

Sample Final Questions.

1. Show using induction that $5 \mid (7^n - 2^n)$ for every positive integer n .
2. Prove that $n! > 2^n$ for all integers $n \geq 4$. [From midterm for Math 3210-2, Sept. 16, 2009]
3. Prove that for all $n \in \mathbb{N}$, $\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}$.
4. Find all pairs of integers x, y which satisfy the diophantine equation

$$50x - 65y = 75.$$

5. Let p be prime. Then for any integer x , $x^p - x$ is divisible by p .
6. Let a_1, a_2, \dots, a_n be positive real numbers. The *Arithmetic Mean* of these numbers is defined to be

$$\mathcal{A}_n = \frac{1}{n} \sum_{i=1}^n a_i.$$

The *Geometric Mean* of these numbers is defined by

$$\mathcal{G}_n = \left(\prod_{i=1}^n a_i \right)^{\frac{1}{n}}.$$

Show that for all $n \geq 1$, $\mathcal{G}_n \leq \mathcal{A}_n$.

7. Show that

$$S_n = \sum_{\{a_1, a_2, \dots, a_k\} \subseteq \{1, 2, \dots, n\}} \frac{1}{a_1 a_2 \cdots a_k} = n$$

where the sum is taken over all nonempty subsets of the set of n smallest positive integers.

8. A knight on a chessboard can move one space horizontally (in either direction) and then two spaces vertically (in either direction) or two spaces horizontally (in either direction) and then one space vertically (in either direction). Suppose that we have an infinite chessboard made up of all squares (m, n) where m and n are nonnegative integers. Show that a knight starting at $(0, 0)$ and travelling within the chessboard can reach any square of the chessboard using a finite number of moves.
9. Determine which amounts of postage can be formed using just 4 cent and 7 cent stamps.
10. Let f_n be the n -th Fibonacci number. Show that for positive integer n we have

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}.$$

11. Let f_n be the n -th Fibonacci number. Show that for positive integer n we have

$$\mathcal{S}_n = f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$$

12. Let f_n be the n -th Fibonacci number. Show that for positive integer n we have

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}.$$

13. Show that the n -th derivative $f_n(x) = \frac{d^n}{dx^n}(xe^x) = (x+n)e^x$ for all positive integers n .
14. Find the number of integers between 1000 and 9999 inclusive that are divisible by 7 or are not divisible by 11.
15. Suppose that there are finite sets A_1, A_2, \dots, A_n . Then the number of elements in the union is given by the inclusion-exclusion formula

$$|A_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|$$

where the second sum is over all k element subsets $\{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, 3, \dots, n\}$. Prove the formula using induction. (Another proof is given in §7.5.)

16. Show that for every positive integer n , $\neg(p_1 \vee p_2 \vee \dots \vee p_n)$ is equivalent to $\neg p_1 \wedge \neg p_2 \wedge \dots \wedge \neg p_n$ where p_1, p_2, \dots, p_n are propositions.
17. Suppose that you begin with a pile of n stones and split the pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile, you multiply the the number of stones in each of the smaller piles you form, so that if the smaller piles have r and s stones in them, you compute rs . Show that no matter how you split the piles, the sum of the products computed at each step equals to $\frac{1}{2}n(n-1)$.
18. Let d be as positive integer. Show that in any group of $d+1$ not necessarily consecutive integers, there must be two with exactly the same remainder when divided by d .
19. Suppose that there are $n \geq 2$ people at a party. Show that there must be at least two of them who know the same number of people at the party.

Questions from Math 3070-1 exam of Jan. 30,2008.

20. A standard deck of 52 cards consists of four suits $\{\clubsuit, \diamond, \heartsuit, \spadesuit\}$. Each suit has 13 different kinds of cards $\{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}$. How many two card hands are there such that both cards are of the same suit or of the same kind?
21. A bag contains 26 scrabble tiles, each labeled by a different letters of the alphabet.
- How many different five letter words can be selected from the bag (without replacement)?
 - How many of these words are in alphabetical order?
 - How many of these contain at least one vowel $\{A, E, I, O, U\}$?

Solutions.

1. Show using induction that $5 \mid (7^n - 2^n)$ for every positive integer n .

Let $P(n)$ denote the statement $5 \mid (7^n - 2^n)$. For the induction, we need to show $P(1)$ and $P(k) \rightarrow P(k+1)$ for all $k \geq 1$.

For $n = 1$, $7^1 - 2^1 = 7 - 2 = 5$ which is divisible by 5. Hence $P(1)$ is true.

Assume for some positive integer k , 5 divides $7^k - 2^k$. Then

$$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k = 5 \cdot 7^k + 2 \cdot (7^k - 2^k)$$

5 divides the first term and by the induction hypothesis, $P(k)$, it divides the second term as well and thus the sum. Hence we conclude $P(k+1)$, and the induction is complete.

2. Prove that $n! > 2^n$ for all integers $n \geq 4$. [From midterm for Math 3210-2, Sept. 16, 2009]

Use induction on n . In the base case $n = 4$, then

$$\text{LHS} = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 > 16 = 2^4 = \text{RHS}.$$

Induction case. Assume that for any $n \geq 4$ we have $n! > 2^n$. Then by the induction hypothesis and $n+1 \geq 2$,

$$(n+1)! = (n+1)n! > (n+1)2^n \geq 2 \cdot 2^n = 2^{n+1}.$$

3. Prove that for all positive integers n ,
$$\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

Proof by induction. In the base case, $n = 1$, the left side is $\sum_{i=1}^1 \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2}$. The right side is $\frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$ hence equality holds.

The induction step is to prove for every $n \geq 1$, the statement for $n+1$ is true assuming it's true for n . But

$$\begin{aligned} \sum_{i=1}^{n+1} \frac{1}{i(i+1)} &= \frac{1}{(n+1)(n+2)} + \sum_{i=1}^n \frac{1}{i(i+1)} && \text{Now use the induction hypothesis.} \\ &= \frac{1}{(n+1)(n+2)} + \frac{n}{n+1} = \frac{1+n(n+2)}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{(n+1)+1}. \end{aligned}$$

4. Find all pairs of integers x, y which satisfy the diophantine equation

$$50x - 65y = 75. \tag{1}$$

Note that $50 = 2 \cdot 5^2$ and $65 = 5 \cdot 13$ so $d = \gcd(50, 65) = 5$. Since 5 divides the left hand side, there is no solution of (1) unless 5 divides the right side too, which it does. Dividing by d we get

$$10x - 13y = 15, \tag{2}$$

whose solutions are the solutions of (1). Now 10 and -13 are relatively prime so there are numbers ℓ and m so that $10\ell - 13m = 1$. To find ℓ and m , we run the Euclidean Algorithm

$$\begin{aligned} 13 &= 1 \cdot 10 + 3 \\ 10 &= 3 \cdot 3 + 1 \\ 3 &= 3 \cdot 1 + 0 \end{aligned}$$

Substituting, we find

$$1 = 10 - 3 \cdot 3 = 10 - 3 \cdot (13 - 10) = 4 \cdot 10 - 3 \cdot 13,$$

so $\ell = 4$ and $m = 3$. Multiplying by 15 we find

$$10 \cdot 60 - 13 \cdot 45 = 15.$$

Thus one solution of the diophantine equation is $x_0 = 60$ and $y_0 = 45$. Suppose (x, y) is another solution of (2). Then

$$10(x - x_0) = 13(y - y_0)$$

Since 10 and 13 are relatively prime, $10 \mid (y - y_0)$ so there is an integer t so that $y = y_0 + 10t$. Then $10(x - x_0) = 130t$ or $x - x_0 = 13t$. It follows that other solutions have the form

$$\begin{aligned} x &= 60 + 13t \\ y &= 45 + 10t. \end{aligned} \tag{3}$$

But by substituting (3) into (2), we see that this is a solution for any t . Therefore all possible solutions of (2), and hence of (1) are given by (3).

5. *Let p be prime. Then for any integer x , $x^p - x$ is divisible by p .*

If $p = 2$ then $x^2 - x = x(x - 1)$ contains an even factor since either x or $x - 1$ is even. Thus we may assume p is an odd prime, and $x \geq 0$, since for odd p , by replacing x by $-x$ yields $-(x^p - x) = (-x)^p - (-x)$ which has the same divisibility by p as $x^p - x$. The statement is true for $x = 0$ and $x = 1$. We argue by induction on x .

Assume the statement is true for any positive integer x . Observe that

$$(x + 1)^p = \sum_{j=0}^p \binom{p}{j} x^j$$

But the binomial coefficients $\binom{p}{1}, \binom{p}{2}, \dots, \binom{p}{p-1}$ are all divisible by p because if $1 \leq j \leq p - 1$ then the integer binomial coefficient

$$\binom{p}{j} = \frac{p(p-1) \cdots (p-j+1)}{j(j-1) \cdots 2 \cdot 1}$$

has a single largest prime factor p in the numerator and no prime factor in the denominator as large as p . Hence $p \mid \binom{p}{j}$. Thus, by the induction hypothesis,

$$(x + 1)^p \equiv x^p + 1 \equiv x + 1 \pmod{p},$$

and the formula holds also for $x + 1$: the induction is complete.

6. *Let a_1, a_2, \dots, a_n be positive real numbers. The Arithmetic Mean of these numbers is defined to be*

$$\mathcal{A}_n = \frac{1}{n} \sum_{i=1}^n a_i.$$

The Geometric Mean of these numbers is defined by

$$\mathcal{G}_n = \left(\prod_{i=1}^n a_i \right)^{\frac{1}{n}}.$$

Show that for all $n \geq 1$, $\mathcal{G}_n \leq \mathcal{A}_n$.

The trick that makes the argument easier is to double the number of terms each induction step rather than just increasing the number by one. Then, by a simple padding argument, the general result follows. We argue by induction that the inequality is true when $n = 2^k$ is a power of two. In the base case $k = 0$ or $n = 1 = 2^0$, $\mathcal{A}_1 = a_1 = \mathcal{G}_1$ and so $\mathcal{G}_1 = \mathcal{A}_1$.

We illustrate the cases $n = 2^1$ and $n = 2^2$ to make the general induction step clearer. If there are two numbers α and β , the basic squaring inequality is

$$0 \leq (\sqrt{\alpha} - \sqrt{\beta})^2 = \alpha - 2\sqrt{\alpha\beta} + \beta$$

from which

$$\mathcal{G}_2 = \sqrt{\alpha\beta} \leq \frac{1}{2}(\alpha + \beta) = \mathcal{A}_2$$

follows.

If there are four numbers α, β, γ and δ , apply the basic squaring inequality twice

$$\begin{aligned} \mathcal{G}_4 &= (\alpha\beta\gamma\delta)^{\frac{1}{4}} = \left(\sqrt{\alpha\beta}\sqrt{\gamma\delta} \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \left(\sqrt{\alpha\beta} + \sqrt{\gamma\delta} \right) \\ &\leq \frac{1}{2} \left(\frac{\alpha + \beta}{2} + \frac{\gamma + \delta}{2} \right) = \frac{1}{4} (\alpha + \beta + \gamma + \delta) = \mathcal{A}_4. \end{aligned}$$

Now, for any $n \geq 0$, assume the result is true for 2^n terms: $\mathcal{G}_{2^n} \leq \mathcal{A}_{2^n}$. Applying first the squaring inequality and then the induction hypothesis to both to the first 2^n numbers and also to the second 2^n numbers

$$\begin{aligned} \mathcal{G}_{2^{n+1}} &= \left(\prod_{i=1}^{2^{n+1}} a_i \right)^{\frac{1}{2^{n+1}}} = \left(\left(\prod_{i=1}^{2^n} a_i \right)^{\frac{1}{2^n}} \left(\prod_{j=2^n+1}^{2^{n+1}} a_j \right)^{\frac{1}{2^n}} \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \left(\left(\prod_{i=1}^{2^n} a_i \right)^{\frac{1}{2^n}} + \left(\prod_{j=2^n+1}^{2^{n+1}} a_j \right)^{\frac{1}{2^n}} \right) \\ &\leq \frac{1}{2} \left(\frac{1}{2^n} \left(\sum_{i=1}^{2^n} a_i \right) + \frac{1}{2^n} \left(\sum_{j=2^n+1}^{2^{n+1}} a_j \right) \right) \\ &= \frac{1}{2^{n+1}} \sum_{i=1}^{2^{n+1}} a_i = \mathcal{A}_{2^{n+1}}. \end{aligned}$$

To illustrate the padding, if there are three numbers, then use \mathcal{A}_3 as a fourth and then the Arithmetic-Geometric inequality for four numbers gives

$$\mathcal{G}_3^{\frac{3}{4}} \mathcal{A}_3^{\frac{1}{4}} = (\alpha\beta\gamma\mathcal{A}_3)^{\frac{1}{4}} \leq \frac{1}{4} (\alpha + \beta + \gamma + \mathcal{A}_3) = \frac{1}{4} (3\mathcal{A}_3 + \mathcal{A}_3) = \mathcal{A}_3$$

from which $\mathcal{G}_3 \leq \mathcal{A}_3$ follows. In general, if $2^k \leq n < 2^{k+1}$, we pad the n numbers with $2^{k+1} - n$ copies of \mathcal{A}_n and use the Arithmetic-Geometric inequality for 2^{k+1} numbers

$$\mathcal{G}_n^{\frac{n}{2^{k+1}}} \mathcal{A}_n^{\frac{2^{k+1}-n}{2^{k+1}}} = \left(\mathcal{A}_n^{2^{k+1}-n} \prod_{i=1}^n a_i \right)^{\frac{1}{2^{k+1}}} \leq \frac{1}{2^{k+1}} \left((2^{k+1} - n)\mathcal{A}_n + n\mathcal{A}_n \right) = \mathcal{A}_n$$

from which $\mathcal{G}_n \leq \mathcal{A}_n$ follows.

7. Show that

$$S_n = \sum_{\{a_1, a_2, \dots, a_k\} \subseteq \{1, 2, \dots, n\}} \frac{1}{a_1 a_2 \cdots a_k} = n$$

where the sum is taken over all nonempty subsets of the set of n smallest positive integers.

We argue by induction on n . For the base case when $n = 1$, the only subset is $\{1\} \subseteq \{1\}$, so the sum is $1/1$ which equals $n = 1$.

Assume for any $n \geq 1$ that the statement is true for sums over subsets of the first n integers. Now, partition the sum into three parts. A subset of $\{1, 2, \dots, n, n+1\}$ either is the singleton set $\{n+1\}$, a subset that contains $n+1$ but has also another element, or is a subset that does not contain $n+1$. In the latter case, the subset may be viewed as a nonempty subset of $\{1, 2, \dots, n\}$. Thus

$$\begin{aligned} S_{n+1} &= \sum_{\{a_1, a_2, \dots, a_k\} \subseteq \{1, 2, \dots, n+1\}} \frac{1}{a_1 a_2 \cdots a_k} \\ &= \sum_{\{n+1\} \subseteq \{1, 2, \dots, n+1\}} \frac{1}{n+1} + \sum_{\{a_1, a_2, \dots, a_k, n+1\} \subseteq \{1, 2, \dots, n+1\}} \frac{1}{a_1 a_2 \cdots a_k (n+1)} \\ &\quad + \sum_{\{a_1, a_2, \dots, a_k\} \subseteq \{1, 2, \dots, n\}} \frac{1}{a_1 a_2 \cdots a_k} \end{aligned}$$

Noting that there is one term in the first sum, that all second sum subsets are of the form of $T \cup \{n+1\}$ where $T = \{a_1, a_2, \dots, a_k\} \subseteq \{1, 2, \dots, n\}$ so that $n+1$ can be factored out of the sum, we see that using the induction hypothesis on the second and third terms

$$\begin{aligned} S_{n+1} &= \frac{1}{n+1} + \frac{1}{n+1} \sum_{\{a_1, a_2, \dots, a_k\} \subseteq \{1, 2, \dots, n\}} \frac{1}{a_1 a_2 \cdots a_k} + \sum_{\{a_1, a_2, \dots, a_k\} \subseteq \{1, 2, \dots, n\}} \frac{1}{a_1 a_2 \cdots a_k} \\ &= \frac{1}{n+1} + \frac{n}{n+1} + n = n+1. \end{aligned}$$

The induction step is complete.

8. A knight on a chessboard can move one space horizontally (in either direction) and then two spaces vertically (in either direction) or two spaces horizontally (in either direction) and then one space vertically (in either direction). Suppose that we have an infinite chessboard made up of all squares (m, n) where m and n are nonnegative integers. Show that a knight starting at $(0, 0)$ and travelling within the chessboard can reach any square of the chessboard using a finite number of moves.

We argue using induction on $m+n$ which runs through all nonnegative integers for chessboard squares. A knight move consists of adding one of eight vectors to the current position. These eight are $(\pm 2, \pm 1)$ or $(\pm 1, \pm 2)$. The knight may not move off of the board. The statement to be verified is

$\mathcal{P}(k)$ = "Starting from $(0, 0)$, the knight can reach the square (m, n) where $m+n = k$ by using at most $\ell_k = 3k$ moves on the chessboard."

The base case is $k = 0$. The knight need not move to reach the only square with $m+n = 0$, namely $(0, 0)$. Hence $\ell_0 = 0$.

For any $k \geq 0$, the induction hypothesis is that we assume that the knight can reach any square (m, n) on the diagonal $m+n = k$ in at most $\ell_k = 3k$ moves. Consider any square (m', n') where $m' + n' = k+1$. Since the sum is positive, one of the numbers must be positive. In case $m' > 0$, we know that $m' - 1$ is nonnegative and $(m' - 1, n')$ is a square on the $(m' - 1) + n' = k$ diagonal. By the induction hypothesis, it takes at most $\ell_k = 3k$

moves to reach the square $(m' - 1, n')$. It takes at most three more moves to reach (m', n') , namely $(1, 2)$ then $(2, -1)$ then $(-2, -1)$. The positions of the knight are

$$(m' - 1, n') \rightarrow (m', n' + 2) \rightarrow (m' + 2, n' + 1) \rightarrow (m', n').$$

Note that the x -coordinates $0 \leq m' - 1 \leq m' \leq m' + 2$ are all nonnegative and the y -coordinates $0 \leq n' \leq n' + 1 \leq n' + 2$ are also nonnegative.

In case $m' = 0$ we have $n' > 0$, thus $n' - 1$ is nonnegative and $(m', n' - 1)$ is a square on the $m' + (n' - 1)' = k$ diagonal. By the induction hypothesis, it takes at most $\ell_k = 3k$ moves to reach the square $(m', n' - 1)$. It takes at most three more moves to reach (m', n') , namely $(2, 1)$ then $(-1, 2)$ then $(-1, -2)$. The positions of the knight are

$$(m', n' - 1) \rightarrow (m' + 2, n') \rightarrow (m' + 1, n' + 2) \rightarrow (m', n').$$

Note that the x -coordinates $0 \leq m' \leq m' + 1 \leq m' + 2$ are all nonnegative and the y -coordinates $0 \leq n' - 1 \leq n' \leq n' + 2$ are also nonnegative.

In both cases, these moves stay in the board. Also, there were three additional moves, so the total is at most $\ell_{k+1} = \ell_k + 3 = 3k + 3 = 3(k + 1)$. The induction is complete.

The number of moves to reach a square may well be less, for example it takes one move to reach $(2, 1)$ where $2 + 1 = 3$ but $\ell_3 = 3 \cdot 3 = 9$. A more complicated induction may yield a sharper estimate.

9. *Determine which amounts of postage can be formed using just 4 cent and 7 cent stamps.*

We solved the Diophantine Equation before, and checked whether any solutions are pairs of nonnegative integers. This time we use strong induction. The first step is to try some sums and guess which values are possible.

We get multiples of four: 0, 4, 8, 12, 16, 20, 24, 28...; with one seven cent stamp: 7, 11, 15, 19, 23, 27...; with two seven cent stamps; 14, 18, 22, 28,...; with three seven cent stamps: 21, 28,...; with four seven cent stamps: 28,... It is plausible to guess that the amounts of postage possible are 0, 4, 7, 8, 11, 12, 14, 15, 16 and all $k \geq 18$. Let us prove the result using strong induction. The smaller values occur in the table, so are possible. It remains to prove for $k \geq 18$ the proposition

$\mathcal{P}(k) =$ “ k cents postage can be formed using just 4 cent and 7 cent stamps.”

The base cases are the four statements $\mathcal{P}(18)$, $\mathcal{P}(19)$, $\mathcal{P}(20)$ and $\mathcal{P}(21)$. The nonnegative solution of $4x + 7y = k$ for $k = 18$ is $(1, 2)$; for $k = 19$ is $(3, 1)$; for $k = 20$ is $(5, 0)$; and for $k = 21$ is $(0, 3)$.

The induction hypothesis for any $k \geq 21$ is $\mathcal{P}(k-3) \wedge \mathcal{P}(k-2) \wedge \mathcal{P}(k-1) \wedge \mathcal{P}(k)$. We show that this implies that $\mathcal{P}(k+1)$ is true. Using the assumption $\mathcal{P}(k-3)$, we make postage for $k-3$ using (x, y) four and seven cent stamps, so that $4x + 7y = k - 3$. Adding one more four cent stamp gives $(x + 1, y)$ stamps which total $4(x + 1) + 7y = (4x + 7y) + 4 = (k - 3) + 4 = k + 1$ cents postage. Thus $\mathcal{P}(k + 1)$ is also true, completing the induction step.

10. *Let f_n be the n -th Fibonacci number. Show that for positive integer n we have*

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}.$$

The Fibonacci numbers are defined by

$$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \text{ for } n = 2, 3, 4, \dots$$

Thus $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, $f_6 = 8$, and so on.

Let's argue by induction. For the base case $n = 1$, the left side is $f_1^2 = 1^2 = 1$ and the right side is $f_1 f_2 = 1 \cdot 1 = 1$ so the equation holds for $n = 1$.

For any $n \geq 1$, assume that the formula holds for n (induction hypothesis). Then for $n + 1$, by the induction hypothesis

$$\sum_{i=1}^{n+1} f_i^2 = \left(\sum_{i=1}^n f_i^2 \right) + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2 = (f_n + f_{n+1}) f_{n+1}.$$

However, the Fibonacci numbers satisfy $f_n + f_{n+1} = f_{n+2}$ so this yields

$$\sum_{i=1}^{n+1} f_i^2 = (f_n + f_{n+1}) f_{n+1} = f_{n+2} f_{n+1} = f_{n+1} f_{(n+1)+1}$$

so the induction step is verified.

11. Let f_n be the n -th Fibonacci number. Show that for positive integer n we have

$$\mathcal{S}_n = f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$$

Let's argue by induction. For the base case $n = 1$, the left side is $f_0 - f_1 + f_2 = 0 - 1 + 1 = 0$ and the right side is $f_1 - 1 = 1 - 1 = 0$ so the equation holds for $n = 1$.

For any $n \geq 1$, assume that the formula holds for n . Then for $n + 1$, by the induction hypothesis

$$\begin{aligned} \mathcal{S}_{n+1} &= f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n} - f_{2n+1} + f_{2n+2} \\ &= (f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n}) - f_{2n+1} + f_{2n+2} \\ &= (f_{2n-1} - 1) - f_{2n+1} + f_{2n+2}. \end{aligned}$$

However, the Fibonacci numbers satisfy $f_{2n-1} + f_{2n} = f_{2n+1}$ and $f_{2n+2} = f_{2n} + f_{2n+1}$ so these yield

$$\mathcal{S}_{n+1} = (f_{2n+1} - f_{2n} - 1) - f_{2n+1} + (f_{2n} + f_{2n+1}) = f_{2n+1} - 1 = f_{2(n+1)-1} - 1.$$

so the induction step is verified.

12. Let f_n be the n -th Fibonacci number. Show that for positive integer n we have

$$A^n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$$

Let's argue by induction. For the base case $n = 1$, the left side is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and

the right side is $\begin{pmatrix} f_2 & f_1 \\ f_1 & f_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ so the equation holds for $n = 1$.

Assume for any $n \geq 1$ that the formula holds for n . Then for $n + 1$, by the induction hypothesis

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+1} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} f_{n+1} + f_n & f_n + f_{n-1} \\ f_{n+1} & f_n \end{pmatrix}. \end{aligned}$$

However, the Fibonacci numbers satisfy $f_{n+1} + f_n = f_{n+2}$ and $f_n + f_{n-1} = f_{n+1}$ so these yield

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+1} = \begin{pmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{pmatrix} = \begin{pmatrix} f_{(n+1)+1} & f_{n+1} \\ f_{n+1} & f_{(n+1)-1} \end{pmatrix}.$$

so the induction step is verified.

13. Show that the n -th derivative $f_n(x) = \frac{d^n}{dx^n}(xe^x) = (x+n)e^x$ for all positive integers n .

We'll use induction and the product rule. Note that f_n is inductively defined: $f_0(x) = xe^x$ and $f_n(x) = (f_{n-1}(x))'$ for n a positive integer.

The base case is $n = 1$. $f_1(x) = (xe^x)' = x'e^x + x(e^x)' = e^x + xe^x = (x+1)e^x$.

Now assume for any $n \geq 1$ that the formula holds for $f_n(x)$. Then, by the recursive definition, induction hypothesis and product rule,

$$f_{n+1}(x) = (f_n(x))' = ((x+n)e^x)' = (x+n)'e^x + (x+n)(e^x)' = e^x + (x+n)e^x = (x+n+1)e^x$$

so the formula holds for $f_{n+1}(x)$ too and the induction is complete.

14. Find the number of integers between 1000 and 9999 inclusive that are divisible by 7 or are not divisible by 11.

Let U , the universal set, be numbers between 1000 and 9999 inclusive. Let A_1 denote the numbers in this range divisible by seven. Let A_2 denote the number divisible by eleven. Note that the numbers divisible by seven or not divisible by eleven may be written $A_1 \cup \overline{A_2} = (A_1 \cap A_2) \cup \overline{A_2}$. But since $A_1 \cap A_2$ and $\overline{A_2}$ are disjoint, by the sum formula

$$|A_1 \cup \overline{A_2}| = |A_1 \cap A_2| + |\overline{A_2}|.$$

Let us count the number both divisible by seven and divisible by eleven, (namely those divisible by 77). Thus we count the number of integer solutions of the inequality $1000 \leq 77t \leq 9999$. Thus $12.99 = 1000/77 \leq t \leq 9999/77 = 129.86$, hence there are $|A_1 \cap A_2| = 129 - 13 + 1 = 117$ solutions.

The to find the number divisible by eleven, we count the number of integer solutions of the inequality $1000 \leq 11t \leq 9999$. Because $90.91 = 1000/11 \leq t \leq 9999/11 = 909$, there are $|A_2| = 909 - 91 + 1 = 819$ solutions. The number of integers from 1000 to 9999 inclusive is $|U| = 9999 - 1000 + 1 = 9000$. Thus the number not divisible by eleven is $|\overline{A_2}| = 9000 - 819 = 8181$.

Thus, substituting in the sum formula

$$|A_1 \cup \overline{A_2}| = 117 + 8181 = 8298.$$

The solution may also be obtained using the inclusion-exclusion formula.

15. Suppose that there are finite sets A_1, A_2, \dots, A_n . Then the number of elements in the union is given by the inclusion-exclusion formula

$$|A_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|$$

where the second sum is over all k element subsets $\{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, 3, \dots, n\}$. Prove the formula using induction.

Let us use strong induction. For one set, the left side is $|A_1|$ and the right side is $(-1)^{1+1}|A_1|$, which are equal, so that the base case is verified.

For two sets, we can write the union as the union of three disjoint sets, namely

$$A_1 \cup A_2 = (A_1 - A_2) \cup (A_1 \cap A_2) \cup (A_2 - A_1).$$

We can write each set as a disjoint union too,

$$A_1 = (A_1 - A_2) \cup (A_1 \cap A_2); \quad A_2 = (A_1 \cap A_2) \cup (A_2 - A_1).$$

Using the sum formula, that we can add the numbers of disjoint sets

$$\begin{aligned} |A_1 \cup A_2| &= |A_1 - A_2| + |A_1 \cap A_2| + |A_2 - A_1| \\ &= (|A_1 - A_2| + |A_1 \cap A_2|) + (|A_1 \cap A_2| + |A_2 - A_1|) - |A_1 \cap A_2| \\ &= |A_1| + |A_2| - |A_1 \cap A_2| \\ &= (-1)^{1+1} (|A_1| + |A_2|) + (-1)^{1+2} |A_1 \cap A_2| \end{aligned}$$

which is the $n = 2$ case.

For the induction step, assume that for some $n \geq 2$ we know that the formula holds for any number k of sets, where $1 \leq k \leq n$. Then, to the union of $n + 1$ sets, apply the union formula for the $n = 2$ case first

$$\begin{aligned} |A_1 \cup \dots \cup A_n \cup A_{n+1}| &= |(A_1 \cup \dots \cup A_n) \cup A_{n+1}| \\ &= |A_1 \cup \dots \cup A_n| + |A_{n+1}| - |(A_1 \cup \dots \cup A_n) \cap A_{n+1}| \\ &= |A_1 \cup \dots \cup A_n| + |A_{n+1}| - |(A_1 \cap A_{n+1}) \cup \dots \cup (A_n \cap A_{n+1})| \end{aligned}$$

where we have used the distributive law for intersections and unions. Now we apply the induction hypothesis using the n set formula twice

$$\begin{aligned} |A_1 \cup \dots \cup A_n \cup A_{n+1}| &= \left(\sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| \right) + |A_{n+1}| \\ &\quad - \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |(A_{i_1} \cap A_{n+1}) \cap (A_{i_2} \cap A_{n+1}) \cap \dots \cap (A_{i_k} \cap A_{n+1})| \\ &= \left(\sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| \right) + (-1)^{1+1} |A_{n+1}| \\ &\quad + \sum_{k=1}^n (-1)^{k+2} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k} \cap A_{n+1}|. \end{aligned}$$

It remains to regroup the terms to recognize the desired right side. Note that the first sum is over all subsets of $\{1, 2, \dots, n+1\}$ not involving $n+1$, the second is the singleton $\{n+1\}$ and the last is over subsets of two or more elements, one of which is $n+1$. This is the same partition we encountered in problem (7). Note that in each sum the power of -1 is one more than the number of sets in the intersection. Combining terms with like number of sets we complete the induction

$$|A_1 \cup \dots \cup A_n \cup A_{n+1}| = \sum_{k=1}^{n+1} (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n+1} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|.$$

16. Show that for every positive integer n , $\neg(p_1 \vee p_2 \vee \dots \vee p_n)$ is equivalent to $\neg p_1 \wedge \neg p_2 \wedge \dots \wedge \neg p_n$ where p_1, p_2, \dots, p_n are propositions.

Let us use strong mathematical induction.

For the base case, the left side is $\neg(p_1)$ and the right side is $\neg p_1$ which are logically equivalent since they are the same statement. For the case $n = 2$, the statement is just De Morgan's Law of Table 6 on p. 24, which is proved using a truth table on p. 22. Thus we have $\neg(p_1 \vee p_2) \equiv \neg p_1 \wedge \neg p_2$.

Let us now assume that the statements are equivalent for some number $n \geq 2$ of propositions. Then for $n + 1$ propositions, using the $n = 2$ equivalence first, and then the inductive n proposition equivalence,

$$\begin{aligned} \neg(p_1 \vee p_2 \vee \dots \vee p_n \vee p_{n+1}) &\equiv \neg([p_1 \vee p_2 \vee \dots \vee p_n] \vee p_{n+1}) \\ &\equiv \neg[p_1 \vee p_2 \vee \dots \vee p_n] \wedge \neg p_{n+1} \\ &\equiv [\neg p_1 \wedge \neg p_2 \wedge \dots \wedge \neg p_n] \wedge \neg p_{n+1} \\ &\equiv \neg p_1 \wedge \neg p_2 \wedge \dots \wedge \neg p_n \wedge \neg p_{n+1} \end{aligned}$$

and the induction step is proved.

Note that we have implicitly used the associative property for conjunction and disjunction of arbitrarily many propositions.

17. Suppose that you begin with a pile of n stones and split the pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile, you multiply the the number of stones in each of the smaller piles you form, so that if the smaller piles have r and s stones in them, you compute rs . Show that no matter how you split the piles, the sum of the products computed at each step equals to $\frac{1}{2}n(n-1)$.

For example, if the original pile has five stones then, the sequence of splittings

$$***** \rightarrow *|***** \rightarrow **|*** \rightarrow *|*|** \rightarrow *|*|*|*$$

results in $1 \cdot 4 = 4$, $2 \cdot 2 = 4$, $1 \cdot 1 = 1$, $1 \cdot 1 = 1$ whose sum is $4 + 4 + 1 + 1 = 10 = \frac{1}{2} \cdot 5 \cdot 4$. For another sequence of splittings

$$***** \rightarrow **|*** \rightarrow **|*|** \rightarrow *|*|*|** \rightarrow *|*|*|*|*$$

results in $2 \cdot 3 = 6$, $1 \cdot 2 = 2$, $1 \cdot 1 = 1$, $1 \cdot 1 = 1$ whose sum is $6 + 2 + 1 + 1 = 10$ also.

Let us use strong induction. For one stone, no splittings are done so the sum of products is zero whereas $\frac{1}{2} \cdot 1 \cdot 0 = 0$ so the numbers are equal.

Now assume for any positive n , the splitting formula holds for any number k of stones, where $1 \leq k \leq n$. Starting with a pile of $n + 1$ stones, assume that an arbitrary first splitting divides the pile into two smaller piles containing r and s stones, where $r + s = n + 1$. This first splitting contributes rs to the sum. Note that both $1 \leq r, s \leq n$ because each small pile is nonempty and therefore, both are smaller than the original pile. Note that further splittings of the whole pile either split the first smaller pile or split the second smaller pile and whichever pile they split, there is no effect on the splittings of the other pile, so may be regarded as splittings of the smaller piles separately. Applying the inductive hypothesis, the further splittings of the first pile contributes $\frac{1}{2}r(r-1)$ and the splittings of the second contributes $\frac{1}{2}s(s-1)$ to the sum. Adding the three

$$rs + \frac{r(r-1)}{2} + \frac{s(s-1)}{2} = \frac{2rs + r^2 - r + s^2 - s}{2} = \frac{(r+s)(r+s-1)}{2} = \frac{(n+1)n}{2}$$

so the induction is complete.

18. Let d be as positive integer. Show that in any group of $d + 1$ not necessarily consecutive integers, there must be two with exactly the same remainder when divided by d .

This is an example of the pigeon hole principle. $d + 1$ “pigeons” (numbers) must reside in d “holes” (congruence classes modulo d) and therefore at least two numbers must belong to the same residue class.

19. Suppose that there are $n \geq 2$ people at a party. Show that there must be at least two of them who know the same number of people at the party.

We assume that knowing is a symmetric relation: if A knows B then B knows A . Let $v(i)$ be the number of other people that the i th person knows. The set of numbers known is $\mathcal{S} = \{v(1), v(2), \dots, v(n)\}$. People may know noone or may know everyone else at the party, so that $0 \leq v(i) \leq n - 1$. If noone knows everybody at the party then $v(i) < n - 1$ so the function takes $v : \{1, 2, \dots, n\} \rightarrow \{0, 1, \dots, n - 2\}$. Since the map is from a larger set to a smaller one, by the pigeon-hole principle there must be two people $i \neq j$ such that $v(i) = v(j)$. On the other hand, there may be somebody, say person i_0 , who knows everybody, $v(i_0) = n - 1$. But then, by symmetry, everyone knows person i_0 , so $v(j) \geq 1$ for all j . This time the function maps $v : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n - 1\}$. Since the map is from a larger set to a smaller one again, by the pigeon-hole principle, there must be two people $i \neq j$ such that $v(i) = v(j)$. In either case, we have shown that there are at least two people who know the same number.

20. A standard deck of 52 cards consists of four suits $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$. Each suit has 13 different kinds of cards $\{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}$. How many two card hands are there such that both cards are of the same suit or of the same kind?

We observe that if the hand has cards of the same kind they cannot be of the same suit and if the cards have the same suit they cannot be of the same kind. Therefore the number of hands is the sum of hands of the same suit plus hands of the same kind. The number of hands of the same suit is the number of suits, 4 times the number of two card combinations chosen from the thirteen cards of that suit, $\binom{13}{2}$. The number of hands of the same kind is number of kinds 13 times the number of two card combinations of the same kind, $\binom{4}{2}$. The total is thus

$$4 \cdot \binom{13}{2} + 13 \cdot \binom{4}{2} = 4 \cdot \frac{13 \cdot 12}{2} + 13 \cdot \frac{4 \cdot 3}{2} = 390.$$

21. A bag contains 26 scrabble tiles, each labeled by a different letters of the alphabet.

- (a) How many different five letter words can be selected from the bag (without replacement)?
 (b) How many of these words are in alphabetical order?
 (c) How many of these contain at least one vowel $\{A, E, I, O, U\}$?

(a.) In choosing all words, order is important. Thus we count the number of permutations of 26 letters, taken 5 at a time. The number of five letter words is

$$P(26, 5) = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7893600.$$

(b.) Given five different letters, there is only one permutation of these in alphabetical order. Thus the number of words in alphabetical order is the number of combinations of 26 letters taken five at a time

$$C(26, 5) = \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22}{5!} = 65780.$$

(c.) The number with at least one vowel is the total number minus the number with no vowel at all. There are 21 non-vowels so the number of words without vowels is the number of permutations of 21 letters taken five at a time

$$P(26, 5) - P(21, 5) = 7893600 - 21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 = 7893600 - 2441880 = 5451720.$$