Math 1310 § 4. **Second Midterm Exam** Name: Solutions
Treibergs $\mathcal{A}\pi$ February 22, 2023

1. *Compute the derivatives of the following functions using the derivative rules.*

   (a) $f(x) = \cos(x)\sqrt{1+e^{(x^2)}}$
   Rewrite
   $$f(x) = \cos(x)\left(1+e^{(x^2)}\right)^{\frac{1}{2}}$$
   By product rule and chain rule
   $$f'(x) = -\sin(x)\left(1+e^{(x^2)}\right)^{\frac{1}{2}} + \cos(x)\cdot\frac{1}{2}\left(1+e^{(x^2)}\right)^{-\frac{1}{2}}\cdot e^{(x^2)}\cdot 2x$$

   (b) $g(x) = \dfrac{\sec(x)}{4+\tan^{-1}(x)}$
   By quotient rule,
   $$g'(x) = \frac{\sec(x)\tan(x)\left(4+\tan^{-1}(x)\right) - \sec(x)\cdot\frac{1}{1+x^2}}{\left(4+\tan^{-1}(x)\right)^2}$$

   (c) $h(x) = x^{\ln(x)}$
   Rewrite
   $$h(x) = \left(e^{\ln(x)}\right)^{\ln(x)} = e^{(\ln(x))^2}$$
   By chain rule
   $$h'(x) = e^{(\ln(x))^2}\cdot 2\ln(x)\cdot\frac{1}{x}$$

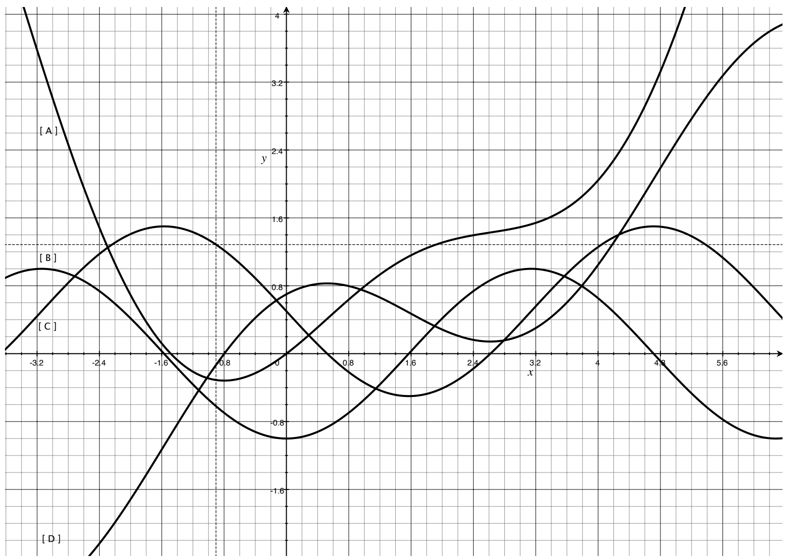
2. (a) *Compute the derivative of* $f(x) = \dfrac{1}{x+3}$ *using the limit based definition.*
   The derivative at $a \neq -3$ is
   $$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\ &= \lim_{h\to 0}\frac{\frac{1}{a+h+3}-\frac{1}{a+3}}{h} \\ &= \lim_{h\to 0}\frac{(a+3)-(a+h+3)}{h(a+h+3)(a+3)} \\ &= \lim_{h\to 0} -\frac{h}{h(a+h+3)(a+3)} \\ &= \lim_{h\to 0} -\frac{1}{(a+h+3)(a+3)} \\ &= -\frac{1}{(a+3)^2} \end{aligned}$$

   (b) *Use the definition of derivative to find* $L = \displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{x}$.

Let $f(x) = \ln(x)$ that has $f'(x) = \dfrac{1}{x}$. Then

$$\begin{aligned} L &= \lim_{h\to 0} \frac{\ln(1+h)}{h} \\ &= \lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h} \\ &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= f'(1) = \frac{1}{1} = 1. \end{aligned}$$

3. (a) *The figure shows graphs of f, f', f'' and f'''. Identify each curve and explain your choices.*



Curve	A, B, C or D ?	Why?
f	A	Starting function.
f'	D	A is decreasing for $x < -.8$, horizontal tangent at $x = -.8$ and increasing for $-.8 < x$. So $A'(x) < 0$ for $x < -.8$, $A'(-.8) = 0$ and $A'(x) > 0$ if $x > -.8$. D is the only function that has these signs.
f''	C	D increases for $x < .5$, decreases for $.5 < x < 2.6$ and then increases. So $D'(x) > 0$ for $x < .5$, $D'(x) < 0$ for $.5 < x < 2.6$, and $D'(x) > 0$ for $2.6 < x$. C is the only function that has these signs.
f'''	B	C increases for $x < -1.6$, decreases for $-1.6 < x < 1.6$, then increases for $1.6 < x < 4.7$ and then decreases. So $C'(x) > 0$ for $x < .5$ or $1.6 < x < 4.7$ and $C'(x) < 0$ for $.5 < x < 2.6$ or $4.7 < x$. B is the only function that has these signs.

(b) *The circumference of a sphere is measured to be* 10 *cm with a possible error or* 0.5 *cm. Use differentials to estimate the maximum error in the calculated volume. What is the relative error?*

$\left[\textit{Hint. Recall that a sphere of circumference } c \textit{ has volume} \quad V = \dfrac{c^3}{6\pi^2}. \right]$

Computing the derivative, remembering that $\dfrac{1}{6\pi^2}$ is constant,

$$\frac{dV}{dc} = \frac{c^2}{2\pi^2}$$

The differential is

$$dV = \frac{dV}{dc}\, dc$$

Thus if the error in circumference is $\Delta c = \pm 0.5$, then at $c = 10$ cm, by using $dc = \Delta c$, the differential gives an approximation

$$\Delta V \approx dV = \frac{dV}{dc}\, dc = \frac{c^2}{2\pi^2} dc = \frac{10^2}{2\pi^2}(\pm 0.5) = \pm\frac{25}{\pi^2} \approx \pm 2.533 \text{ cm}^3 =$$

The relative error is

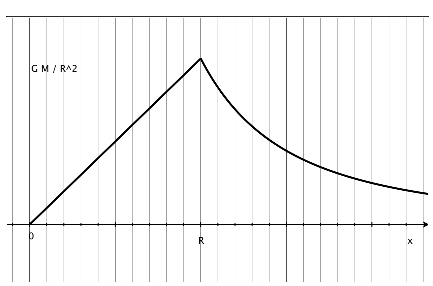
$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{\frac{c^2}{2\pi^2} dc}{\frac{c^3}{6\pi^2}} = \frac{3dc}{c} = \frac{3\cdot(\pm 0.5)}{10} = \pm\frac{3}{20}$$

Thus a 5% relative error on measuring the circumference results in about a 15% relative error in the volume.

4. *The gravitational force exerted by the earth on a unit mass at a distance x from the center of the planet is*

$$F(x) = \begin{cases} \dfrac{GMx}{R^3} & \text{if } x < R \\ \\ \dfrac{GM}{x^2} & \text{if } x \ge R \end{cases}$$

where M is the mass of the earth, R is the radius and G is the gravitational constant. Sketch the graph of $y = F(x)$*. Is* $F(x)$ *a continuous function of* x*? Why? Is* $F(x)$ *differentiable function of* x*? Why?*



Plot of gravitational force function $F(x)$.

$F(x)$ is continuous. $F(x)$ is a linear function for $x < R$ (the region inside the earth) and then is inverse quadratic if $R < x$ outside the earth. Hence to the left and right of $x = R$

the functions are continuous. At $x = R$ the left and right limits agree, so $F(x)$ is also continuous at $x = R$. To see this, compute the left and right limits

$$\lim_{x \to R-} F(x) = \lim_{x \to R-} \frac{GMx}{R^3} = \frac{GM}{R^2}; \qquad \lim_{x \to R+} F(x) = \lim_{x \to R+} \frac{GM}{x^2} = \frac{GM}{R^2}.$$

$F(x)$ is not differentiable. Because $F(x)$ is a linear function for $x < R$ and inverse quadratic if $R < x$, to the left and right of $x = R$ the functions are dfferentiable. But there is a kink in $F(x)$ at $x = R$ because the slopes to the left and to the right are different so F is not differentiable at $x = R$. The limits of the left and right derivatives differ. To see this, compute the left and right limits

$$\lim_{x \to R-} F'(x) = \lim_{x \to R-} \frac{GM}{R^3} = \frac{GM}{R^3}; \qquad \lim_{x \to R+} F'(x) = \lim_{x \to R+} -\frac{2GM}{x^3} = -\frac{2GM}{R^3}.$$

5. (a) *Show that* $(2, 1)$ *is a point on the lemniscate curve. Find the slope of the tangent line at the point* $(2, 1)$.

$$3(x^2 + y^2)^2 = 25(x^2 - y^2). \tag{1}$$

At the point $(2, 1)$ the left and right sides evaluate to

$$\begin{aligned} \text{LHS} &= 3(x^2 + y^2)^2 = 3(2^2 + 1^2)^2 = 3 \cdot 5^2 = 75; \\ \text{RHS} &= 25(x^2 - y^2) = 25(2^2 - 1^2) = 25 \cdot 3 = 75. \end{aligned}$$

They are equal so (1) holds at $(x, y) = (2, 1)$.
Differentiating implicitly with respect to x and remembering that $y = y(x)$ is a function of x we get dividing by two,

$$\begin{aligned} 3 \cdot 2(x^2 + y^2) \cdot \left(2x + 2y\frac{dy}{dx}\right) &= 25 \left(2x - 2y\frac{dy}{dx}\right) \\ 6(x^2 + y^2)x + 6(x^2 + y^2)y\frac{dy}{dx} &= 25x - 25y\frac{dy}{dx} \\ \left(6(x^2 + y^2) + 25\right) y\frac{dy}{dx} &= \left(25 - 6(x^2 + y^2)\right) x \end{aligned}$$

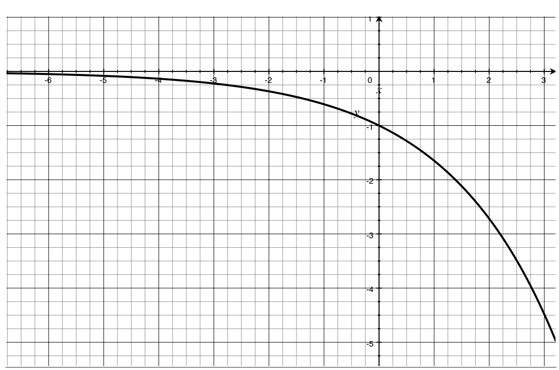
At $(2, 1)$ this yields

$$55\frac{dy}{dx} = \left(6(2^2 + 1^2) + 25\right) \frac{dy}{dx} = \left(25 - 6(2^2 + 1^2)\right) \cdot 2 = -10,$$

hence

$$\frac{dy}{dx} = -\frac{10}{55} = -\frac{2}{11}.$$

(b) *Sketch the graph of a function whose first and second derivatives are negative.*
Consider $f(x) = -e^{x/2}$. Then $f'(x) = -\frac{1}{2}e^{x/2} < 0$ and $f''(x) = -\frac{1}{4}e^{x/2} < 0$.



Graph of $f(x) = -e^{x/2}$ whose $f' < 0$ and $f'' < 0$.