) = (i,j)$ where $k = i + 4j - 4$ thus f is onto. On the other hand if $f(k) = (i,j) = (i',j') = f(k')$ then $k = i + 4j - 4 = i' + 4j' - 4 = k'$ and f is one-to-one. Thus f is a one-to-one correspondence.

3. *(a) Find the base* 6 *expansion of* $(1421)_{10}$.

   (b) Which is greater: $x = (ABCD)_{16}$ *given in base* 16 *or* $y = (22233033)_4$ *given in base* 4*? Why?*

   Repeatedly dividing by 6 we find

$$\begin{aligned} 1421 &= 236 \cdot 6 + 5 \\ 236 &= 39 \cdot 6 + 2 \\ 39 &= 6 \cdot 6 + 3 \\ 6 &= 1 \cdot 6 + 0 \\ 1 &= 0 \cdot 6 + 1. \end{aligned}$$

   Hence $(1421)_{10} = (10325)_6$.

   To compare the numbers, we convert to the same base. But since $(100)_4 = (10)_{16}$, each pair of digits in base four corresponds to a single hexadecimal digit. Thus, x is

$$\begin{aligned} (ABCD)_{16} &= 10 \cdot 16^3 + 11 \cdot 16^2 + 12 \cdot 16 + 13 \\ &= (2 \cdot 4 + 2) \cdot 4^6 + (2 \cdot 4 + 3) \cdot 4^4 + (3 \cdot 4 + 0) \cdot 4^2 + (3 \cdot 4 + 1) \\ &= 2 \cdot 4^7 + 2 \cdot 4^6 + 2 \cdot 4^5 + 3 \cdot 4^4 + 3 \cdot 4^3 + 0 \cdot 4^2 + 3 \cdot 4 + 1 = (22233031)_4 \end{aligned}$$

   which is less than $y = (22233033)_4$, as seen in the last digit.

   Equivalently, we just replace $(A)_{16} = (22)_4$, $(B)_{16} = (23)_4$, $(C)_{16} = (30)_4$, $(D)_{16} = (31)_4$ to get $(A\ B\ C\ D)_{16} = (22\ 23\ 30\ 31)_4$.

4. *Let* a, b, c, m *be positive integers with* $m \ge 3$. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) If $ab \equiv 0 \pmod{m}$ *then* $a \equiv 0 \pmod{m}$ *or* $b \equiv 0 \pmod{m}$.

   (b) There are integers x, y *that solve the equation* $ax + by = c$.

   (c) Let $[a]$ *denote the equivalence class of the integer* a *for congruence modulo* m. *Suppose* $1 \le b < m$. *Then the* $m - 1$ *equivalence classes* $[b], [2b], [3b], \ldots, [(m-1)b]$ *are pairwise disjoint.*

   (a.) FALSE. *e.g.,* take $a = 2$, $b = 3$ and $m = 6$. Then $2 \not\equiv 0 \pmod 6$ and $3 \not\equiv 0 \pmod 6$ but $2 \cdot 3 = 6 \equiv 0 \pmod 6$.

   (b.) FALSE. *e.g.,* take $a = 2$, $b = 4$ and $c = 3$. Then there is no integer solution of $2x + 4y = 3$. To see it, for any pair of integers x, y, the left side is divisible by two $2 \mid 2x + 4y$, but the right side is not divisible by 2, $2 \nmid 3$.

   (c.) FALSE. *e.g.,* take $b = 2$, $m = 4$. Then $3 \cdot 2 = 6 \equiv 1 \cdot 2 \pmod 4$ so that $[1 \cdot 2] = [3 \cdot 2]$.

5. *State the definition:* $\sim$ *is an* equivalence relation *on the set* S.

   Let $\sim$ *be defined on the set of real numbers* $\mathbf{R}$ *by* $x \sim y$ *if and only if* $x - y$ *is an even integer. Determine whether* $\sim$ *is an equivalence relation. Prove your answer.*

   For any $x, y \in S$, let $\sim$ be a relation, that is, $x \sim y$ is a two variable predicate or statement about x and y. (Equivalently, a relation is a subset $\mathcal{R} \subseteq S \times S$ such that $x \sim y \leftrightarrow (x,y) \in \mathcal{R}$.) It is an *equivalence relation* if three axioms hold:

(i.) $\sim$ is reflexive: $(\forall x \in S)\ x \sim x$;

(ii.) $\sim$ is symmetric: $(\forall x \in S)\,(\forall y \in S)\ x \sim y \to y \sim x$;

(iii.) $\sim$ is transitive: $(\forall x \in S)\,(\forall y \in S)\,(\forall z \in S)\ (x \sim y) \wedge (y \sim z) \to x \sim z$.

The relation $x \sim y$ if and only if $x - y$ is an even integer is an equivalence relation on $\mathbf{R}$. We verify the three axioms:

(i.) Choose $x \in \mathbf{R}$. Since $x - x = 0$ which is an even integer we have $x \sim x$ for every x: $\sim$ is reflexive.

(ii.) Choose $x, y \in \mathbf{R}$ such $x \sim y$. This means that $x - y = 2j$ for some integer j. However, $y - x = -2j$ is also an even integer. Thus $y \sim x$ holds: $\sim$ is symmetric.

(iii.) Choose $x, y, z \in \mathbf{R}$ such $x \sim y$ and $y \sim z$. This means that $x - y = 2j$ and $y - z = 2k$ for some integers j, k. However, $x - z = (z - y) + (y - z) = 2j + 2k = 2(j + k)$ which is also an even integer. Thus $x \sim z$ holds: $\sim$ is transitive.

6. *(a) Find* $\gcd(29, 13)$.

(b) Find all integers x such that $13x \equiv 11 \pmod{29}$.

(c) Find all integers x such that

$$x \equiv 2 \pmod{3}$$
$$x \equiv 4 \pmod{5}$$
$$x \equiv 6 \pmod{7}$$

(a.) The Euclidean Algorithn gives

$$29 = 2 \cdot 13 + 3$$
$$13 = 4 \cdot 3 + 1$$
$$3 = 3 \cdot 1 + 0$$

Hence $\gcd(29, 13) = 1$.

(b.) Substituting from the Euclidean Algorithm we find

$$1 = 13 - 4 \cdot 3 = 13 - 4 \cdot (29 - 2 \cdot 13) = 9 \cdot 13 - 4 \cdot 29.$$

so $13 \cdot 9 \equiv 1 \pmod{29}$. Multiplying by 11 we find the solution $13 \cdot 99 \equiv 11 \pmod{29}$. Thus all solutions satisfy $x \equiv 99 \equiv 12 \pmod{29}$. Equivalently, all solutions are given by $x = 12 + 29t$ where t is any integer.

(c.) We find the solution using the Chinese Remainder Theorem. For this theorem to apply, we require that $m_1 = 3$, $m_2 = 5$ and $m_3 = 7$ be pairwise relatively prime. But since m_i are distinct primes, this is true. The solution is unique modulo $m = m_1 m_2 m_3 = 3 \cdot 5 \cdot 7 = 105$. Since each residue is as large as possible, we may guess the answer to be 104 (and we would be correct! Check!)

Proceeding in the usual manner, we compute $M_i = \frac{m}{m_i}$. We get $M_1 = m_2 m_3 = 35 \equiv 2 \pmod{3}$. Hence its inverse $y_1 = 2$ since $2 \cdot 2 \equiv 1 \pmod{3}$. Similarly, $M_2 = m_1 m_3 = 21 \equiv 1 \pmod{5}$. Hence its inverse $y_2 = 1$ since $1 \cdot 1 \equiv 1 \pmod{5}$. Finally, $M_3 = m_1 m_2 = 15 \equiv 1 \pmod{7}$. Hence its inverse $y_3 = 1$ since $1 \cdot 1 \equiv 1 \pmod{7}$. Then the solution is given by

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + a_3 M_3 y_3 = 2 \cdot 35 \cdot 2 + 4 \cdot 21 \cdot 1 + 6 \cdot 15 \cdot 1 = 314.$$

It follows that $x \equiv 314 \equiv 104 \pmod{105}$. In other words, all solutions are given by $x = 104 + 105t$ where $t \in \mathbf{Z}$.