Math 1310

Problem Bank for Midterm 3

1. An electrical circuit contains two resistors in series, with resistances $R_1\Omega$ and $R_2\Omega$. It is known that the total resistance R may be expressed as

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{1}$$

The first resistance R_1 is equal to $\frac{4}{3}\Omega$, and is decreasing at a rate of $4\Omega/hr$. The second resistance R_2 is equal to 4Ω and is increasing at a rate of $8\Omega/hr$.

(i) What is the total resistance R?

(ii) What is the rate of change of R in Ω/hr?

Solution: (i) We first evaluate R.

$$\frac{1}{R} = \frac{3}{4} + \frac{1}{4} = 1. \tag{2}$$

This means that $R = 1\Omega$.

(ii) By the Chain Rule,

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt}. \tag{3}$$

Substituting in the values for R, R_1 and R_2,

$$\begin{aligned}\frac{dR}{dt} &= \frac{9}{16}\frac{dR_1}{dt} + \frac{1}{16}\frac{dR_2}{dt}\\ &= -\frac{9\times 4}{16} + \frac{8}{16}\\ &= -\frac{7}{4}.\end{aligned}$$

This means that the total resistance is decreasing at a rate of $\frac{7}{4}$ Ω/hr.

2. Let $f(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 - 2x + 10$.

(i) Find the extreme points of f. Identify which of these are local maxima and which are local minima, and explain your answer.

(ii) Find the intervals over which f is increasing, and the intervals where f is decreasing.

(iii) Find any inflection points, and intervals where f is concave upwards, and intervals where f is concave downwards.

(iv) Sketch the curve.

Solution: Differentiating,

$$f'(x) = 2x^2 - 3x - 2$$
$$= (x-2)(2x+1).$$

Since f is differentiable everywhere, its extreme values are only where its derivative is zero. These points are 2 and $-\frac{1}{2}$.

There are two ways you can determine whether these extreme values are maxima or minima. The first method is to check the second derivative. The second derivative is

$$f''(x) = 4x - 3. \tag{4}$$

Since $f'' > 0$ at $x = 2$, this is a local minimum. Since $f'' < 0$ at $x = -\frac{1}{2}$, this is a local maximum.

The other method is to look at the signs of f' either side of the extreme value. It can be checked that $f' > 0$ for $x > 2$, and $f'(0) = -2 < 0$. This means that $x = 2$ is a local minimum. Also $f'(-1) = 3$, and since $f'(0) < 0$, this means that $x = -\frac{1}{2}$ is a local minimum.

(ii) f is increasing over the interval $(2, \infty)$, because $x = 2$ is a local minimum. f is decreasing over the interval $(-1/2, 2)$, because this interval has a local maximum on the left and a local minimum on the right. f is increasing over the interval $(-\infty, -1/2)$, since this interval has a local maximum on the right.

(iii) Inflection points can only occur when $f'' = 0$. The only point where $f'' = 0$ is $x = \frac{3}{4}$. This is an inflection point because $f'' < 0$ for $x < \frac{3}{4}$, and $f'' > 0$ for $x > \frac{3}{4}$.

The function is concave upwards on the interval $(\frac{3}{4}, \infty)$, and it is concave downwards over the interval $(-\infty, \frac{3}{4})$.

3. Let $f(x) = x^{\frac{1}{3}}(x-1)^2$. Identify the critical numbers of f.

Solution: The critical numbers are where $f' = 0$, or the derivative doesn't exist. Differentiating,

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}(x-1)^2 + 2(x-1)x^{\frac{1}{3}}$$
$$= x^{\frac{1}{3}}(x-1)\left(\frac{x-1}{3x} + 2\right).$$

Since $|f'(x)| \to \infty$ as $x \to 0$, the derivative does not exist at $x = 0$, so $x = 0$ is a critical number.

The derivative is zero when either $x-1=0$, in which case $x=1$, or

$$\frac{x-1}{3x}+2=0,$$

in which case $x=\frac{1}{7}$.

In summary, the critical numbers $0,1$ and $\frac{1}{7}$.

4. Let $f(x)=\cos(x)+1-x$. Find the absolute maximum and absolute minimum values of f over the interval $[-\pi,2\pi]$.

Solution: We first identify the local maxima and minima through differentiating. We find that

$$f'(x)=-\sin(x)-1.$$

Over the interval, $f'(x)=0$ when $x=-\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$.

Evaluating the function at these points, we find that $f\left(-\frac{\pi}{2}\right)=1+\frac{\pi}{2}$ and $f\left(\frac{3\pi}{2}\right)=1-\frac{3\pi}{2}$.

At the ends of the interval, $f(-\pi)=\pi$, and $f(2\pi)=2-2\pi$.

Of these four values, the biggest is π, so the absolute maximum is π.

The smallest is $2-2\pi$, so the absolute minimum is $2-2\pi$.