Math 1310

Midterm Exam 3

Name and Unid (print clearly): ______________________________

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1. Suppose a sphere's radius is growing at a rate 3 cm/min. At what rate is the surface area growing when the radius is 5cm? Note the surface area of a sphere is $A = 4\pi r^2$.

Solution: $\frac{\mathrm{d}r}{\mathrm{d}t} = 3$ cm/min is given, the question asks for $\frac{\mathrm{d}A}{\mathrm{d}t}$ when $r = 5$. Use implicit differentiation:

$$\frac{dA}{dt} = \frac{d}{dt}4\pi r^2 = 8\pi r\frac{dr}{dt}.$$

Plugging in the values, the derivative A' is obtained:

$$\frac{dA}{dt} = 8\pi(5 \times 3) = 8 \times 15 \times \pi.$$

2. Consider the function $f(x) = -x^4 - 4x^3 + 20x^2 + 1$.

(a) Find the intervals, where f is increasing and the intervals where f is decreasing.

Solution: Find the critical points and identify the intervals between them:

$$f'(x) = -4x^3 - 12x^2 + 40x = -4x(x^2 + 3x - 10) = -4x(x-2)(x+5).$$

the critical points are $c = -5, 0, 2$; and the intervals between are $(-\infty, -5)$, $(-5, 0)$, $(0, 2)$, and $(2, \infty)$ So f is increasing on $(-\infty, -5)$ and $(0, 2)$ and decreasing on $(-5, 0)$ and $(2, \infty)$.

(b) Find the x-locations of the local maxima and local minima.

Solution:
Local maxima: $x = -5$, $x = 2$.
Local minima: $x = 0$.

3. (20 points) Identify the indeterminate form and evaluate the following limits using l'Hospitals rule.

(a) $\lim_{x\to 1} \frac{e^{x-1}-1}{x^2}$

> **Solution:** The indeterminate form is "$\frac{0}{0}$".
>
> $$\lim_{x\to 1} \frac{e^{x-1}-1}{x^2} \overset{l'H}{=} \lim_{x\to 1} \frac{e^{x-1}}{2x} = \frac{1}{2}$$

(b) $\lim_{x\to 0} \sec(x)^{\frac{1}{x}}$, Note: $\sec(x) = \frac{1}{\cos(x)}$, and $(\sec(x))' = \sec(x)\tan(x)$.

> **Solution:** The indeterminate form is "1^∞". Taking ln of the limit gives
>
> $$\lim_{x\to 0} \frac{1}{x} \ln(\sec(x)) = \lim_{x\to 0} \frac{\ln(\sec(x))}{x} \overset{l'H}{=} \lim_{x\to 0} \frac{\frac{\sec(x)\tan(x)}{\sec(x)}}{1} = \lim_{x\to 0} \frac{\tan(x)}{1} = 0$$
>
> Thus
>
> $$\lim_{x\to 0} \sec(x)^{\frac{1}{x}} = e^0 = 1$$

4. (20 points) Suppose a soup manufacturer want's to stand out from their competition by packaging their new line of organic soups in rectangular cartons with a base depth w that is X-times its base width w. The soup volume is fixed at 400 mL (cm^3). The rectangular carton shape must be chosen with a given height h, width w, and depth d cm. In order to minimize packaging costs, the manufacturer wants to find the rectangular shape that minimizes surface area. Your goal, find the dimensions (in cm) of the minimum-surface area carton. You may leave your calculations for w, h, and d as algebraic expressions.

Solution:

$$V = w \times Xw \times h = 400 = Xw^2 h$$

$$h = \tfrac{400}{Xw^2}$$

$$S = 2hw + 2hXw + 2Xw^2$$

$$S = \tfrac{800}{Xw} + \tfrac{800}{w} + 2Xw^2$$

$$S = \left(\tfrac{1}{X} + 1\right)\tfrac{800}{w} + 2Xw^2$$

$$S'(w) = -\left(\tfrac{1}{X} + 1\right)\tfrac{800}{w^2} + 4Xw = 0$$

$$4Xw^3 = 800\left(\tfrac{1}{X} + 1\right)$$

$$w^3 = \frac{200\left(\tfrac{1}{X} + 1\right)}{X} = 200\left(\tfrac{1+X}{X^2}\right)$$

$$w_* = \sqrt[3]{200\left(\tfrac{1+X}{X^2}\right)}$$

$$h_* = \tfrac{400}{Xw_*^2}$$

$$d_* = Xw_*.$$

5. (20 points) Integration via summation

(a) Write down the integral equalling the following Riemann Sum:

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} \left[x_n e^{x_n} - \cos(x_n) \right] \frac{14}{N},$$

where $x_n = -11 + \frac{14n}{N}$. Is this a left-endpoint, right-endpoint or midpoint rule?

(b) Write down the N-point, right-endpoint Riemann Sum approximation to the following integral. Express your answer in terms of the points $x_n = -3 + \frac{6n}{N}$.

$$\int_{-3}^{3} \left(x^2 + \sin(x) \right) dx.$$

Solution:

a) This is the left-endpoint rule, and the integral is

$$\int_{-11}^{3} \left(xe^x - \cos(x) \right) dx.$$

b)

$$R_N = \frac{6}{N} \sum_{n=1}^{N} \left(x_n^2 + \sin(x_n) \right)$$

6. (20 points) Compute the (a) definite integral, (b) indefinite integral.

(a)

$$\int_{-2}^{2} \left(x^2 - \cos(x)\right) dx$$

(b)

$$\int \left(e^x + 4x^{-1} - x^2\right) dx$$

Solution: a) This is $\left[\frac{x^3}{3} - \sin(x)\right]_{-2}^{2}$, which evaluates to $\frac{16}{3} - \sin(2) + \sin(-2) = \frac{16}{3}$.

b) This is

$$e^x + 4\log(x) - \frac{x^3}{3} + C.$$