

Sample First Midterm Questions. Some questions from Math 3210 Midterms of Sept. 17, 2008 and Sept. 16, 2009.

1. From Lewis Carroll: extract a pair of premises and finish the conclusion.

“It was most absurd of you to offer it! You might have known, if you had any sense, that no old sailors ever like gruel.”

“But I thought, as he was an uncle of yours—”

“An uncle of mine, indeed! Stuff!”

“You may call it stuff if you like. All I know is *my uncles* are all old men: and they like gruel like anything!”

“Well then your uncles are—”

2. Prove the logical identity $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$.
3. Express the following English sentence in terms of predicates, quantifiers, logical connectives and mathematical operators. Negate the expression. Render the resulting expression as an English sentence.

“Every positive integer is the sum of squares of three integers.”

4. Show that the following expression is a tautology.

$$(\neg p \wedge (p \rightarrow q)) \rightarrow \neg p.$$

5. Show that the following statements are inconsistent.

“If Kira takes the job offer, then she will get a signing bonus.”

“If Kira takes a job offer, then she will receive a higher salary.”

“If Kira gets a signing bonus, then she will not receive a higher salary.”

“Kira takes the job offer.”

6. Consider the statement about numbers.

$$(\exists x)(\exists y)[(x \neq y) \wedge (\forall z)(z = x \vee z = y)]$$

(a) Find a domain for the quantifiers in this statement such that the statement is false.

(b) Find a domain for the quantifiers in this statement such that the statement is true.

7. Prove that if x^3 is irrational, then x is irrational.
8. Prove that given any nonnegative integer n , there is a unique nonnegative integer m such that

$$m^2 \leq n < (m + 1)^2. \tag{1}$$

9. Prove that there are no solutions in positive integers for the equation

$$x^4 + y^4 = 625. \quad (2)$$

10. The geometric and harmonic means of two positive real numbers are given by

$$\mathcal{G}(x, y) = \sqrt{xy}, \quad \mathcal{H}(x, y) = \frac{1}{\frac{1}{2} \left(\frac{1}{x} + \frac{1}{y} \right)} = \frac{2xy}{x+y}.$$

Formulate a conjecture relating the magnitudes of these quantities and prove your conjecture.

11. Let A , B and C be subsets of a universal set U . Show using set builder notation and logical manipulation only that

$$(A - B) - C = (A - C) - (B - C).$$

12. Let A , B and C be subsets of a universal set U . Draw the Venn Diagram for

$$(A - B) - C$$

13. When are the following true?

(a) $A \cup (B \cap C) = (A \cup C) \cap (A \cup B)$.

(b) $A \cap (B \cap C) = (A \cap B) \cap C$.

(c) $A \cup (B \cup C) = A - (B - C)$.

(d) $A - (B - C) = (A - B) - C$.

(e) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$.

(f) $A - (B \cap C) = (A - B) \cup (A - C)$.

(g) $A - (B \cup C) = (A - B) \cap (A - C)$.

14. Suppose that f is a one-to-one function from the set A to the set B . Let S and T be subsets of A . Show that $f(S \cap T) = f(S) \cap f(T)$.

15. Suppose A , B and C are finite subsets of U . Show that the cardinality satisfies the inclusion-exclusion formula

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

16. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) **Statement.** If $A, B, C \subset X$ are subsets then $A \setminus B = C$ implies $A = B \cup C$.

(b) **Statement.** Suppose that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are functions such that the composite $g \circ f : X \rightarrow Z$ is one-to-one. Then $f : X \rightarrow Y$ is one-to-one.

(c) **Statement.** If $f : X \rightarrow Y$ is onto, then for all subsets $A, B \subset X$ we have $f(A \cap B) = f(A) \cap f(B)$.

17. Let $f : X \rightarrow Y$ be a function and $V_\alpha \subset Y$ be a subset for each $\alpha \in A$. Show

$$f^{-1} \left(\bigcap_{\alpha \in A} V_\alpha \right) = \bigcap_{\alpha \in A} f^{-1}(V_\alpha).$$

18. Let $f : X \rightarrow Y$ be a function. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

- (a) If $X = f^{-1}(Y)$ then f is onto.
- (b) Suppose that for all $x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$ we have $x_1 \neq x_2$. Then f is one-to-one.
- (c) **Statement C.** If $A \subset X$ and $f(A) = Y$ then $A = X$.

19. Give as simple a description as possible of the set \mathcal{S} in terms of intervals of the real numbers

$$\mathcal{S} = \{x \in \mathbb{R} : (\exists m \in \mathbb{N})(\forall \epsilon \in \mathbb{R} \text{ such that } \epsilon > 0) \quad m - \epsilon < x\}.$$

Show that set you describe equals \mathcal{S} .

20. Give an example of a function from \mathbf{N} to \mathbf{N} that is onto but not one-to-one.

Solutions.

1. *From Lewis Carroll: extract a pair of premises and finish the conclusion.*

“It was most absurd of you to offer it! You might have known, if you had any sense, that no old sailors ever like gruel.”

“But I thought, as he was an uncle of yours—”

“An uncle of mine, indeed! Stuff!”

“You may call it stuff if you like. All I know is *my uncles* are all old men: and they like gruel like anything!”

“Well then your uncles are—”

Assuming (in a non-P.C. fashion) that sailors are men, then the essential premises may be rendered

“All men who are old and sailors don’t like gruel.”

“All men who are your uncles are old and like gruel.”

“All men who are your uncles—”

Let the domain be “men,” $P(x)$ = “ x is old,” $S(x)$ = “ x is a sailor,” $U(x)$ = “ x is your uncle.” and $G(x)$ = “ x likes gruel.”. Symbolically, this becomes

1. $(\forall x) (P(x) \wedge S(x) \rightarrow \neg G(x))$
2. $(\forall x) (U(x) \rightarrow P(x) \wedge G(x))$

To conclude the universal generalization, let c be any arbitrary man. Then

3.	$(\forall x) (P(x) \wedge S(x) \rightarrow \neg G(x))$	hypothesis (1)
4.	$P(c) \wedge S(c) \rightarrow \neg G(c)$	universal instantiation of (3)
5.	$(\forall x) (U(x) \rightarrow P(x) \wedge G(x))$	hypothesis (2)
6.	$U(c) \rightarrow P(c) \wedge G(c)$	universal instantiation of (5)
7.	$P(c) \wedge G(c) \rightarrow G(c)$	simplification
8.	$U(c) \rightarrow G(c)$	hypothetical syllogism of (6) and (7)
9.	$G(c) \rightarrow \neg(P(c) \wedge S(c))$	contrapositive of (4)
10.	$G(c) \rightarrow \neg P(c) \vee \neg S(c)$	de Morgan's law in (9)
11.	$U(c) \rightarrow \neg P(c) \vee \neg S(c)$	hypothetical syllogism of (8) and (10)
12.	$U(c) \rightarrow (P(c) \wedge G(c)) \wedge (\neg P(c) \vee \neg S(c))$	equivalence of conditional statements $(a \rightarrow b) \wedge (a \rightarrow c) \equiv a \rightarrow (b \wedge c)$
13.	$U(c) \rightarrow (P(c) \wedge G(c) \wedge \neg P(c)) \vee (P(c) \wedge G(c) \wedge \neg S(c))$	distributive in (12)
13.	$U(c) \rightarrow \text{FALSE} \vee (P(c) \wedge G(c) \wedge \neg S(c))$	negation law
14.	$U(c) \rightarrow P(c) \wedge G(c) \wedge \neg S(c)$	identity law
15.	$(\forall x)(U(x) \rightarrow P(x) \wedge G(x) \wedge \neg S(x))$	universal generalization

Thus the conclusion may be phrased

“All men who are old and sailors don't like gruel.”

“All men who are your uncles are old and like gruel.”

“All men who are your uncles are not sailors but are old and like gruel. ”

As Carroll may have put it,

“Well then your uncles are gruel liking old men who are not sailors. ”

2. *Prove the logical identity* $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$.

let us write the truth tables for both sides.

p	q	r	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \wedge (p \rightarrow r)$	$q \wedge r$	$p \rightarrow (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

The $(p \rightarrow q) \wedge (p \rightarrow r)$ and $p \rightarrow (p \wedge r)$ columns are the same so that they are logically equivalent expressions.

3. Express the following English sentence in terms of predicates, quantifiers, logical connectives and mathematical operators. Negate the expression. Render the resulting expression as an English sentence.

Every positive integer is the sum of squares of three integers.

If the domain is the integers, the statement can be rendered

$$(\forall x)[(x > 0) \rightarrow (\exists y)(\exists z)(\exists w)(x = y^2 + z^2 + w^2)].$$

The negation is

$$(\exists x)[(x > 0) \wedge (\forall y)(\forall z)(\forall w)(x \neq y^2 + z^2 + w^2)],$$

which is equivalent to

$$(\exists x)(\forall y)(\forall z)(\forall w)[(x > 0) \wedge (x \neq y^2 + z^2 + w^2)].$$

We may render this in English as

There is a positive integer that is the not sum of squares of any three integers.

4. Show that the following expression is a tautology.

$$(\neg p \wedge (p \rightarrow q)) \rightarrow \neg p.$$

One way to do this is by truth tables. Another is to manipulate using the basic logical equivalences.

- | | | |
|----|--|---------------------------|
| 1. | $(\neg p \wedge (p \rightarrow q)) \rightarrow \neg p$ | given |
| 2. | $\neg(\neg p \wedge (p \rightarrow q)) \vee \neg p$ | equivalent of conditional |
| 3. | $\neg[(\neg p \wedge (p \rightarrow q)) \wedge p]$ | De Morgan |
| 4. | $\neg[(p \rightarrow q) \wedge \neg p] \wedge p$ | commutative |
| 5. | $\neg[(p \rightarrow q) \wedge (\neg p \wedge p)]$ | associative |
| 6. | $\neg[(p \rightarrow q) \wedge \text{FALSE}]$ | negation law |
| 7. | $\neg[\text{FALSE}]$ | domination law |
| 8. | TRUE | negation |

Thus the given statement is a tautology.

5. Show that the following statements are inconsistent.

“If Kira takes the job offer, then she will get a signing bonus.”

“If Kira takes a job offer, then she will receive a higher salary.”

“If Kira gets a signing bonus, then she will not receive a higher salary.”

“Kira takes the job offer.”

The sentences are consistent if there are truth values for which all conditions are true. Let us assign names to the statements. j = “Kira takes the job offer.” , s = “Kira takes gets a signing bonus.” and h = “Kira receives a higher salary.” . The statements are thus

- | | |
|----|------------------------|
| 1. | $j \rightarrow s$ |
| 2. | $j \rightarrow h$ |
| 3. | $s \rightarrow \neg h$ |
| 4. | j |

Can they be true simultaneously? (4) says that j is true. (1) implies s is true. (3) implies $\neg h$ is true. The contrapositive of (2) is $\neg h \rightarrow \neg j$, hence $\neg j$ is true, or j is false. This is a contradiction, so the statements are inconsistent so cannot be simultaneously true.

6. Consider the statement about numbers.

$$(\exists x)(\exists y)[(x \neq y) \wedge (\forall z)(z = x \vee z = y)]$$

- (a) Find a domain for the quantifiers in this statement such that the statement is false.
 (b) Find a domain for the quantifiers in this statement such that the statement is true.

If we take the domain $D = \mathbf{R}$ to be all real numbers, then the statement is false. For all choices of $x, y \in \mathbf{R}$ such that $z \neq y$ we may find a third real number z such that $z \neq x$ and $z \neq y$. Thus the negation holds

$$(\forall x)(\forall y)[(x \neq y) \wedge (\exists z)(z \neq x \wedge z \neq y)].$$

But if we take the domain $D = \{\alpha, \beta\}$ to be a set of two distinct ($\alpha \neq \beta$) real numbers, then the statement is true. For the choice $x = \alpha$ and $y = \beta$ we have $x \neq y$ and, because z can only be one of the two then it must be that $z = x$ or $z = y$. Hence the statement holds.

7. Prove that if x^3 is irrational, then x is irrational.

Arguing by contrapositive, we have to show that if x is rational then x^3 is rational. By definition, x is rational if and only if there are integers $p, q \in \mathbf{Z}$ with $q \neq 0$ such that $x = p/q$. Cubing both sides,

$$x^3 = \left(\frac{p}{q}\right)^3 = \frac{p^3}{q^3} = \frac{k}{\ell},$$

where $k = p^3$ and $\ell = q^3$ are integers, because multiplication of integers are integers. We also have $\ell \neq 0$ because $q \neq 0$. (If $\ell = q^3 = 0$, then either $q = 0$ or $q^2 = 0$ by Theorem 6 of Appendix A2. Because $q \neq 0$ we must have $q^2 = 0$. Again, by Theorem 6, this implies $q = 0$, which is not the case, so $q^2 \neq 0$.) Thus we have shown that $x^3 = k/\ell$ where $k, \ell \in \mathbf{Z}$ with $\ell \neq 0$. By definition, x^3 is rational. Thus we have established the contrapositive: if x^3 is irrational then x is irrational.

8. Prove that given any nonnegative integer n , there is a unique nonnegative integer m such that

$$m^2 \leq n < (m+1)^2. \quad (3)$$

This problem depends on a property of the integers we have not discussed, and will be discussed in Math 3210. The property, which seems intuitively obvious, is called the Well Ordering Principle (see Appendix 2). Choose an arbitrary $n \in \mathbf{N}$ and consider the set

$$\mathcal{S} = \{x \in \mathbf{N} : n < x^2\}.$$

The set is nonempty because it contains $x = n + 1$. This is because

$$x^2 = (n+1)^2 = n^2 + 2n + 1 > 0 + 2n + 0 \geq 1n = n.$$

By the Well ordering property, \mathcal{S} has a least element, call it j . In other words $j \in \mathcal{S}$, $x \geq j$ for all $x \in \mathcal{S}$ and $j-1 \notin \mathcal{S}$. Note that $j \in \mathcal{S}$ implies $1 \leq n < j^2$ so $j > 1$. Hence $m = j-1$ is a positive integer and is the desired number to satisfy (3). $j = m+1 \in \mathcal{S}$ means $n < (m+1)^2$. Since $m = j-1 \notin \mathcal{S}$ we have $\neg(n < m^2)$ so $m^2 \leq n$. Thus we have proved the existence of m .

Let m be a positive integer that satisfies the condition (3). To prove the uniqueness, we argue as suggested in the text that if there were another number $p \neq m$ then the condition

$$p^2 \leq n < (p+1)^2 \quad (4)$$

would fail. Since $p \neq m$ either $p < m$ or $p > m$. In the first case, $p+1 \leq m$ so that by (3),

$$(p+1)^2 \leq m^2 \leq n$$

so that (4) fails. In the second case, $m+1 \leq p$ so that by (3),

$$n < (m+1)^2 \leq p^2$$

so that (4) fails also. Because (4) failed whenever $p \neq m$, then m is unique.

9. Prove that there are no solutions in positive integers for the equation

$$x^4 + y^4 = 625. \quad (5)$$

We prove the result in three cases: under (i.) $x \geq 5$, under (ii.) $y \geq 5$ or under (iii.) $x < 5$ and $y < 5$. These cases exhaust all possible pairs $x, y \in \mathbf{N}$.

Under case (i.), we have for any $x, y \in \mathbf{N}$ with $x \geq 5$,

$$x^4 + y^4 \geq 5^4 + 1 \geq 625 + 1 > 625,$$

so (5) fails.

Under case (ii.), we have for any $x, y \in \mathbf{N}$ with $y \geq 5$,

$$x^4 + y^4 \geq 1 + 5^4 \geq 1 + 625 > 625,$$

so (5) fails.

Under case (iii.), we have to show that for any $x, y \in \mathbf{N}$ such that $1 \leq x, y \leq 4$ that $x^4 + y^4 \neq 625$. In this case

$$x^4 + y^4 \leq 4^4 + 4^4 = 512 < 625.$$

so (5) fails also. Since (5) failed in all three cases, there are no pairs of positive integers that satisfy (5).

10. The geometric and harmonic means of two positive real numbers are given by

$$\mathcal{G}(x, y) = \sqrt{xy}, \quad \mathcal{H}(x, y) = \frac{1}{\frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)} = \frac{2xy}{x + y}.$$

Formulate a conjecture relating the magnitudes of these quantities and prove your conjecture.

A few test cases give $\mathcal{G}(1, 1) = 1$, $\mathcal{H}(1, 1) = 1$; $\mathcal{G}(1, 2) = \sqrt{2} = 1.414\dots$, $\mathcal{H}(1, 2) = 4/3 = 1.333\dots$; $\mathcal{G}(1, 3) = \sqrt{3} = 1.732$, $\mathcal{H}(1, 3) = 3/2 = 1.5$; $\mathcal{G}(1, 9) = 3$, $\mathcal{H}(1, 9) = 9/5 = 1.8$. We conjecture

Theorem. For any pair of positive real numbers x, y , the geometric mean exceeds the harmonic mean $\mathcal{G}(x, y) \geq \mathcal{H}(x, y)$, and equality holds if and only if $x = y$.

Here is a proof of our conjecture. Note that both $\mathcal{G}(x, y)$ and $\mathcal{H}(x, y)$ are positive as they are the root of a product or the quotient of a product and a sum of positive numbers. Hence the result follows if we can show $\mathcal{G}(x, y)^2 \geq \mathcal{H}(x, y)^2$ for all $x, y > 0$ and equality holds if and only if $x = y$. Computing

$$\begin{aligned} \mathcal{G}(x, y)^2 - \mathcal{H}(x, y)^2 &= xy \left(1 - \frac{4xy}{(x + y)^2} \right) \\ &= xy \frac{(x + y)^2 - 4xy}{(x + y)^2} \\ &= xy \frac{(x - y)^2}{(x + y)^2} \end{aligned}$$

The right side is nonnegative as it is a square multiplied by a positive, so $\mathcal{G}(x, y) \geq \mathcal{H}(x, y)$ follows. If $x = y$ then $\mathcal{G}(x, y) = \mathcal{H}(x, y)$. Moreover if $\mathcal{G}(x, y) = \mathcal{H}(x, y)$ then $(x - y)^2 = 0$ so $x = y$ follows.

11. Let A , B and C be subsets of a universal set U . Show using set builder notation and logical manipulation only that

$$(A - B) - C = (A - C) - (B - C).$$

We work from both ends to meet in middle.

- | | | |
|-----|---|----------------|
| 1. | $(A - B) - C$ | left hand side |
| 2. | $= \{x x \in (A - B) \wedge x \notin C\}$ | set equivalent |
| 3. | $= \{x [x \in A \wedge x \notin B] \wedge x \notin C\}$ | set difference |
| 4. | $= \{x [x \in A \wedge \neg(x \in B)] \wedge \neg(x \in C)\}$ | negation |
| 5. | $= \{x ([x \in A \wedge \neg(x \in C)] \wedge \neg(x \in B)) \vee \text{FALSE}\}$ | identity |
| 6. | $= \{x ([x \in A \wedge \neg(x \in C)] \wedge \neg(x \in B))$
$\vee (x \in A \wedge \text{FALSE})\}$ | domination |
| 7. | $= \{x ([x \in A \wedge \neg(x \in C)] \wedge \neg(x \in B))$
$\vee (x \in A \wedge [\neg(x \in C) \wedge x \in C])\}$ | negation |
| 8. | $= \{x ([x \in A \wedge \neg(x \in C)] \wedge \neg(x \in B))$
$\vee ([x \in A \wedge \neg(x \in C)] \wedge x \in C)\}$ | associative |
| 9. | $= \{x [x \in A \wedge \neg(x \in C)] \wedge [\neg(x \in B) \vee x \in C]\}$ | distributive |
| 10. | $= \{x [x \in A \wedge \neg(x \in C)] \wedge \neg[x \in B \wedge \neg(x \in C)]\}$ | De Morgan |
| 11. | $= \{x [x \in A \wedge x \notin C] \wedge \neg[x \in B \wedge x \notin C]\}$ | negation |
| 12. | $= \{x x \in (A - C) \wedge \neg[x \in (B - C)]\}$ | set equivalent |
| 13. | $= \{x x \in (A - C) \wedge x \notin (B - C)\}$ | negation |
| 14. | $= (A - C) - (B - C)$ | set equivalent |

12. Let A , B and C be subsets of a universal set U . Draw the Venn Diagram for

$$(A - B) - C$$

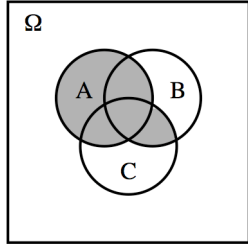
13. When are the following true?

- (a) $A \cup (B \cap C) = (A \cup C) \cap (A \cup B)$.
 (b) $A \cap (B \cap C) = (A \cap B) \cap C$.
 (c) $A \cup (B \cup C) = A - (B - C)$.
 (d) $A - (B - C) = (A - B) - C$.
 (e) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$.

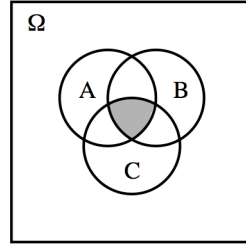
$$(f) A - (B \cap C) = (A - B) \cup (A - C).$$

$$(g) A - (B \cup C) = (A - B) \cap (A - C).$$

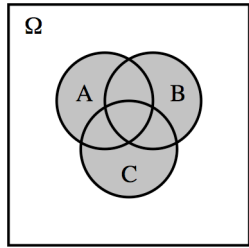
We indicate the named sets by shading in the corresponding Venn diagrams. Let Ω denote the universal set. The symmetric difference is here denoted by $A \Delta B$ (our text uses \oplus) which denotes elements that are in A or in B but not in both. If the Venn diagrams are the same for both sets, then the equality holds always. If there are different regions shaded in the diagram, then equality implies that the regions that are unshaded in one diagram and shaded in the other should be empty.



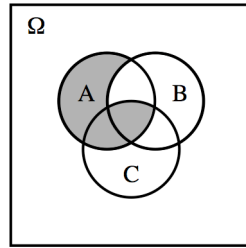
(a.) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
holds always.



(b.) $A \cap (B \cap C) = (A \cap B) \cap C$
holds always.



(c.) $A \cup (B \cup C)$



$A - (B - C)$

Equality holds if all four conditions hold:

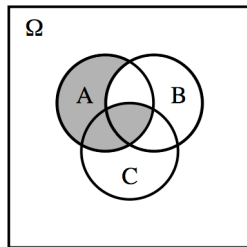
$$A \cap B \cap \bar{C} = \emptyset, \quad \bar{A} \cap B \cap \bar{C} = \emptyset, \quad \bar{A} \cap B \cap C = \emptyset, \quad \bar{A} \cap \bar{B} \cap C = \emptyset.$$

Equivalently, using $E \cap F = \emptyset$ and $E \cap F^c = \emptyset$ if and only if $E = \emptyset$, the first two and the last two say

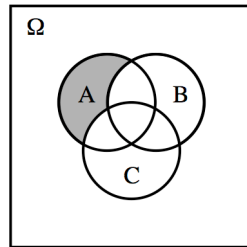
$$B \cap \bar{C} = \emptyset, \quad \bar{A} \cap C = \emptyset$$

which are equivalent to

$$B \subseteq C \subseteq A.$$



(d.) $A - (B - C)$



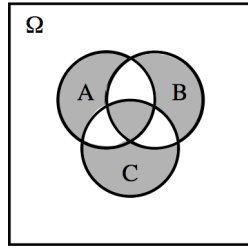
$(A - B) - C$

Equality holds if the two conditions hold:

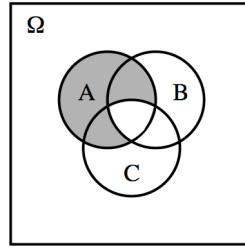
$$A \cap B \cap C = \emptyset, \quad A \cap \bar{B} \cap C = \emptyset.$$

Equivalently,

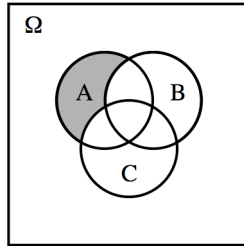
$$A \cap C = \emptyset.$$



(e.) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$
holds always.



(f.) $A - (B \cap C) = (A - B) \cup (A - C)$
holds always.



(g.) $A - (B \cup C) = (A - B) \cap (A - C)$
holds always.

14. Suppose that f is a one-to-one function from the set A to to the set B . Let S and T be subsets of A . Show that $f(S \cap T) = f(S) \cap f(T)$.

Equality is shown in two steps, that $f(S \cap T) \subseteq f(S) \cap f(T)$ and then that $f(S \cap T) \supseteq f(S) \cap f(T)$.

To show $f(S \cap T) \subseteq f(S) \cap f(T)$, we show that any arbitrary $y \in f(S \cap T)$ also satisfies $y \in f(S) \cap f(T)$. For $y \in f(S \cap T)$, there is $x \in S \cap T$ so that $y = f(x)$ by definition of the image of a set. Because $x \in S \cap T$ we have that $x \in S$ and $x \in T$. It follows that $y = f(x) \in f(S)$ and $y = f(x) \in f(T)$. As y is in both sets we conclude $y \in f(S) \cap f(T)$, as to be shown. Note that one-to-one was not needed in this direction.

To show $f(S \cap T) \supseteq f(S) \cap f(T)$, we show that any arbitrary $y \in f(S) \cap f(T)$ also satisfies $y \in f(S \cap T)$. Since $y \in f(S) \cap f(T)$, it follows that $y \in f(S)$ and $y \in f(T)$. It follows that there is $p \in S$ so that $y = f(p)$ and that there is a $q \in T$ so that $y = f(q)$ by definition of the image of sets. Thus we have $f(p) = y = f(q)$. Since f is one-to-one, it follows that $p = q$. Let $x = p = q$. As $x = p \in S$ and $x = q \in T$, it follows that $x \in S \cap T$. Hence it follows that $y = f(x) \in f(S \cap T)$ as to be shown.

Since we have shown both containments, we conclude that the sets are equal $f(S \cap T) = f(S) \cap f(T)$.

15. Suppose A, B and C are finite subsets of U . Show that the cardinality satisfies the inclusion-exclusion formula

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

The formula for three sets follows from the formula for two sets, namely,

$$|A \cup D| = |A| + |D| - |A \cap D|.$$

Thus, putting $D = B \cup C$ we get

$$\begin{aligned}
 |A \cup B \cup C| &= |A \cup D| \\
 &= |A| + |D| - |A \cap D| \\
 &= |A| + |B \cup C| - |A \cap (B \cup C)| \\
 &= |A| + (|B| + |C| - |B \cap C|) - |(A \cap B) \cup (A \cap C)| \\
 &= |A| + |B| + |C| - |B \cap C| - [|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|] \\
 &= |A| + |B| + |C| - |B \cap C| - [|A \cap B| + |A \cap C| - |A \cap B \cap C|] \\
 &= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|.
 \end{aligned}$$

De Morgan's formula was used in the fourth equation.

16. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) **Statement.** If $A, B, C \subset X$ are subsets then $A \setminus B = C$ implies $A = B \cup C$.

FALSE. *e.g.*, take $X = \mathbb{R}$, $A = [0, 2]$, $B = [1, 3]$ so $C = [0, 1)$. But then $A \neq B \cup C = [0, 3]$.

(b) **Statement.** Suppose that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are functions such that the composite $g \circ f : X \rightarrow Z$ is *one-to-one*. Then $f : X \rightarrow Y$ is *one-to-one*.

TRUE. Choose $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Apply g to both sides $g \circ f(x_1) = g \circ f(x_2)$. But $g \circ f$ is one-to-one so $x_1 = x_2$. Thus f is one-to-one.

(c) **Statement.** If $f : X \rightarrow Y$ is *onto*, then for all subsets $A, B \subset X$ we have $f(A \cap B) = f(A) \cap f(B)$.

FALSE. *e.g.*, take $X = Y = \mathbb{R}$ and $f(x) = x^2(x-1)$ which is onto. But for $A = (-1, 0)$ and $B = (0, 1)$ we have $A \cap B = \emptyset$ so $f(A \cap B) = \emptyset$ but $f(A) \cap f(B) = (-2, 0) \cap [-\frac{4}{27}, 0) = [-\frac{4}{27}, 0) \neq \emptyset = f(A \cap B)$.

17. Let $f : X \rightarrow Y$ be a function and $V_\alpha \subset Y$ be a subset for each $\alpha \in A$. Show

$$f^{-1}\left(\bigcap_{\alpha \in A} V_\alpha\right) = \bigcap_{\alpha \in A} f^{-1}(V_\alpha).$$

Proof. We show x is in the left set iff x is in the right set.

$$\begin{aligned}
 x \in f^{-1}\left(\bigcap_{\alpha \in A} V_\alpha\right) &\iff f(x) \in \bigcap_{\alpha \in A} V_\alpha \\
 &\iff (\forall \alpha \in A) (f(x) \in V_\alpha) \\
 &\iff (\forall \alpha \in A) (x \in f^{-1}(V_\alpha)) \\
 &\iff x \in \bigcap_{\alpha \in A} f^{-1}(V_\alpha).
 \end{aligned}$$

18. Let $f : X \rightarrow Y$ be a function. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) If $X = f^{-1}(Y)$ then f is onto.

FALSE. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2$ then $f^{-1}(\mathbb{R}) = \mathbb{R}$ but f is not onto since $-4 \in \mathbb{R}$ is not in the image since $f(x) \geq 0$ for all $x \in \mathbb{R}$. In fact, $X = f^{-1}(Y)$ is true for every function.

(b) Suppose that for all $x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$ we have $x_1 \neq x_2$. Then f is one-to-one.

FALSE. Same example as in A. The logically equivalent contrapositive statement is $x_1 = x_2$ implies $f(x_1) = f(x_2)$ which is true for every function, not just one-to-one functions. Thus for $f(x) = x^2$ we have $x_1^2 \neq x_2^2$ implies $x_1 \neq x_2$ but f is not one-to-one since $f(-2) = 4 = f(2)$.

(c) **Statement C.** If $A \subset X$ and $f(A) = Y$ then $A = X$.

FALSE. Define $f : \mathbb{R} \rightarrow [0, \infty)$ by $f(x) = x^2$. Let $X = \mathbb{R}$ and $A = Y = [0, \infty)$. Then $f(A) = Y$ but $A \neq X$.

19. Give as simple a description as possible of the set \mathcal{S} in terms of intervals of the real numbers

$$\mathcal{S} = \{x \in \mathbb{R} : (\exists m \in \mathbb{N})(\forall \epsilon \in \mathbb{R} \text{ such that } \epsilon > 0) \quad m - \epsilon < x\}.$$

Show that set you describe equals \mathcal{S} .

$$\mathcal{S} = \left\{ x \in \mathbb{R} : (\exists m \in \mathbb{N}) \left(x \in \bigcap_{\epsilon > 0} (m - \epsilon, \infty) \right) \right\} = \bigcup_{m \in \mathbb{N}} \bigcap_{\epsilon > 0} (m - \epsilon, \infty) = \bigcup_{m \in \mathbb{N}} [m, \infty) = [1, \infty).$$

To show $\mathcal{S} = [0, \infty)$ we prove “ \subseteq ” and “ \supseteq .”

To show “ \subseteq ”, we choose any $y \in \mathcal{S}$ to show $y \in [1, \infty)$. But $y \in \mathcal{S}$ means for some $m_0 \in \mathbb{N}$ we have for all $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$ there holds $m_0 - \epsilon < y$. Hence, $y \geq m_0$ since otherwise, $y < m_0$ implies that for some $\epsilon_0 \in \mathbb{R}$ such that $0 < \epsilon_0 < m_0 - y$ we have $m_0 - \epsilon_0 > y$ contrary to $m - \epsilon < y$ for all $\epsilon > 0$. Finally, since $y \geq m_0 \geq 1$ we have $y \in [1, \infty)$.

To show “ \supseteq ,” we choose $y \in [1, \infty)$ to show $y \in \mathcal{S}$. Then for $m_0 = 1$ we have $y \geq m_0$. Thus, for every $\epsilon > 0$ we have $y > m_0 - \epsilon$. In other words $(\forall \epsilon \in \mathbb{R} : \epsilon > 0)(m_0 - \epsilon < y)$. Since $m_0 \in \mathbb{N}$ we also have

$$(\exists m \in \mathbb{N})(\forall \epsilon \in \mathbb{R} : \epsilon > 0)(m - \epsilon < y).$$

Thus y satisfies the condition to belong to \mathcal{S} .

20. Give an example of a function from \mathbf{N} to \mathbf{N} that is onto but not one-to-one.

There are many examples. The one that comes to mind first is

$$f(x) = \begin{cases} 1, & \text{if } x = 1; \\ x - 1, & \text{if } x \geq 2. \end{cases}$$

f is not one-to-one because $f(1) = f(2) = 1$. However, f is onto because, for any arbitrary $y \in \mathbf{N}$, we let $x = y + 1$ which is also in \mathbf{N} and $f(x) = f(y + 1) = (y + 1) - 1 = y$.