Math 2200 § 2. **First Midterm Exam** Name: Solutions
Treibergs February 10, 2011

1. *Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$.*

   (a) *Find the power set $P(B)$.*
   The power set is the set of all subsets. Thus
   $$\mathcal{P}(B) = \Big\{\emptyset, \{3\}, \{4\}, \{5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{3,4,5\}\Big\}.$$

   (b) *Find the Cartesian Product $A \times B$.*
   The Cartesian Product is the set of ordered pairs
   $$A \times B = \Big\{(1,3), (2,3), (3,3), (4,3), (1,4), (2,4), (3,4), (4,4), (1,5), (2,5), (3,5), (4,5)\Big\}.$$

   (c) *Find the difference $B - A$.*
   The difference set is the set of elements of B that are not in A, so
   $$B - A = \Big\{5\Big\}.$$

   (d) *Suppose $f : A \to B$ is given by $f(x) = \left\lfloor \frac{x+9}{3} \right\rfloor$. Find $f(\{2,3\})$.*
   The floor function gives the greatest integer less than or equal to. The image of a set is the set of images
   $$f(\{2,3\}) = \Big\{f(2), f(3)\Big\} = \left\{\left\lfloor \frac{2+9}{3} \right\rfloor, \left\lfloor \frac{3+9}{3} \right\rfloor\right\} = \left\{\left\lfloor \frac{11}{3} \right\rfloor, \lfloor 4 \rfloor\right\} = \Big\{3, 4\Big\}.$$

2. (a) *Use universal and existential quantifiers and predicates to express the following statement. Be sure to state the domains of your quantifiers.*

   "Everybody doesn't like something, but nobody doesn't like Sara Lee."

   Let S denote the set of people and T denote the set of things. "Sara Lee" is a thing, a coffee cake, or a big bakery. The quotation is from a famous jingle, *e.g.,*

   `http://www.youtube.com/watch?v=qEL4MALB8Ig`

   Let $L(s,t)$ = "s likes t.". Then "nobody" can be rendered "it is not the case that there is someone." Or you can use its equivalent. Both versions are acceptable.
   $$\begin{aligned}&\big[(\forall s \in S)\ (\exists t \in T)\ \neg L(s,t)\big] \wedge \neg(\exists s \in S)\ \neg L(s, \text{Sara Lee})\\ \equiv\ &\big[(\forall s \in S)\ (\exists t \in T)\ \neg L(s,t)\big] \wedge (\forall s \in S)\ L(s, \text{Sara Lee})\end{aligned}$$

   (b) *Find the negation of your expression from (a) in a form such that the negators come after the quantifiers.*
   The negation of the second expression is
   $$\begin{aligned}&\neg\big[(\forall s \in S)\ (\exists t \in T)\ \neg L(s,t)\big] \vee \neg(\forall s \in S)\ L(s, \text{Sara Lee})\\ \equiv\ &\big[(\exists s \in S)\ (\forall t \in T)\ L(s,t)\big] \vee (\exists s \in S)\ \neg L(s, \text{Sara Lee})\\ \equiv\ &(\exists s \in S)\ \big[\big((\forall t \in T)\ L(s,t)\big) \vee \neg L(s, \text{Sara Lee})\big].\end{aligned}$$

(c) *Render your negated statement in (b) as an English sentence.*

"Somebody likes everything or doesn't like Sara Lee."

3. (a) *By constructing a truth table, show that $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ are logically equivalent.*

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Since the $p \leftrightarrow q$ and the $(p \to q) \wedge (q \to p)$ columns are the same, the two expressions are logically equivalent.

(b) *For the following premises, what relevant conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.*

"If I want to do well in analysis, I study discrete math."
"I study discrete math if I'm interested in computer science."
"I did not study discrete math."

Let $P =$ "I want to do well in analysis," $Q =$ "I study discrete math," and $R =$ "I'm interested in computer science." The hypotheses are

h1. $P \to Q$

h2. $R \to Q$

h3. $\neg Q$.

We may draw some conclusions

1. $P \to Q$ hypothesis (h1)
2. $\neg Q$ hypothesis (h3)
3. $\neg P$ modus tollens from (1) and (2)
4. $R \to Q$ hypothesis (h2)
5. $\neg R$ modus tollens from (2) and (4)
6. $\neg P \wedge \neg R$ conjunction of (3) and (5)
7. $\neg(P \vee Q)$ De Morgan on (6)

The conclusion may be rendered

"I neither want to do well in analysis nor am interested in computer science."

4. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

   (a) *If $A, B \subset U$ are subsets then $A \cup \overline{B} = \overline{B - A}$.*

   TRUE. $\overline{B - A} = \overline{B \cap \overline{A}} = \overline{B} \cup A$.

   (b) *Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions such that the composite $g \circ f : X \to Z$ is onto. Then $f : X \to Y$ is onto.*

   FALSE. For example, let $X = \{1\}$, $Y = \{2, 3\}$ and $Z = \{4\}$ and the function $f : X \to Y$ given by $f(1) = 2$, $g : Y \to Z$ be given by $g(2) = g(3) = 4$. f is not onto because $f(X) = \{2\} \neq Y$, but $g \circ f : X \to Z$ takes $g \circ f(1) = g(f(1)) = g(2) = 4$ so $g \circ f(X) = Z$ so $g \circ f$ is onto.

   (c) *If $f, g : \mathbf{R} \to \mathbf{R}$ are both one-to-one, then $f + g : \mathbf{R} \to \mathbf{R}$ is one-to-one.*

   FALSE. Let $f(x) = x$ and $g(x) = -x$. Then f is one-to-one because $f(x) = f(y)$ implies $x = y$. g is one-to-one because $g(x) = g(y)$ implies $-x = -y$ implies $x = y$. However $(f + g)(x) = x - x = 0$ which is not one-to-one, because, for example $(f + g)(1) = 0 = (f + g)(2)$.

5. *Let $f : X \to Y$ be a function which is one-to-one. Let $A, B \subseteq X$ be two subsets of X. Show that $f(A \cap B) = f(A) \cap f(B)$.*

   We argue in two steps, first, to show $f(A \cap B) \subseteq f(A) \cap f(B)$ and second, to show $f(A \cap B) \supseteq f(A) \cap f(B)$.

   To see that $f(A \cap B) \subseteq f(A) \cap f(B)$, we choose an arbitrary $y \in f(A \cap B)$ to show that also $y \in f(A) \cap f(B)$. Since $y \in f(A \cap B)$, by the definition of being in the image, there is an $x \in A \cap B$ so that $y = f(x)$. But, $x \in A \cap B$ implies $x \in A$ and $x \in B$. Because $x \in A$ we have $y = f(x) \in f(A)$. Also, because $x \in B$ we have $y = f(x) \in f(B)$. As y is in both $f(A)$ and $f(B)$ we conclude that it is in the intersection $y \in f(A) \cap f(B)$, finishing the first step. Note that one-to-one was not used in this step.

   To see that $f(A \cap B) \supseteq f(A) \cap f(B)$, we choose an arbitrary $y \in f(A) \cap f(B)$ to show that also $y \in f(A \cap B)$. Since $y \in f(A) \cap f(B)$, by the definition of intersection, $y \in f(A)$ and $y \in f(B)$. By the definition of being in the image $y \in f(A)$, there is an $x_1 \in A$ so that $y = f(x_1)$. By the definition of being in the image $y \in f(B)$, there is also a $x_2 \in B$ so that $y = f(x_2)$. We do not yet know if we can take x_1 and x_2 to be the same—that is where one-to-one is used. Since f is assumed to be one-to-one, and we have shown that $f(x_1) = y = f(x_2)$, it follows that $x_1 = x_2$ by the definition of one-to-one. Let us call this point $x = x_1 = x_2$. Note that $x = x_1 \in A$ and $x = x_2 \in B$. Since x is in both, by the definition of intersection, $x \in A \cap B$. Finally, by the definition of the image, $y = f(x) \in f(A \cap B)$. This finishes the second step and thus the argument.

6. (a) *State the definition: the real number x is rational.*

   The real number x is rational if there are two integers $p, q \in \mathbf{Z}$ with $q \neq 0$ such that $x = p/q$. If the number is not rational it is called irrational.

   (b) *Prove that if x is a rational number then $y = x + \sqrt{2}$ is irrational. [Hint: you may assume that $\sqrt{2}$ is irrational.]*

   Arguing by contradiction we assume that the hypothesis is true (x is rational) and the conclusion is false (y is rational.) By the definition of rational, there are four integers, $p, q, r, s \in \mathbf{Z}$ with $q \neq 0$ and $s \neq 0$ such that $x = p/q$ and $y = r/s$. It follows that

   $$\sqrt{2} = y - x = \frac{p}{q} - \frac{r}{s} = \frac{ps - qr}{qs} = \frac{t}{u},$$

   where $t = ps - qr$ is an integer and $u = qs$ is a nonzero integer because both q and s are nonzero. Thus we have shown $\sqrt{2} = t/u$ where $t, u \in \mathbf{Z}$ with $u \neq 0$. In other words $\sqrt{2}$ is rational, contradicting the fact that $\sqrt{2}$ is irrational. □