Math 2200 § 2. **Final Exam** Name:________________________
Treibergs May 2, 2011

1. *Prove that there are no solutions in integers x and y to the equation*

$$2x^2 + 3y^2 = 9.$$

Proof. Let $(x, y) \in \mathbf{Z}^2$ be an ordered pair of integers. We show that (x, y) is not a solution. We consider three cases: if $|x| \ge 3$; if $|y| \ge 2$; or if $|x| < 3$ and $|y| < 2$.

In the first case $|x| \ge 3$ the left side is bigger than the right so there is no equality. We have

$$2x^2 + 3y^3 \ge 2|x|^2 + 0 \ge 2 \cdot 3^2 = 18 > 9.$$

In the second case $|y| \ge 2$ the left side is bigger than the right also. We have

$$2x^2 + 3y^3 \ge 0 + 3|y|^2 \ge 3 \cdot 2^2 = 12 > 9.$$

In the third case, $|x| < 3$ and $|y| < 2$, there are only fifteen remaining possibilities

$$(x, y) \in \Big\{ (0, 0),\ (\pm 1, 0),\ (\pm 2, 0),\ (0, \pm 1),\ (\pm 1, \pm 1),\ (\pm 2, \pm 1) \Big\}$$

For these, $f(x, y) = 2x^2 + 3y^2$ takes the values

$$f(0, 0) = 0,\ f(\pm 1, 0) = 2,\ f(\pm 2, 0) = 8,\ f(0, \pm 1) = 3,\ f(\pm 1, \pm 1) = 5,\ f(\pm 2, \pm 1) = 11.$$

Since none take the value $f(x, y) = 9$, there are no solutions in this case either.

Since every choice of (x, y) fits at least one of the cases, we have shown that $f(x, y) \ne 9$ for all $(x, y) \in \mathbf{Z}^2$. □

2a. *How many integers between* 100 *and* 1000 *inclusive are divisible by* 4 *or* 5*?*

Let A denote the integers divisible by four between 100 and 1000 inclusive. Then $4j \in A$ for some integer j iff $100 \le 4j \le 1000$ so $25 \le j \le 250$. It follows that $|A| = 250 - 25 + 1 = 226$.

Let B denote the integers divisible by five between 100 and 1000 inclusive. Then $5k \in B$ for some integer k iff $100 \le 5j \le 1000$ so $20 \le j \le 200$. It follows that $|B| = 200 - 20 + 1 = 181$.

Let $A \cap B$ denote the integers divisible by both four and five (so divisible by 20) between 100 and 1000 inclusive. Then $20\ell \in A \cap B$ for some integer ℓ iff $100 \le 20\ell \le 1000$ so $5 \le \ell \le 50$. It follows that $|A \cap B| = 50 - 5 + 1 = 46$.

By the inclusion exclusion formula, the number divisible by either four or five is

$$|A \cup B| = |A| + |B| - |A \cap B| = 226 + 181 - 46 = 361.$$

2b. *What is the minimum number of students, each of whom comes from one of the* 29 *counties of Utah, who must be enrolled in the university to guarantee that at least* 8 *come from the same county and are born in the same month?*

In the worst case, there are seven students both coming from the same county and having the same birthmonth, for all pairs of county and month. By the pigeon hole principle, if there is one more student then at least eight share the same county and month. This number is

$$7 \cdot 29 \cdot 12 + 1 = 2437.$$

3a. *Show that if* a, b *and* c *are integers with* $c \geq 2$ *then* $a \equiv b \pmod{c}$ *implies* $a^3 \equiv b^3 \pmod{c}$.

Proof. Suppose that a and b are integers such that $a \equiv b \pmod{c}$. Then there is an integer k so that $a - b = ck$. By factoring the difference of cubes,

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) = ck(a^2 + ab + b^2).$$

Since the right side is an integer multiple of c, it follows that $a^3 \equiv b^3 \pmod{c}$. □

3b. *Find all integer solutions of* $169x \equiv 3 \pmod{176}$.

Using the Euclidean Algorithm, we find $\gcd(176, 169) = 1$.

$$\begin{aligned} 176 &= 1 \cdot 169 + 7 \\ 169 &= 24 \cdot 7 + 1 \\ 7 &= 7 \cdot 1 + 0. \end{aligned}$$

Reading the equations backward

$$1 = 169 - 24 \cdot 7 = 169 - 24(176 - 169) = 25 \cdot 169 - 24 \cdot 176.$$

Hence 25 is the inverse of 169 modulo 176. Multiplying by 3,

$$3 - 75 \cdot 169 = 72 \cdot 176.$$

It follows that $169 \cdot 75 \equiv 3 \pmod{176}$.

Thus, all solutions x are congruent to 75 modulo 176: $x \in \{75 + 176k : k \in \mathbf{Z}\}$.

4. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

a. *Suppose that a pair of dice is rolled. Let* A *be the event that the sum is seven, and* B *be the event that "boxcars," a double six is rolled. Then the events* A *and* B *are independent.*

FALSE. These events are mutually exclusive (disjoint), not independent. There are six ways for dice to add to seven, and 36 possible rolls, so assuming each roll is equally likely, $P(A) = \frac{6}{36} = \frac{1}{6}$. There is only one way to roll boxcars so $P(B) = \frac{1}{36}$. However, a roll can't both sum to seven an be a pair of sixes, thus $P(A \cap B) = 0$. Thus

$$P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{1}{36} \neq 0 = P(A \cap B)$$

which would have to be equal for A and B to be independent.

b. *The relation* $\mathcal{R}$ *on the set of integers defined by* $x\mathcal{R}y \leftrightarrow 13 \mid (x - y)$ *is an equivalence relation.*

TRUE. The relation is congruence modulo 13. We show that it satisfies the three properties of an equivalence relation.

Because $x - x = 0$ so $13 \mid (x - x)$, for all x we have $x\mathcal{R}x$, so $\mathcal{R}$ is reflexive.

Because $13 \mid (x-y)$ implies $13 \mid (y-x)$, for all x, y we have $x\mathcal{R}y \to y\mathcal{R}x$, so $\mathcal{R}$ is symmetric.

Because $13 \mid (x-y)$ and $13 \mid (y-z)$ implies $13 \mid (x-z)$ since $(x-z) = (x-y) + (y-z)$, for all x, y, z we have $(x\mathcal{R}y) \wedge (y\mathcal{R}z) \to x\mathcal{R}z$, so $\mathcal{R}$ is transitive.

c. *Let $C \subseteq \mathbf{R}$ be a set of real numbers, $\overline{C}$ be its complement and $f : \mathbf{R} \to \mathbf{R}$ be a function. Then $f\left(\overline{C}\right) = \overline{f(C)}$.*

FALSE. Any function that is not one-to-one will do. Let $f(x) = 0$ for all x. Let $C = (0, \infty)$. Then $f(C) = \{0\}$ and $\overline{f(C)} = \mathbf{R} - \{0\}$. However, $\overline{C} = (-\infty, 0]$ so that $f\left(\overline{C}\right) = \{0\}$.

4. *Define a sequence using the following recursion*

$$a_1 = 1; \qquad a_2 = 2; \qquad a_3 = 3; \qquad a_{n+1} = a_n + a_{n-1} + a_{n-2} \qquad \text{for all } n \geq 3.$$

What are a_4, a_5 and a_6? Show that for all $n \geq 1$, $a_n \leq 2^n$.

$a_4 = a_3 + a_2 + a_1 = 3 + 2 + 1 = 6$,
$a_5 = a_4 + a_3 + a_2 = 6 + 3 + 2 = 11$,
$a_6 = a_5 + a_4 + a_3 = 11 + 6 + 3 = 20$.

Proof. For each $n \geq 1$, we must prove $a_n \leq 2^n$. We argue by strong induction.

For the base case, $a_1 = 1 \leq 2 = 2^1$. $a_2 = 2 \leq 4 = 2^2$. $a_3 = 3 \leq 8 = 2^3$ so that the inequality holds for $n = 1$, $n = 2$ and $n = 3$ and the base case is verified.

For the induction case, we assume that for any given $n \geq 3$ that $a_k \leq 2^k$ for every $1 \leq k \leq n$ (the induction hypothesis). To show that the statement is true for $n + 1$, we use the recursion

$$a_{n+1} = a_n + a_{n-1} + a_{n-2}.$$

By the induction hypothesis when $k = n$, $k = n - 1$ and $k = n - 2$,

$$\begin{aligned} a_{n+1} &\leq 2^n + 2^{n-1} + 2^{n-2} \\ &\leq 2^n + 2^{n-1} + 2 \cdot 2^{n-2} \\ &= 2^n + 2^{n-1} + 2^{n-1} \\ &= 2^n + 2 \cdot 2^{n-1} \\ &= 2^n + 2^n \\ &= 2 \cdot 2^n \\ &= 2^{n+1}. \end{aligned}$$

Thus the inequality holds for $n + 1$ and the induction case is complete.

By mathematical induction, since both the base and induction cases are verified, we have shown that $a_n \leq 2^n$ for every $n \geq 1$. □

6. *A standard deck of 52 cards consists of four suits $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$. Each suit has 13 different kinds of cards $\{2, 3, 4, 5, 6, 7, 8, 9, 10, \mathrm{J}, \mathrm{Q}, \mathrm{K}, \mathrm{A}\}$. Suppose that a thirteen card bridge hand is randomly dealt from the deck without replacement. What is the probability that the hand is balanced (four cards of one suit and three cards of each of the other suits, e.g., $\{\clubsuit AKQJ, \diamondsuit 10\,98, \heartsuit 765, \spadesuit 432\}$.)?*

By the multiplication principle, we choose one of four suits to be the long suit, we choose which combination of four cards is taken from the long suit, and which combinations of three are taken from the three short suits. We divide by the number of bridge hands, which is the number of combinations of thirteen chosen from 52 cards.

$$P = \frac{4\binom{13}{4}\binom{13}{3}^3}{\binom{52}{13}} = \frac{66905856160}{635013559600} = 0.1053613.$$

7. *Prove that x is a positive integer, where*

$$x = \left(1+\sqrt{2}\right)^{11} + \left(1-\sqrt{2}\right)^{11}.$$

By the binomial formula,

$$\begin{aligned}
\left(1+\sqrt{2}\right)^{11} + \left(1-\sqrt{2}\right)^{11} &= \sum_{j=0}^{11}\binom{11}{j}1^{11-j}(\sqrt{2})^j + \sum_{j=0}^{11}\binom{11}{j}1^{11-j}(-\sqrt{2})^j \\
&= \sum_{j=0}^{11}\binom{11}{j}2^{j/2} + \sum_{j=0}^{11}\binom{11}{j}(-1)^j 2^{j/2} \\
&= \sum_{j=0}^{11}\binom{11}{j}\left(1+(-1)^j\right)2^{j/2} \\
&= 2\sum_{k=0}^{5}\binom{11}{2k}2^k \\
&= 16238.
\end{aligned}$$

We observe that the odd terms cancel and that in the even terms, $\sqrt{2}$ is raised to an even power, thus is an integer. By substituting $j = 2k$ and summing over the surviving (even) terms, x is the sum of positive integers, hece is itself a positive integer.

7. *Determine which amounts of postage can be formed using just 3-cent and 5-cent stamps. Prove your answer.*

With 3-cent stamps only we may form 0, 3, 6, 9, 12, 15,.... With one 5-cent and the rest 3-cent stamps we may form 5, 8, 11, 14, 17, 20,.... With two 5-cent stamps and the rest 3-cent stamps we may form 10, 13, 16, 19, 22, 25,.... We have the following theorem.

Theorem. *With just 3-cent and 5-cent stamps, one can form 0, 3, 5, 6, and any integer $x \ge 8$ cents of postage.*

Proof. Let $f(x,y) = 3x+5y$. If (x,y) denotes the nonnegative integer number of 3-cent and 5-cent stamps, respectively, then we may take $(0,0)$, $(1,0)$, $(0,1)$ and $(2,0)$ stamps to make $f(0,0) = 0$, $f(1,0) = 3$, $f(0,1) = 5$ and $f(2,0) = 6$ cents worth of postage.

To show that we may form postage for any integer $x \ge 8$, we argue by strong induction.

In the base case, by choosing (x,y) stamps to be $(1,1)$, $(3,0)$ and $(0,2)$ we make $f(1,1) = 8$, $f(3,0) = 9$ and $f(0,2) = 10$ cents worth of postage. This completes the base case.

In the induction case, we assume that for any given $n \ge 10$ that we can make postage for all k cents where $8 \le k \le n$ (the induction hypothesis). To make $n+1$ cents of postage, by the induction hypothesis, since $n \ge 10$, we can form $n-2 \ge 8$ cents with some choice (x,y) of three and five sent stamps $f(x,y) = n-2$. Then by adding another three cent stamp,

$$f(x+1,y) = 3(x+1)+5y = 3+3x+5y = 3+f(x,y) = 3+(n-2) = n+1,$$

we can make $n+1$ cents of postage. Thus the induction case is complete.

Since both the base and induction cases are verified, by strong mathematical induction, we can make postage for all integers $n \ge 8$ using just 3-cent and 5-cent stamps. □