Math 2200 § 2. **Second Midterm Exam** Name: Sample
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**Sample Second Midterm Midterm Questions.**

1. *Consider the sequence given by*
$$a_1 = 1; \qquad a_2 = 1; \qquad a_{n+2} = a_n + a_{n+1} \text{ for } n \ge 2.$$
   (a) *Calculate* $a_3$, $a_4$, $a_5$, $a_6$.
   (b) *Compute*
$$\sum_{j=2}^{5} (3a_j + 2).$$
2. *Find the value of the sum*
$$\sum_{j=6}^{11} \left(3 + 4j + 7^j\right).$$
3. *Let* $m$ *and* $n$ *be two consecutive positive integers. Show that* $m$ *and* $n$ *are relatively prime.*
4. *Solve the congruence*
$$462x \equiv 371 \pmod{13}$$
5. *Find the hexadecimal representation of the number* $(1234567)_{10}$. *Find the decimal representation of* $(43210)_5$.
6. *Find all integers that solve the simultaneous congruences:*
$$\begin{aligned} x &\equiv 2 \pmod 2 \\ x &\equiv 1 \pmod 3 \\ x &\equiv 1 \pmod 5 \\ x &\equiv 3 \pmod 7 \end{aligned}$$
7. *Let* $\sim$ *be the relation on the set of all functions from* $\mathbf{Z}$ *to* $\mathbf{Z}$ *given by*
$$f \sim g \qquad \leftrightarrow \qquad (\exists c \in \mathbf{Z})\,(\forall x \in \mathbf{Z})\, f(x) - g(x) = c.$$
   *Determine whether* $\sim$ *is an equivalence relation. Prove your statement.*
8. *Let* $R$ *be the relation on the set* $A = \{1, 2, 3, 4\}$ *given by*
$$\mathcal{R} = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,3), (3,4), (4,3), (4,4)\}$$
   *Determine whether* $R$ *is and equivalence relation. Prove your statement.*
9. *Let* $A$ *be a countably infinite set. Determine whether the power set* $\mathcal{P}(A)$, *the set of all subsets, is countable. Prove your answer.*
10. *Let* $A$ *and* $B$ *be countably infinite sets. Then the union* $A \cup B$ *is countably infinite.*
11. (a) *Let* $a_0, a_1, \ldots$ *be a sequence of real numbers. Show that* $\displaystyle\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0$.

(b) *Find a formula for* $\sum_{k=1}^{n} k^2$ *using (a.)*

(c) *Find* $\sum_{k=51}^{100} k^2$

12. Compute $5^{1237} \,\mathbf{mod}\, 13$, $5^{1237} \,\mathbf{mod}\, 3$ and $5^{1237} \,\mathbf{mod}\, 39$.

13. *Let $a$ and $b$ be odd integers. Show that* $\gcd(a, b) = \gcd(a - b, b)$.

14. *Show that if* $ac \equiv bc \pmod{m}$*, then* $a \equiv b \pmod{m/d}$ *where* $d = \gcd(m, c)$.

15. *Show that an inverse of $a$ modulo $m$ does not exist if* $\gcd(a, m) > 1$.

16. *Suppose that you have unlimited supplies of 18¢ stamps and 22¢ stamps. Is it possible to make exactly $5.00 worth of postage just using these stamps? In how many ways can this be done? (Find how many pairs of nonnegative integers $(x, y)$ there are so that* $22x + 18y = 500$.*)*

**Solutions.**

1. *Consider the sequence given by*

$$a_1 = 1; \qquad a_2 = 1; \qquad a_{n+2} = a_n + a_{n+1} \text{ for } n \geq 2.$$

   (a) *Calculate* $a_3$, $a_4$, $a_5$, $a_6$.

   (b) *Compute*

$$\sum_{j=2}^{5} (3a_j + 2).$$

   The recursion tells us $a_3 = a_1 + a_2 = 1 + 1 = 2$, $a_4 = a_2 + a_3 = 1 + 2 = 3$, $a_5 = a_3 + a_4 = 2 + 3 = 5$ and $a_6 = a_4 + a_5 = 3 + 5 = 8$.

   The sum reduces to

$$\begin{aligned} \sum_{j=2}^{5} (3a_j + 2) &= 3\left(\sum_{j=2}^{5} a_j\right) + 2\sum_{j=2}^{5} 1 \\ &= 3(a_2 + a_3 + a_4 + a_5) + 2(1 + 1 + 1 + 1) \\ &= 3(1 + 2 + 3 + 5) + 2 \cdot 4 = 41. \end{aligned}$$

2. *Find the value of the sum*

$$\sum_{j=6}^{11} \left(3 + 4j + 7^j\right).$$

   One adds the sum of an arithmetic sequence and the sum of a geometric sequence using the formulas (for $r \neq 1$ and $k \leq n$)

$$\sum_{j=k}^{n} 1 = n - k + 1; \qquad \sum_{j=0}^{n} j = \frac{n(n+1)}{2}; \qquad \sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r}.$$

   Thus

$$\begin{aligned} \sum_{j=6}^{11} \left(3 + 4j + 7^j\right) &= \sum_{j=0}^{11} \left(3 + 4j + 7^j\right) - \sum_{j=0}^{5} \left(3 + 4j + 7^j\right) \\ &= 3\left(\sum_{j=6}^{11} 1\right) + 4\left(\sum_{j=0}^{11} j - \sum_{j=0}^{5} j\right) + \left(\sum_{j=0}^{11} 7^j - \sum_{j=0}^{5} 7^j\right) \\ &= 3(11 - 6 + 1) + 4\left(\frac{11 \cdot 12}{2} - \frac{5 \cdot 6}{2}\right) + \left(\frac{1 - 7^{12}}{1 - 7} - \frac{1 - 7^6}{1 - 7}\right) \\ &= 18 + 306 + 13841169552 = 13841169876. \end{aligned}$$

3. *Let* $m$ *and* $n$ *be two consecutive positive integers. Show that* $m$ *and* $n$ *are relatively prime.*

   We argue by contraposition. Assume that $m$ and $n$ are not relatively prime. Then their greatest common divisor $d = \gcd(m, n) > 1$. Since $d$ is a factor in both, we may assume $n = ad$ and $m = bd$ where $a, b$ are different positive integers. Hence $n - m = ad - bd = (a - b)d$ where $a \neq b$.Thus $|m - n| = d|a - b| \geq d > 1$ which means that $m$ and $n$ are not consecutive because consecutive integers satisfy $|m - n| = 1$.

4. *Solve the congruence*

$$462x \equiv 371 (\bmod 13)$$

We compute $d = \gcd(462, 13)$ using the Euclidean Algorithm:

$$\begin{aligned} 462 &= 35 \cdot 13 + 7 \\ 13 &= 1 \cdot 7 + 6 \\ 7 &= 1 \cdot 6 + 1 \\ 6 &= 6 \cdot 1 + 0 \end{aligned}$$

so $d = 1$. Substituting back,

$$1 = 7 - 6 = 7 - (13 - 7) = 2 \cdot 7 - 13 = 2 \cdot (462 - 35 \cdot 13) - 13 = 2 \cdot 462 - 71 \cdot 13.$$

It follows that $2 \cdot 462 \equiv 1 (\bmod 13)$, hence 2 is the inverse of 462 modulo 13. Multiplying by 371, we find that $x = 742$ is a solution because $13 \mid (462 \cdot 742 - 371)$.

Equivalently, because $7 = 462 \mathbf{\,mod\,} 13$ and $7 = 371 \mathbf{\,mod\,} 13$, the congruence is equivalent to $7x \equiv 7 (\bmod 13)$. But $2 \cdot 7 \equiv 1 (\bmod 13)$ so multiplying 2 by the right side, $2 \cdot 7 \equiv 1$, so the solutions are $x \equiv 1 (\bmod 13)$ or all $x = 13t + 1$ for integral $t$. On can check that $462 \cdot (13t + 1) \equiv 371 (\bmod 13)$ because $13 \mid (462 \cdot (13t + 1) - 371)$.

5. *Find the hexadecimal representation of the number* $(1234567)_{10}$. *Find the decimal representation of* $(43210)_5$.

Using repeated division by 16 we find

$$\begin{aligned} 123456 &= 77160 \cdot 16 + 7 \\ 77160 &= 4822 \cdot 16 + 8 \\ 4822 &= 301 \cdot 16 + 6 \\ 301 &= 18 \cdot 16 + 13 \\ 18 &= 1 \cdot 16 + 2 \end{aligned}$$

It follows that

$$\begin{aligned} 1234567 &= ((((1 \cdot 16 + 2) \cdot 16 + 13) \cdot 16 + 6) \cdot 16 + 8) \cdot 16 + 7 \\ &= 1 \cdot 16^5 + 2 \cdot 16^4 + 13 \cdot 16^3 + 6 \cdot 16^2 + 8 \cdot 16 + 7 \end{aligned}$$

so that $(1234567)_{10} = (12D687)_{16}$ where $(D)_{16} = (13)_{10}$.

The decimal representation of

$$(43210)_5 = 4 \cdot 5^4 + 3 \cdot 5^3 + 2 \cdot 5^2 + 1 \cdot 5 = 4 \cdot 625 + 3 \cdot 125 + 2 \cdot 25 + 5 = (2930)_{10}.$$

6. *Find all integers that solve the simultaneous congruences:*

$$\begin{aligned} x &\equiv 2 (\bmod 2) \\ x &\equiv 1 (\bmod 3) \\ x &\equiv 1 (\bmod 5) \\ x &\equiv 3 (\bmod 7) \end{aligned}$$

The moduli 2, 3, 5, 7 are all different primes and thus are pairwise relatively prime. The Chinese Remainder Theorem guarantees that there is a unique solution modulo $m = 2 \cdot 3 \cdot 5 \cdot$

$7 = 210$. We follow the construction of the solution as in the text. Let $M_k = m/m_k$ so that $M_1 = 105$, $M_2 = 70$, $M_3 = 42$ and $M_4 = 30$. Let $y_k$ be the inverse of $M_k$ modulo $m_k$, which is possible since $\gcd(M_k, m_k) = 1$. Now $M_1 \,\mathbf{mod}\, m_1 = 105 \,\mathbf{mod}\, 2 = 1$, $70 \,\mathbf{mod}\, 3 = 1$, $42 \,\mathbf{mod}\, 5 = 2$ and $30 \,\mathbf{mod}\, 7 = 2$. Thus, their inverses are $y_1 = 1$, $y_2 = 1$, $y_3 = 3$ and $y_4 = 4$ (*e.g.*, $4 \cdot 2 \equiv 1 \pmod 7$). Thus a solution is

$$x = a_1 M_1 y_1 + \cdots a_4 M_4 y_4 = 2 \cdot 105 \cdot 1 + 1 \cdot 70 \cdot 1 + 1 \cdot 42 \cdot 3 + 3 \cdot 30 \cdot 4 = 766$$

Thus all solutions are congruent to $x \equiv 766 \equiv 136 \pmod{210}$.

7. *Let $\sim$ be the relation on the set of all functions from $\mathbf{Z}$ to $\mathbf{Z}$ given by*

$$f \sim g \qquad \leftrightarrow \qquad (\exists c \in \mathbf{Z})\,(\forall x \in \mathbf{Z})\, f(x) - g(x) = c.$$

*Determine whether $\sim$ is an equivalence relation. Prove your statement.*

We show $\sim$ is an equivalence relation by verifying the three axioms.

Since $(\forall x)\, f(x) - f(x) = 0$, the condition follows by taking $c = 0$, hence $\sim$ is reflexive: $f \sim f$ for all $f : \mathbf{Z} \to \mathbf{Z}$.

If $f \sim g$ there is $c_1 \in \mathbf{Z}$ such that $(\forall x)\, f(x) - g(x) = c_1$. Hence $(\forall x)\, g(x) - f(x) = -c_1$, and the condition follows by taking $c = -c_1$, hence $\sim$ is symmetric: $f \sim g \to g \sim f$ for all $f, g : \mathbf{Z} \to \mathbf{Z}$.

If $f \sim g$ and $g \sim h$ there are $c_1, c_2 \in \mathbf{Z}$ such that $(\forall x)\, f(x) - g(x) = c_1$ and $(\forall x)\, g(x) - h(x) = c_2$. Because $f(x) - h(x) = [f(x) - g(x)] + [g(x) - h(x)]$, we have $(\forall x)\, f(x) - h(x) = c_1 + c_2$, and the condition follows by taking $c = c_1 + c_2$, hence $\sim$ is transitive: $(f \sim g) \wedge (g \sim h) \;\to\; f \sim h$ for all $f, g, h : \mathbf{Z} \to \mathbf{Z}$.

8. *Let $R$ be the relation on the set $A = \{1, 2, 3, 4\}$ given by*

$$\mathcal{R} = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,3), (3,4), (4,3), (4,4)\}$$

*Determine whether $R$ is and equivalence relation. Prove your statement.*

The set $\mathcal{R}$ includes the diagonal and is symmetric about the diagonal so $R$ is reflexive and symmetric. But the relation $R$ is not an equivalence relation because transitivity fails: $1R2$ and $2R3$ hold but $1R3$ does not hold.

9. *Let $A$ be a countably infinite set. Determine whether the power set $\mathcal{P}(A)$, the set of all subsets, is countable. Prove your answer.*

The power set is uncountable, as can be seen by a Cantor Diagonalization argument. Let $A = \{a_1, a_2, \ldots\}$ be an enumeration of $A$ (so the map $i \to a_i$ is a one-to-one correspondence with $\{1, 2, 3, \ldots\}$.) For every subset $S \subseteq A$, we can encode $S$ as a sequence of digits $d_{S,i} = 1$ if $a_i \in S$ and $d_{S,i} = 0$ if $a_i \notin S$. Thus $\mathcal{P}(S)$ is in one-to-one correspondence with all infinite zero-one sequences given by $f(S) = (d_{S,1} d_{S,2} d_{S,3} \ldots)$. Assume that $\mathcal{P}(A)$ is countable for contradiction. Then there is a one-to-one correspondence from $\{1, 2, 3, \ldots\}$ to $\mathcal{P}(A)$, or what is the same, infinite sequences of ones and zeros. Writing $d_{j,i}$ for $d_{S_j,i}$ we have

$$\begin{aligned} f(S_1) &= (d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}, \ldots) \\ f(S_2) &= (d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}, \ldots) \\ f(S_3) &= (d_{3,1}, d_{3,2}, d_{3,3}, d_{3,4}, \ldots) \\ f(S_4) &= (d_{4,1}, d_{4,2}, d_{4,3}, d_{4,4}, \ldots) \\ \cdots & \qquad \cdots \end{aligned}$$

Now consider the sequence $b_i = 1 - d_{i,i}$ that replaces the diagonal ones by zeros and zeros by ones. It corresponds to a subset of $A$, namely $B = \big\{a_j | j \in \{1, 2, 3, \ldots\} \;\wedge\; b_j = 1\big\}$. But $f(B) \neq f(S_i)$ for any $i$ because the sequences differ at the $i$-th digit, the set $B$ is not included in the initial enumeration, which is the contradiction.

10. *Let $A$ and $B$ be countably infinite sets. Then the union $A \cup B$ is countably infinite.*

Since $A \cup B$ contains the infinite set $A$, it must be infinite. We construct a surjective function $f : \{1, 2, 3, \ldots\} \to A \cup B$. It follows that $A \cup B$ is in one-to-one correspondence with a subset of $\{1, 2, 3, \ldots\}$ (by throwing out repetitions) and hence $A \cup B$ is at most countable.

Now, since $A$ and $B$ are countable, they have enumerations

$$A = \{a_1, a_2, a_3, \ldots\}, \qquad B = \{b_1, b_2, b_3, \ldots\}$$

so that $i \mapsto a_i$ and $j \mapsto b_j$ are one-to-one correspondences from $\{1, 2, 3, \ldots\}$ to $A$ and to $B$, respectively. We construct $f : \{1, 2, 3, \ldots\} \to A \cup B$ as follows

$$f(1) = a_1,\ f(2) = b_1,\ f(3) = a_2,\ f(4) = b_2, \ldots, f(2k-1) = a_k,\ f(2k) = b_k, \ldots$$

This function is onto, for example if $y \in A \cup B$ then $y \in A$ or $y \in B$. If $y \in B$ then $y = b_j$ for some $j$. But $b_j = f(2j)$. If $y \in A$, $y = a_j = f(2j-1)$ for some $j$. In both cases, $y$ is in the image of $f$ so $f$ is onto.

11. (a) *Let $a_0, a_1, \ldots$ be a sequence of real numbers. Show that* $\displaystyle\sum_{k=1}^{n}(a_k - a_{k-1}) = a_n - a_0$.

(b) *Find a formula for* $\displaystyle\sum_{k=1}^{n} k^2$ *using (a.)*

(c) *Find* $\displaystyle\sum_{k=51}^{100} k^2$

$$\begin{aligned}\sum_{k=1}^{n}(a_k - a_{k-1}) &= (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) \\ &= -a_0 + (a_1 - a_1) + (a_2 - a_2) + \cdots + (a_{n-1} - a_{n-1}) + a_n \\ &= -a_0 + 0 + 0 + \cdots + 0 + a_n.\end{aligned}$$

Take $a_k = k^3$. Then $a_k - a_{k-1} = k^3 - (k-1)^3 = [k - (k-1)][k^2 + k(k-1) + (k-1)^2] = k^2 + k^2 - k + k^2 - 2k + 1 = 3k^2 - 3k + 1$. Hence, by (a),

$$\sum_{k=1}^{n}(3k^2 - 3k + 1) = \sum_{k=1}^{n} k^3 - (k-1)^3 = n^3.$$

Thus

$$\sum_{k=1}^{n} k^2 = \left(\sum_{k=1}^{n} k\right) - \frac{n}{3} + \frac{n^3}{3} = \frac{n(n+1)}{2} - n + \frac{n^3}{3} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

where we used $\sum_{k=1}^{n} 1 = n$ and $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

Thus

$$\sum_{k=51}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{50} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{50 \cdot 51 \cdot 101}{6} = 2030094 - 42925 = 1987169.$$

12. Compute $5^{1237} \ \mathbf{mod}\ 13$, $5^{1237} \ \mathbf{mod}\ 3$ and $5^{1237} \ \mathbf{mod}\ 39$.

By Fermat's Little theorem, since 5 and 13 are relatively prime,

$$5^{12} \equiv 1 \pmod{13}$$

so

$$5^{1237} \equiv (5^{12})^{102} 5 \equiv (1)^{102} 5 \equiv 5 \pmod{13},$$

so $5 = 5^{1237} \mathbf{mod}\, 13$. Also

$$5^2 \equiv 1 \pmod 3$$

so

$$5^{1237} \equiv (5^2)^{618} 5 \equiv (1)^{618} 5 \equiv 5 \equiv 2 \pmod 3,$$

so $2 = 5^{1235} \mathbf{mod}\, 3$. Finally, $x = 5^{1237} \mathbf{mod}\, 39$ satisfies

$$\begin{aligned} x &\equiv 2 \pmod 3, \\ x &\equiv 5 \pmod{13}. \end{aligned}$$

By the Chinese Remainder Theorem, there is a unique solution modulo $m = 3 \cdot 13 = 39$. We let $M_1 = m/3 = 13 \equiv 1 \pmod 3$ so $y_1 = 1$ is its inverse modulo 3 and $M_2 = m/13 = 3$ so $y_2 = 9$ is its inverse modulo 13. Then the common solution is

$$x = 2 \cdot 13 \cdot 1 + 5 \cdot 3 \cdot 9 = 161.$$

Thus $x = 5^{1237} \mathbf{mod}\, 39 = 161 \,\mathbf{mod}\, 39 = 5$.

13. *Let $a$ and $b$ be odd integers. Show that* $\gcd(a, b) = \gcd(a - b, b)$.

    Let $d = \gcd(a, b)$, to show that it is $\gcd(a - b, b)$. The greatest common divisor $d > 0$ of two numbers $p$, $q$, not both zero, satisfies two conditions: (i.) $d \mid p$ and $d \mid q$; (ii.) If $x > 0$ satisfies $x \mid p$ and $x \mid q$ then $x \mid d$. Lets check the conditions on $a - b$ and $b$. $b \neq 0$ since it is odd. Since $d \mid a$ and $d \mid b$ it follows that $d \mid a - b$ and $d \mid b$, so the first condition holds. If $x > 0$ satisfies $x \mid a - b$ and $x \mid b$ then because $a = (a - b) + b$ we have $x \mid a$ and $x \mid b$. But by the second condition for $\gcd(a, b)$, we have $x \mid d$ so the second condition holds for $\gcd(a - b, b)$. As both conditions hold, $d = \gcd(a - b, b)$.

14. *Show that if $ac \equiv bc \pmod m$, then $a \equiv b \pmod{m/d}$ where $d = \gcd(m, c)$.*

    We have to show that $m/d \mid a - b$. Since $d \mid m$ and $d \mid c$, there are nonzero integers $e$, $f$ such that $c = de$ and $m = df$ so $f = m/d$. Also, since $ac \equiv bc \pmod m$, there is an integer $k$ such that

    $$ac - bc = km.$$

    It follows that

    $$ade - bde = kdf$$

    so

    $$(a - b)e = kf.$$

    Using the lemma that the gcd can be expressed as a linear combination, there are integers $p, q$ such that

    $$d = pc + qm = pde + qdf$$

    so

    $$1 = pe + qf.$$

    Multiplying by $a - b$ we see that

    $$a - b = (a - b)ep + (a - b)qf = kfp + (a - b)qf$$

    which is divisible by $f$ as to be shown.

15. *Show that an inverse of a modulo m does not exist if* $\gcd(a, m) > 1$.

Let $d = \gcd(a, m)$. Since $d \mid a$ and $d \mid m$, there are nonzero integers $e$, $f$ such that $a = de$ and $m = df$. Suppose that there were an inverse $b$ for contradiction. This inverse satisfies $ba \equiv 1 \pmod m$. That means that $m \mid ba - 1$, or there is a nonzero integer $k$ so that

$$ba - 1 = mk$$

or

$$1 = ba - mk = bde - kdf.$$

But this says that 1 is divisible by $d > 1$, which is false for the integers and is a contradiction

16. *Suppose that you have unlimited supplies of 18¢ stamps and 22¢ stamps. Is it possible to make exactly \$5.00 worth of postage just using these stamps? In how many ways can this be done? (Find how many pairs of nonnegative integers* $(x, y)$ *there are so that* $22x + 18y = 500$*.)*

Since $18 = 2 \cdot 3^2$ and $22 = 2 \cdot 11$, we have $d = \gcd(18, 22) = 2$. Because $d \mid 500$, there are integer solutions. Let's find all pairs of integer solutions $(x, y)$ and determine how many of these correspond to solutions of the stamp problem ($x \geq 0$ and $y \geq 0$.) The Euclidean Algorithm tells us that

$$\begin{aligned} 22 &= 1 \cdot 18 + 4 \\ 18 &= 4 \cdot 4 + 2 \\ 4 &= 2 \cdot 2 + 0. \end{aligned}$$

Thus $\gcd(22, 18) = 2$. Working backwards, we find integers $p$ and $q$ so that $22p + 18q = d$.

$$2 = 18 - 4 \cdot 4 = 18 - 4(22 - 18) = 5 \cdot 18 - 4 \cdot 22.$$

One solution is gotten by multiplying by 250

$$22 \cdot (-1000) + 18 \cdot 1250 = 500.$$

Let $(x_0, y_0) = (-1000, 1250)$. The general solution $(x, y)$ satisfies $22x + 18y = 500$. Subtracting and dividing by $d$,

$$11(1000 + x) = 9(1250 - y).$$

It follows that $11 \mid 1250 - y$ so that there is an integer $t$ so that

$$1250 - y = 11t.$$

Substituting, $11(1000 + x) = 99t$. Thus a general solution has the form

$$\begin{aligned} x &= -1000 + 9t, \\ y &= 1250 - 11t. \end{aligned}$$

For any $t \in \mathbf{Z}$ these are solutions of $22x + 18y = 500$, so this parameterizes all solutions. Which, if any, correspond to the postage stamp problem? We must have $x \geq 0$ or $1000/9 \leq t$ which means $112 \leq t$. Also we must have $y \geq 0$ which means $11t \leq 1250$ or $t \leq 113$. Thus there are TWO solutions for the stamp problem, namely for $t = 112$ which is $(x, y) = (8, 18)$ and for $t = 113$ which is $(x, y) = (17, 7)$.