Practice Problems for Final Exam, Math 1310-4, Fall 2016

1. Consider the function

$$f(x) = \frac{1}{\sqrt{2-x}}$$

   (a) Find the domain of $f(x)$.

   Solution: The square root is defined for nonnegative numbers, but the denominator cannot be zero, therefore we have

   $$\text{Domain} = \{x : x < 2\}$$

   (b) Find the inverse function $f^{-1}(x)$.

   Solution: Solve $y = f(x)$ for x:

   $$y = \frac{1}{\sqrt{2-x}}, \quad \sqrt{2-x} = \frac{1}{y}, \quad 2 - x = \frac{1}{y^2},$$
   $$x = 2 - \frac{1}{y^2}$$

   The inverse function is

   $$f^{-1}(x) = 2 - \frac{1}{x^2}.$$

2. Find each limit if it exists, and explain the reason if it does not exist. If you use l'Hospital's rule, make sure that you justify the use by stating the type of the limit. (i.e. $\infty/\infty,\ 0/0,\ 0 \cdot \infty$, etc.)

   (a)

   $$\lim_{x \to -2} \frac{x^2 - 4}{x + 2}$$

   (b)

   $$\lim_{x \to \infty} \frac{x^2 + 1}{2x^3 - 2x - 4}$$

   (c)

   $$\lim_{x \to 0} \frac{\tan x}{\sin 3x}$$

(d)

$$\lim_{x \to 3} \frac{\sin(x-3)}{x^2-9}$$

(e)

$$\lim_{x \to 9} \frac{x^2-81}{\sqrt{x}-3}$$

(f)

$$\lim_{x \to \infty} \frac{1}{x-\sqrt{x^2+x}}$$

(g)

$$\lim_{x \to 0} \frac{1-\cos(x^2)}{5x^5}$$

(h)

$$\lim_{x \to 2} \frac{|x-2|}{2x-4}$$

(i)

$$\lim_{x \to 0^+} \sqrt{x} \sin\left(\frac{1}{x^2}\right)$$

(j)

$$\lim_{x \to 0^+} x(\ln x)^2$$

Solution:

(a)

$$\lim_{x \to -2} \frac{x^2-4}{x+2} = \lim_{x \to -2} \frac{(x+2)(x-2)}{x+2} = -4$$

(b)

$$\lim_{x \to \infty} \frac{x^2+1}{2x^3-2x-4} = \lim_{x \to \infty} \frac{\frac{1}{x}+\frac{1}{x^3}}{2-\frac{2}{x^2}-\frac{4}{x^3}} = \frac{0+0}{2-0-0} = 0$$

(c)

$$\lim_{x\to 0}\frac{\tan x}{\sin 3x}=\lim_{x\to 0}\frac{\sin x}{\sin 3x}\cdot\frac{1}{\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{3x}{\sin 3x}\cdot\frac{1}{3\cos x}=1\cdot 1\cdot 1\cdot\frac{1}{3}=\frac{1}{3}$$

This limit is of type 0/0 and we can also use l'Hospital's rule:

$$\lim_{x\to 0}\frac{\tan x}{\sin 3x}=\lim_{x\to 0}\frac{\sec^2 x}{3\cos 3x}=\frac{1}{3}$$

(d)

$$\lim_{x\to 3}\frac{\sin(x-3)}{x^2-9}=\lim_{x\to 3}\frac{\sin(x-3)}{x-3}\frac{1}{x+3}=1\cdot\frac{1}{6}=\frac{1}{6}$$

(e)

$$\lim_{x\to 9}\frac{x^2-81}{\sqrt{x}-3}=\lim_{x\to 9}\frac{(x+9)(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3}=18\times 6=108$$

(f)

$$\begin{aligned}\lim_{x\to\infty}\frac{1}{x-\sqrt{x^2+x}}&=\lim_{x\to\infty}\frac{x+\sqrt{x^2+x}}{(x-\sqrt{x^2+x})(x+\sqrt{x^2+x})}\\&=\lim_{x\to\infty}\frac{x+\sqrt{x^2+x}}{x^2-(x^2+x)}=\lim_{x\to\infty}\frac{x+\sqrt{x^2+x}}{-x}\\&=\lim_{x\to\infty}\left(-1-\sqrt{1+\frac{1}{x}}\right)=-1-1\\&=-2\end{aligned}$$

(g) Using the double angle formula

$$\cos 2\theta=1-2\sin^2\theta$$

we have

$$\frac{1-\cos(x^2)}{5x^5}=\frac{2\sin^2\left(\frac{x^2}{2}\right)}{4\cdot 5(\frac{x^2}{2})^2 x}=\frac{\sin^2\left(\frac{x^2}{2}\right)}{(\frac{x^2}{2})^2}\cdot\frac{1}{10x}$$

The first part tends to 1, but the second part does not have a limit, so the limit in this question does not exist.

We can also use l'Hospital's rule as this is of type 0/0:

$$\lim_{x\to 0}\frac{1-\cos(x^2)}{5x^5}=\lim_{x\to 0}\frac{2x\sin(x^2)}{25x^4}=\frac{2}{25}\lim_{x\to 0}\left(\frac{\sin(x^2)}{x^2}\cdot\frac{1}{x}\right)=\frac{2}{25}\lim_{x\to 0}\frac{1}{x}$$

Now it is obvious that the limit does not exist, as $1/x$ tends to ∞ as $x\to 0^+$ and to $-\infty$ as $x\to 0^-$.

(h) Since

$$\frac{|x-2|}{2x-4}=\begin{cases}\frac{1}{2} & x>2\\ -\frac{1}{2} & x<2\end{cases}$$

the left and right limits do not agree. The limit in question does not exist.

(i) As $|\sin\left(\frac{1}{x^2}\right)|\le 1$ and $\sqrt{x}$ tends to 0 as $x\to 0^+$, the limit is 0 based on the squeeze theorem.

(j) We use l'Hospital's rule as this is of type $0\cdot\infty$,

$$\lim_{x\to 0^+}x(\ln x)^2=\lim_{x\to 0^+}\frac{(\ln x)^2}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{2\ln x\cdot\frac{1}{x}}{-\frac{1}{x^2}}=-2\lim_{x\to 0^+}x\ln x$$

and we also use l'Hospital's rule to calculate

$$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=-\lim_{x\to 0^+}x=0$$

Therefore the original limit is also 0.

3. Find the discontinuities of the function

$$f(x)=\frac{\sin x^2}{x(x+2)}$$

and determine for each one if it is removable or nonremovable.

Solution: $x=0$ and $x=-2$ are discontinuities. However the limit of f as $x\to 0$ exists so it is removable, while $x=-2$ is not.

4. Use the limit definition of the derivative to find $f'(2)$ where

(a)

$$f(x) = \frac{3}{x-1}$$

(b)

$$f(x) = \sin(\pi(x-2))$$

Solution:

(a)

$$\lim_{h\to 0} \frac{\frac{3}{2+h-1} - \frac{3}{2-1}}{h} = 3\lim_{h\to 0} \frac{1-(1+h)}{h(1+h)} = 3\lim_{h\to 0} \frac{-1}{(1+h)} = -3$$

(b)

$$\lim_{h\to 0} \frac{\sin(\pi h) - \sin(\pi \cdot 0)}{h} = \pi \lim_{h\to 0} \frac{\sin(\pi h)}{\pi h} = \pi$$

5. Sketch the graph of

$$f(x) = \begin{cases} 1 & \text{if } x < -1 \\ |x| & \text{if } -1 \leq x < 1 \\ \frac{1}{2}x & \text{if } x \geq 1 \end{cases}$$

(a) Determine if $f(x)$ is continuous at $x = -1, 0,$ and 1. Justify your answer using the three conditions for continuity.

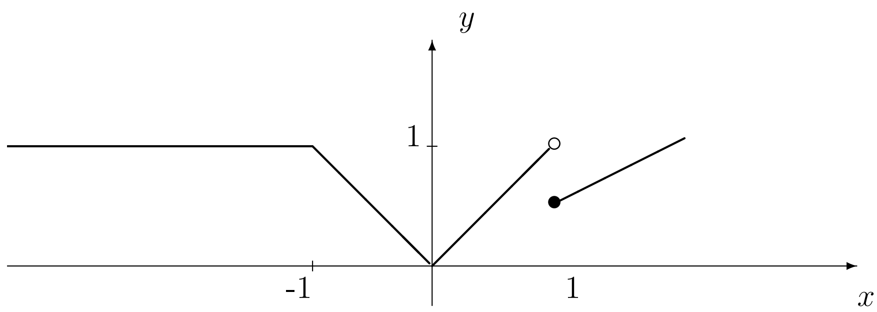
(b) Determine if $f(x)$ is continuous on $(-1, 1)$.

(c) Determine if $f(x)$ is continuous on $[-1, 1]$.

(d) Find each of the following limits if it exists.

$$\lim_{x\to\infty} f(x), \qquad \lim_{x\to-\infty} f(x).$$

Solution:



(a) f is continuous at $x = -1$ and 0, but discontinuous at $x = 1$, at which the left-handed limit is 1 but the right-handed limit is 1/2.

(b) Yes.

(c) No, because it is not left continuous at $x = 1$.

(d) $\lim_{x\to\infty} = +\infty$, and $\lim_{x\to-\infty} f(x) = 1$.

6. Let

$$f(t) = \begin{cases} t^2 - c & \text{if } t \in (-\infty, 2] \\ c\,t + 1 & \text{if } t \in (2, \infty) \end{cases}$$

For what value c is the function f continuous on $(-\infty, \infty)$?

Solution: The only point that could be a discontinuity is $x = 2$ and we take the left and right limits:

$$\lim_{x\to 2^-} = 4 - c, \quad \lim_{x\to 2^+} = 2c + 1.$$

If we choose $c = 1$ then $f(x)$ will be continuous at $x = 2$ and elsewhere as well, so it will be continuous on $(-\infty, \infty)$.

7. A particle moves along a coordinate line and s, its directed distance in centimeters from the origin after t seconds, is given by $s = f(t) = \sqrt{t^2 + 16}$. Find the instantaneous velocity of the particle after 3 seconds, and the average velocity over the time interval $[0, 3]$. Can you argue that there exists a time $0 < t < 3$ such that the velocity at this time is exactly the average velocity over $[0, 3]$? What theorem can you apply to guarantee the existence of such a moment?

Solution:

$$v(t) = f'(t) = \frac{t}{\sqrt{t^2+16}}, \quad \text{so } v(3) = \frac{3}{\sqrt{9+16}} = \frac{3}{5} \text{ (cm/sec)}$$

The average velocity over $[0,3]$ is

$$v_{ave} = \frac{s(3)-s(0)}{3} = \frac{5-4}{3} = \frac{1}{3}$$

Using the mean value theorem for derivatives, we can find a time $t \in (0,3)$ (actually $t = \sqrt{2}$) such that

$$v(t) = \frac{t}{\sqrt{t^2+16}} = \frac{1}{3}$$

8. Find the indicated derivative of the given functions.

(a)

$$y = x^4 - 2x^3 + 4, \quad \text{find } y''$$

(b)

$$D_x\left(\sqrt{x^2+2\sqrt{x}}\right)$$

(c)

$$D_x\left(\frac{(x^3+\cos x)^3}{2+\sqrt[3]{2x}}\right)$$

(d)

$$y = x^3 - \sin(\cos x^2), \quad \text{find } y''$$

(e)

$$\frac{d}{dx}\left[\frac{x^2+2x-1}{x^2+1}\right]$$

(f)

$$f(x) = \frac{\tan x + x\cot x}{\cos 2x}, \quad \text{find } f'(x)$$

(g)

$$D_x^{49}\left((x^7-3x+1)^4\right)$$

(h)

$$\frac{dy}{dx}, \quad \text{given } y^3 - xy = 2x^3 + 2x^2 + 1$$

(i)

$$y = (\sin x)^x, \quad \text{find } y'$$

(j)

$$y = x^{x^2}, \quad \text{find } y'$$

Solution:

(a)

$$y' = 4x^3 - 6x^2, \quad y'' = 12x^2 - 12x$$

(b)

$$\frac{2x + \frac{1}{\sqrt{x}}}{2\sqrt{x^2 + 2\sqrt{x}}}$$

(c)

$$\frac{3(x^3+\cos x)^2(3x^2-\sin x)(2+\sqrt[3]{2x}) + \frac{2}{3}(x^3+\cos x)^3(2x)^{-2/3}}{(2+\sqrt[3]{2x})^2}$$

(d)

$$y' = 3x^2 + 2x\cos(\cos x^2)\sin x^2,$$
$$y'' = 6x + 2\cos(\cos x^2)\sin x^2 + 4x^2\left[\sin(\cos x^2)\sin^2 x^2 + \cos(\cos x^2)\cos x^2\right]$$

(e) First we can simplify the function to

$$1 + \frac{2(x-1)}{x^2+1}$$

so the derivative is

$$\frac{2(x^2+1) - 4x(x-1)}{(x^2+1)^2} = \frac{-2x^2+4x+2}{(x^2+1)^2}$$

(f)

$$f'(x) = \frac{(\sec^2 x + \cot x - x\csc^2 x)\cos 2x + 2\sin 2x(\tan x + x\cot x)}{\cos^2 2x}$$

(g) The function is a polynomial of degree 28, so the 49th derivative will be zero.

(h) Implicit differentiation gives

$$3y^2y' - y - xy' = 6x^2 + 4x,$$

so

$$y' = \frac{6x^2 + 4x + y}{3y^2 - x}$$

(i) We use logarithmic differentiation. Let $y = (\sin x)^x$. Then $\ln y = x\ln(\sin x)$. Differentiate this and we get:

$$\frac{y'}{y} = \ln(\sin x) + x\frac{\cos x}{\sin x}$$
$$y' = y\left(\ln(\sin x) + x\frac{\cos x}{\sin x}\right)$$
$$y' = (\sin x)^x \ln(\sin x) + x\ \cos x(\sin x)^{x-1}.$$

(j) Let $y = x^{x^2}$ so $\ln y = x^2 \ln x$.

$$\frac{y'}{y} = 2x\ln x + x^2\frac{1}{x} = 2x\ln x + x$$

So

$$y' = y(2x\ln x + x) = x^{x^2}(2x\ln x + x)$$

9. Given a function

$$y = f(x) = \frac{x^2 - 2x + 4}{(x+1)^2}$$

(a) Determine its natural domain;

(b) Find the limits $\lim_{x\to\pm\infty} f(x)$ and all the horizontal and vertical asymptotes;

(c) Find all the local maximum and minimum points;

(d) Find all the inflection points;

(e) Sketch the graph of $f(x)$.

Solution: First we note

$$f(x) = 1 - \frac{4x-3}{(x+1)^2}$$

and we calculate

$$f'(x) = \frac{4x-10}{(x+1)^3}, \quad f''(x) = \frac{34-8x}{(x+1)^4}$$

(a) The natural domain is $(-\infty, -1) \bigcup (-1, \infty)$.

(b) $y = 1$ is the horizontal asymptote and $x = -1$ is the vertical asymptote.

(c) $x = 5/2$ is the only critical point and $f''(5/2) > 0$ so it is a local minimum.

(d) $x = 17/4$ is the only inflection point.

(e)

10. An object moves along a horizontal coordinate line in such a way that its position at time t is specified by $s = t^3 - 6t^2 + 9t + 9$. Here s is measured in feet and t in seconds.

(a) When is the velocity 0?

(b) When is the velocity positive?

(c) When is the object moving to the left (that is, in the negative direction)?

(d) When is the acceleration positive?

Solution:

(a)

$$v(t) = 3(t^2 - 4t + 3) = 3(t-1)(t-3)$$

So v is zero at $t = 1$ and $t = 3$.

(b) When $t < 1$ or $t > 3$.

(c) When $1 < t < 3$.

(d) Since the acceleration is $a(t) = 6t - 12$, it is positive when $t > 2$.

11. Use differential to approximate the increase in the volume of the circular cone when the height h is increased from 10 inches to 10.1 inches. This cone comes with a shape that the diameter $(2r)$ at the opening is equal to the height of the cone. Hint: the volume of this cone is $V = \frac{1}{3}\pi r^2 h$.

Solution: We have $2r = h$ so $V = \frac{1}{12}\pi h^3$, and $dV = \frac{1}{4}\pi h^2 dh$. When $h = 10$,

$$\Delta V \approx \frac{1}{4}\pi \cdot 10^2 \cdot \Delta h = \frac{1}{4}\pi 10^2 \times 0.1 = 2.5 \times \pi \ \ \text{in}^3.$$

12. A farmer is planning to fence off three identical adjoining rectangular pens, each with 200 square feet of area. What is the optimal way to choose the width and length of each pen so that the least amount of fence is required?

Solution: The total length of the fence is $L = 6x + 4y$, and each pen has area $xy = 200$ so $y = 200/x$. We can write

$$L = 6x + \frac{800}{x}$$

The minimum is achieved at $x = 20/\sqrt{3}$ and $y = 10\sqrt{3}$.

13. For all rectangles with a diagonal of 12 inches, find the dimensions of the one with the maximum area.

Solution: Let the dimensions be x and y so $x^2+y^2 = 144$. The objective function is

$$A = xy = x\sqrt{144 - x^2}$$

and the maximum is achieved at $x = y = 12/\sqrt{2}$.

14. Find the equation of the tangent line to the curve (an ellipse)

$$\frac{x^2}{36} + \frac{y^2}{4} = 1$$

at the point $(-3, \sqrt{3})$. The equation of this tangent line can be written in the form $y = mx + b$.

Solution: Use implicit differentiation,

$$\frac{x}{18} + \frac{y}{2}y' = 0,$$

Plug in $x = -3,\ y = \sqrt{3}$, we have $y' = \frac{1}{3\sqrt{3}}$. The equation of the tangent line is therefore

$$y - \sqrt{3} = \frac{1}{3\sqrt{3}}(x+3) = \frac{1}{3\sqrt{3}}x + \frac{1}{\sqrt{3}}$$

15. As the sun sets behind a 40-foot building, the building's shadow grows. How fast is the shadow growing (in feet per second) when the sun's ray make an angle of 45°?

Solution: Suppose the shadow length is x and the angle at the shadow edge is θ, we have $\tan\theta = \frac{40}{x}$. Differentiating this relation,

$$-\frac{40}{x^2}\frac{dx}{dt} = \sec^2\theta\,\frac{d\theta}{dt}, \quad \text{or} \quad \frac{dx}{dt} = -\frac{40}{\sin^2\theta}\frac{d\theta}{dt}.$$

At $\theta = \pi/4$, $\sin^2\theta = \frac{1}{2}$, so

$$\frac{dx}{dt} = -80\frac{d\theta}{dt} \approx 0.0058 \text{ ft/sec}$$

Here we assume a constant $d\theta/dt = 2\pi/(24 \times 60 \times 60)$ rad/sec.

16. Sand is pouring from a pipe at the rate of 16 cubic feet per second. If the falling sand forms a conical pile on the ground whose height is always 1/4 the diameter of the base, how fast is the height increasing when the pile is 4 feet high? (Note: volume of a right circular cone is $V = 1/3\pi r^2 h$.)

Solution: We have $h = D/4 = r/2$,

$$V = \frac{1}{3}\pi r^2 h = \frac{4}{3}\pi h^3, \quad V' = 4\pi h^2 h'$$

When $V' = 16$ and $h = 4$,

$$h' = \frac{16}{4^3\pi} = \frac{1}{4\pi} \ \text{ ft/sec}$$

17. Find the number c guaranteed by the Mean Value Theorem for $f(x) = 2\sqrt[3]{x}$ on $[1, 8]$. That is, find the number c, with $1 < c < 8$, such that

$$f'(c) = \frac{f(8) - f(1)}{8 - 1}$$

Solution: We have $f'(x) = \frac{2}{3}x^{-\frac{2}{3}}$ so need to solve

$$\frac{2}{3}c^{-\frac{2}{3}} = 2\frac{8^{\frac{1}{3}} - 1^{\frac{1}{3}}}{7} = \frac{2}{7}$$

so $c = (7/3)^{3/2}$.

18. Evaluate the Riemann sum for $f(x) = 3x^2 - 2$ on the interval $[-1, 2]$ using the partition of three subintervals (of equal length), with the sample point $\bar{x}_i$ being the midpoint of the i-th interval. You may need the following formulas:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \quad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution: Use $x_1 = -0.5, x_2 = 0.5, x_3 = 1.5$, we find the Riemann sum

$$3\left[(-0.5)^2 + 0.5^2 + 1.5^2\right] - 6$$

19. Evaluate the following integrals. If it is an improper integral, determine the type and compute the integral if it is convergent.

(a)

$$\int \frac{2t^3}{\sqrt{t^4 - 6}}\, dt$$

(b)

$$\int \left(u^{5/4} - 2u + 10\right) du$$

(c)

$$\int x^2 \left(2x^2 + \frac{1}{x}\right) dx$$

(d)

$$\int (x^2+4)^6 x \, dx$$

(e)

$$\int_1^4 \frac{2x}{(2x^2+1)^2} \, dx$$

(f)

$$\int_0^2 (2-|x-1|) \, dx$$

(g)

$$\int_0^{\pi/2} \sin^3 x \cos x \, dx$$

(h)

$$\int_3^5 \frac{x\cos(\sqrt{x^2-9})}{\sqrt{x^2-9}} \, dx$$

(i)

$$\int_{-3}^3 \frac{x\cos x}{x^2+1} \, dx$$

(j)

$$\int_0^5 \frac{1}{x-1} \, dx$$

(k)

$$\int_1^\infty \frac{1}{1+x^2} \, dx$$

Solution:

(a)

$$\sqrt{t^4-6}+C$$

(b)

$$\frac{4}{9}u^{\frac{9}{4}} - u^2 + 10u + C$$

(c)

$$\frac{2}{5}x^5 + \frac{1}{2}x^2 + C$$

(d)

$$\frac{1}{14}(x^2+4)^7 + C$$

(e)

$$\frac{1}{2}\int_3^{33} \frac{du}{u^2} = -\frac{1}{66} + \frac{1}{6} = \frac{5}{33}$$

(f)

$$\int_0^2 (2 - |x-1|)\,dx = \int_0^1 (1+x)\,dx + \int_1^2 (3-x)\,dx = 3$$

(g) Let $u = \sin x$, we have

$$\int_0^1 u^3\,du = \frac{1}{4}$$

(h) Let $u = \sqrt{x^2-9}$, $\quad du = \frac{x\,dx}{\sqrt{x^2-9}}$, so the integral is

$$\int_0^4 \cos u\,du = \sin 4$$

(i) The integrand is an odd function of x and we are integrating from -3 to 3. The integral from -3 to 0 and the integral from 0 to 3 shall cancel out and the integral from -3 to 3 is therefore 0.

(j) This is an improper integral of type 2 where the integrand is discontinuous at $x = 1$. We need to consider separately and the first one is for c close to 1

$$\int_0^c \frac{1}{x-1}\,dx = \ln|x-1|\Big|_0^c = \ln|c-1| - \ln 1$$

and the limit of the above as $c \to 1^-$ does not exist. So one of the integrals is divergent and the integral from 0 to 5 is also divergent.

(k)

$$\int_1^c \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_1^c = \tan^{-1} c - \tan^{-1} 1 = \tan^{-1} c - \frac{\pi}{4}$$

As $c \to \infty$, $\tan^{-1} c \to \pi/2$ so

$$\int_1^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

20. Find the average value of $f(x) = 5x^4 - 3x^2 + 4$ on $[0, 1]$.

Solution:

$$f_{ave} = \int_0^1 \left(5x^4 - 3x^2 + 4\right)\,dx = 1 - 1 + 4 = 4$$

21. Find $G'(x)$ given $G(x) = \int_{-1}^{x^2} t\sin(t^2+1)\,dt$.

Solution: Let $G = \int_{-1}^{u} t\sin(t^2+1)\,dt$, so

$$G'(x) = \frac{d}{du}G \cdot \frac{du}{dx} = u\sin(u^2+1)(2x) = 2x^3\sin(x^4+1)$$

22. Find the area of the region bounded by $y = (x-1)^2 + 2,\ y = 0,\ x = 0$, and $x = 3$.

Solution: This function is above zero for $0 \le x \le 3$, so the area is

$$\int_0^3 \left[(x-1)^2 + 2\right]\,dx = \frac{1}{3}\left[2^3 - (-1)^3\right] + 6 = 9$$

23. Find the area of the region bounded by $y = x$ and $y = 5x - x^2$.

Solution: We can find that the curves meet at $(0,0)$ and $(4,4)$. On $[0,4]$, $y = 5x - x^2$ is above $y = x$ so the area is given by

$$\int_0^4 (5x - x^2 - x)\,dx = \int_0^4 (4x - x^2)\,dx = \left(2x^2 - \frac{1}{3}x^3\right)\Big|_0^4 = \frac{32}{3}$$

24. Find the volume of the solid generated by revolving about the x-axis the region bounded by $y = x^2/3,\ x = 1$, and $y = 0$.

Solution: Using the method of disks,

$$V = \int_0^1 \pi\left(\frac{x^2}{3}\right)^2 dx = \frac{\pi}{9}\int_0^1 x^4\,dx = \frac{\pi}{45}$$

25. Find the volume of the solid generated by revolving about the y-axis the region bounded by $y = \sqrt{x}/3,\ x = 9$, and $y = 0$.

Solution: Using the method of shells,

$$V = \int_0^9 2\pi x\frac{\sqrt{x}}{3}\,dx = \frac{2\pi}{3}\int_0^9 x^{\frac{3}{2}}\,dx = \frac{4\pi}{15}3^5 = \frac{324\pi}{5}$$

26. Find the volume of the solid generated by revolving about the x-axis the region bounded by $y = 2x$ and $y = 2x^4$.

Solution: Using the method of washers,

$$V = \int_0^1 \pi\left[(2x)^2 - (2x^4)^2\right]\,dx = 4\pi\int_0^1 (x^2 - x^8)\,dx = 4\pi\left(\frac{1}{3} - \frac{1}{9}\right) = \frac{8\pi}{9}$$

27. Find the volume of the solid generated by revolving about the y-axis the region bounded by $y = 2x$ and $y = 2x^4$.

Solution: Using the method of shells,

$$V = \int_0^1 2\pi x(2x - 2x^4)\,dx = 4\pi \int_0^1 (x^2 - x^5)\,dx = 4\pi \left(\frac{1}{3} - \frac{1}{6}\right) = \frac{2\pi}{3}$$