Math 1310 § 4. **Third Midterm Exam** Solutions
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1. *Use the midpoint rule with $n = 4$ subintervals to approximate the integral $I = \int_1^2 \frac{1}{x}\,dx$. You do not need to simplify your answer.*

   With $n = 4$ intervals, the width of each interval and the function is

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}, \qquad f(x) = \frac{1}{x}.$$

   Then the equally spaced partition points are

$$x_0 = 1, \qquad x_1 = \frac{5}{4}, \qquad x_2 = \frac{3}{2}, \qquad x_3 = \frac{7}{4}, \qquad x_4 = 2.$$

   The general formula is $x_i = a + i\Delta x = 1 + \frac{i}{4}$. The midpoints are

$$x_1^* = \frac{9}{8}, \qquad x_2^* = \frac{11}{8}, \qquad x_3^* = \frac{13}{8}, \qquad x_4^* = \frac{15}{8}.$$

   The general formula for the midpoint is is

$$x_i^* = \frac{x_{i-1} + x_i}{2} = \frac{1 + \frac{i-1}{4} + 1 + \frac{i}{4}}{2} = \frac{7}{8} + \frac{i}{4}.$$

   The midpoint rule is the Riemann sum using midpoints

$$R_n = \sum_{i=1}^{n} f\left(x_i^*\right)\,\Delta x$$

   Substituting the given information, the midpoint rule with four intervals is

$$R_4 = \sum_{i=1}^{n} \frac{1}{\frac{7}{8} + \frac{i}{4}} \cdot \frac{1}{4} = \left\{ \frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15} \right\} \frac{1}{4}$$

2. *Use Newton's Method to approximate the cube root $\sqrt[3]{3}$. Find a polynomial function with integer coefficients, for which $\sqrt[3]{3}$ is a root. Give the iteration formula. Perform two iterations by hand using the initial approximation $x_1 = 1$.*

   Because $\left(3^{\frac{1}{3}}\right)^3 = 3$, a polynomiql function for which $\sqrt[3]{3}$ is a root is

$$f(x) = x^3 - 3.$$

   Newton's Method starts from an initial guess x_1 and proceeds by the recursion for $i \ge 1$,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

   For our $f(x)$ this simplifies to

$$x_{i+1} = x_i - \frac{x_i^3 - 3}{3x_i^2} = \frac{2}{3}x_i + \frac{1}{x_i^2}.$$

Running this by hand,

$$
\begin{aligned}
x_1 &= 1\\
x_2 &= \frac{2}{3}\cdot 1 + \frac{1}{1^2} = \frac{5}{3}\\
x_3 &= \frac{2}{3}\cdot\frac{5}{3} + \frac{1}{\left(\frac{5}{3}\right)^2} = \frac{10}{9} + \frac{9}{25} = \frac{250+81}{225} = \frac{331}{225}
\end{aligned}
$$

It's not part of the answer, but you may be amused by what this looks like run on a computer. Here is an **R**© program.

```
> x[1] = 1; for(i in 1:9) {x[i+1] = 2*x[i]/3 + 1/(x[i]^2)}
> print(x[1:10], digits=15)
    1.00000000000000
    1.66666666666667
    1.47111111111111
    1.44281209824934
    1.44224978959900
    1.44224957030744
    1.44224957030741
    1.44224957030741
    1.44224957030741
    1.44224957030741
```

3. (a) *Find two numbers whose difference is* 5 *and whose product is a minimum.*

Let x and y be the numbers. They satisfy the constraint that their difference is five,

$$x - y = 5.$$

The objective is to minimize their product

$$P = xy.$$

Solving the constraint equation for one of the variables

$$x = 5 + y.$$

Substituting into the objective makes it a function of one variable

$$P(y) = (5+y)y = 5y + y^2.$$

This is a parabola opening upward so there is only one minimum at the vertex. The minimum occurs at the stationary point. Differentiate

$$\frac{dP}{dy} = 5 + 2y.$$

Set equal to zero and solve.

$$
\begin{aligned}
5 + 2y &= 0\\
y &= -\frac{5}{2}.
\end{aligned}
$$

The other number is

$$x = 5 + y = \frac{5}{2}.$$

Thus the two numbers whose difference is five which have minimum product are

$$\boxed{x = \frac{5}{2}, \quad y = -\frac{5}{2}.}$$

(b) *Two cars are moving from the same point. One travels east at 72 mi/h and the other travels north at 30 mi/h. At what rate is the distance between the cars increasing two hours later?*

Starting from the origin, one car is at $x(t)$ along the x-axis and the other is $y(t)$ along the y-axis after t hours. Their velocities are $x' = 72$ mph and $y' = 30$ mph so after $t = 2$ hours, the cars are at $x = 144$ mi and $y = 60$ mi. Let $r(t)$ be the distance between the two cars. By the Pythagorean Theorem

$$x(t)^2 + y(t)^2 = r(t)^2.$$

Note that $x = 12 \cdot 12$ and $y = 12 \cdot 5$ after two hours. This is a $5 - 12 - 13$ triangle so $r = 12 \cdot 13 = 156$ mi after two hours. Indeed

$$144^2 + 60^2 = 20736 + 3600 = 24336 = 156^2.$$

Differentiating with respect to time

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$

Solving for $\frac{dr}{dt}$ and substituting the values at two hours,

$$\frac{dr}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{r} = \frac{144 \cdot 72 + 60 \cdot 30}{156} = \frac{10368 + 1800}{156} = \frac{12168}{156} = \boxed{78 \text{ mph.}}$$

4. *Consider the function $f(x)$. Find the critical numbers for f. Find the intervals over which f is increasing, and the intervals where f is decreasing. Determine whether each critical number is local maximum or a local minimum, and explain your answer. Sketch the curve.*

$$f(x) = x^5 - 10x^4 + 20x^3.$$

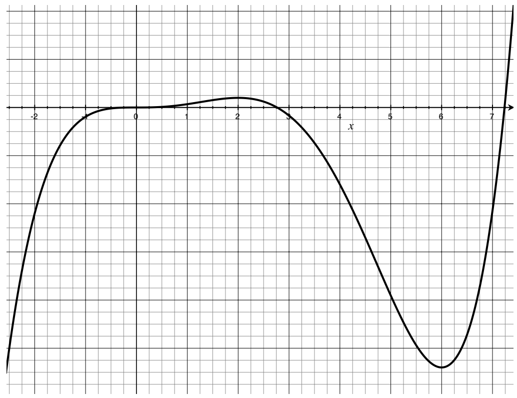
Differentiating,

$$f'(x) = 5x^4 - 40x^3 + 60x^2 = 5x^2(x^2 - 8x + 12) = 5x^2(x-2)(x-8).$$

The derivative exists at all real numbers and there are no boundary points, thus the only critical numbers are the stationary points $\boxed{x = 0, 2, 6.}$ On the intervals between the stationary points we have

$$\begin{aligned} f' > 0 &\text{ on } (-\infty, 0) \text{ where } f \text{ is increasing;} \\ f' > 0 &\text{ on } (0, 2) \text{ where } f \text{ is increasing;} \\ f' < 0 &\text{ on } (2, 6) \text{ where } f \text{ is decreasing;} \\ f' > 0 &\text{ on } (6, \infty) \text{ where } f \text{ is increasing.} \end{aligned}$$

Using the first order criterion, f is increasing on $(-\infty, 0)$ and $(0, 2)$ so 0 is neither a local minimum nor a local maximum; f is increasing on $(0, 2)$ and decreasing on $(2, 6)$ so 2 is a local maximum; f is decreasing on $(2, 6)$ and increasing on $(6, \infty)$ so 6 is a local minimum.



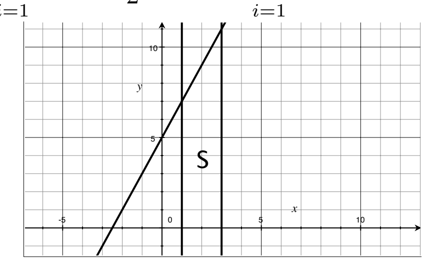
Plot of $f(x) = x^5 - 10x^4 + 20x^3$.

5.

$$J = \int_1^3 2x + 5\,dx$$

Find the integral J using geometry. Suppose you approximate the area J with n rectangles of equal width. Using the right endpoint rule, what are the coordinates x_i, x_i^* and what is Δx? Write a Riemann Sum expression $R_n = \sum_{?}^{?} (?)\,\Delta x$ for the total area of these rectangles. Find the limit $\lim\limits_{n\to\infty} R_n$.

$$\left[\text{Hint: } \sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.\right]$$



Plot of $f(x) = 2x + 5$.

The region S bounded by $x = 1$, $x = 3$, $y = 0$ and $y = 2x + 5$ is a trapezoid. Its area is

$$J = \text{Area}(S) = \frac{h(b_1 + b_2)}{2} = \frac{2(7 + 11)}{2} = 18$$

where the height is $h = 3 - 1 = 2$ and bases are $b_1 = f(1) = 7$ and $b_2 = f(3) = 11$.

Now suppose that the interval $[a, b] = [1, 3]$ is subdivided into n equal subintervals, each of width

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}.$$

Then the equally spaced partition points are

$$x_i = a + i\Delta x = 1 + \frac{2i}{n}, \qquad \text{where } i = 0, 1, 2, \ldots, n.$$

The ith subinterval is $I_i = [x_{i-1}, x_i]$. The right endpoint rule takes the sample point to be the rightmost point in each subinterval

$$x_i^* = x_i = 1 + \frac{2i}{n}.$$

The ith rectangle at I_i contributes an area height times width $f(x_i^*)\Delta x$. Adding up all n rectangles gives the Riemann Sum

$$R_n = \sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} \left\{ 2\left(1 + \frac{2i}{n}\right) + 5 \right\} \frac{2}{n}$$

Use the hint to rewrite the Riemann Sum:

$$\begin{aligned} R_n &= \sum_{i=1}^{n} \left\{ 7 + \frac{4i}{n} \right\} \frac{2}{n} \\ &= \frac{14}{n} \sum_{i=1}^{n} 1 + \frac{8}{n^2} \sum_{i=1}^{n} i \\ &= \frac{14}{n} \cdot n + \frac{8}{n^2} \cdot \frac{n(n+1)}{2} \\ &= 14 + \frac{4(n+1)}{n} \end{aligned}$$

Taking the limit gives the integral

$$J = \lim_{n\to\infty} R_n = 14 + 4 = 18,$$

as expected.