Math 1310 § 4. **First Midterm Exam** Name: Solutions
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1. *Find the equation $y = mx + b$ of the secant line for the function $y = 2^x$ through the points $x = -1$ and $x = 2$.*

   Let $f(x) = 2^x$. The secant line passes through the points at $x_1 = -1$ where $y_1 = f(-1) = 2^{-1} = \frac{1}{2}$ ands $x_2 = 2$ where $y_2 = f(2) = 2^2 = 4$. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - \frac{1}{2}}{2 - (-1)} = \frac{\frac{7}{2}}{3} = \frac{7}{6}.$$

   The line has the formula

$$y = mx + b = \frac{7}{6}x + b.$$

   It passes through the point $(2, 4)$ thus

$$4 = \frac{7}{6} \cdot 2 + b$$

   or

$$b = 4 - \frac{7}{6} \cdot 2 = \frac{12}{3} - \frac{7}{3} = \frac{5}{3}.$$

   Thus the equation of the secant line is

$$y = \frac{7}{6}x + \frac{5}{3}.$$

2. *Find the limit if it exists. Explain your answer.*

   (a) $\displaystyle\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$

   Simplifying the expression when $x \neq -3$ and using the difference and constant multiple laws for limits,

$$\begin{aligned}\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} &= \lim_{x \to -3} \frac{(x + 3)(x - 2)}{x + 3} \\ &= \lim_{x \to -3} (x - 2) \\ &= \lim_{x \to -3} x - \lim_{x \to -3} 2 = -3 - 2 = -5.\end{aligned}$$

   (b) $\displaystyle\lim_{x \to 4} \sqrt{x^2 + 9}$

   In order to use the square root law, we need to make sure that the radicand is positive. Using the power and sum laws we see that

$$\lim_{x \to 4} (x^2 + 9) = \lim_{x \to 4} x^2 + \lim_{x \to 4} 9 = \left(\lim_{x \to 4} x\right)^2 + \lim_{x \to 4} 9 = 4^2 + 9 = 25$$

   is positive. Thus the square root law applies

$$\lim_{x \to 4} \sqrt{x^2 + 9} = \sqrt{\lim_{x \to 4} (x^2 + 9)} = \sqrt{25} = 5$$

   which is nonzero. Thus the quotient law applies. Using also the product rule yields

$$\begin{aligned}\lim_{x \to 4} \frac{x(x + 1)}{\sqrt{x^2 + 9}} &= \frac{\lim_{x \to 4} x(x + 1)}{\lim_{x \to 4} \sqrt{x^2 + 9}} = \frac{\lim_{x \to 4} x \cdot \lim_{x \to 4} (x + 1)}{5} \\ &= \frac{4 \cdot \left(\lim_{x \to 4} x + \lim_{x \to 4} 1\right)}{5} = \frac{4 \cdot (4 + 1)}{5} = 4.\end{aligned}$$

(c) $\lim_{x\to 5} \frac{\left[\left[\frac{x}{2}\right]\right]}{|x+5|}$. *[Hint: $[[\bullet]]$ is the greatest integer function.]*

For every x is near 5 we have $\frac{x}{2}$ is near 2.5 and $[[\frac{x}{2}]] = 2$, the nearest integer less than $\frac{x}{2}$. Also, for the same x near 5, the sum $x+5$ is near 10, which is positive. Thus for these x, $|x+5| = x+5$. Since the denominator limit

$$\lim_{x\to 5}(x+5) = \lim_{x\to 5} x + \lim_{x\to 5} 5 = 5+5 = 10$$

is nonzero we may apply the quotient rule

$$\lim_{x\to 5} \frac{\left[\left[\frac{x}{2}\right]\right]}{|x+5|} = \lim_{x\to 5} \frac{2}{x+5} = \frac{\lim_{x\to 5} 2}{\lim_{x\to 5}(x+5)} = \frac{2}{10} = \frac{1}{5}.$$

3. *Find the limit if it exists and explain. Assume that the real functions $f(x)$ and $g(x)$ have limits*

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M.$$

(a) $\lim_{x\to a} (2g(x)+3)^4$

One uses the power law, the sum law and the constant multiple law

$$\begin{aligned}
\lim_{x\to a} (2g(x)+3)^4 &= \left(\lim_{x\to a} (2g(x)+3)\right)^4 \\
&= \left(\lim_{x\to a} 2g(x) + \lim_{x\to a} 3\right)^4 \\
&= \left(2\lim_{x\to a} g(x) + \lim_{x\to a} 3\right)^4 \\
&= (2L+3)^4
\end{aligned}$$

(b) $\lim_{x\to\infty} \frac{g(x)}{f(x)+g(x)}$

The denominator limit is by the sum rule

$$\lim_{x\to\infty} (f(x)+g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x) = L+M.$$

If we assume that $L+M \neq 0$ we may apply the quotient rule.

$$\lim_{x\to\infty} \frac{g(x)}{f(x)+g(x)} = \frac{\lim_{x\to\infty} g(x)}{\lim_{x\to\infty}(f(x)+g(x))} = \frac{L}{L+M}$$

(c) $\lim_{x\to a} \sqrt{f(x)} \cdot \sqrt[3]{g(x)}$.

The square root limit law requires that we assume that $L > 0$. Then

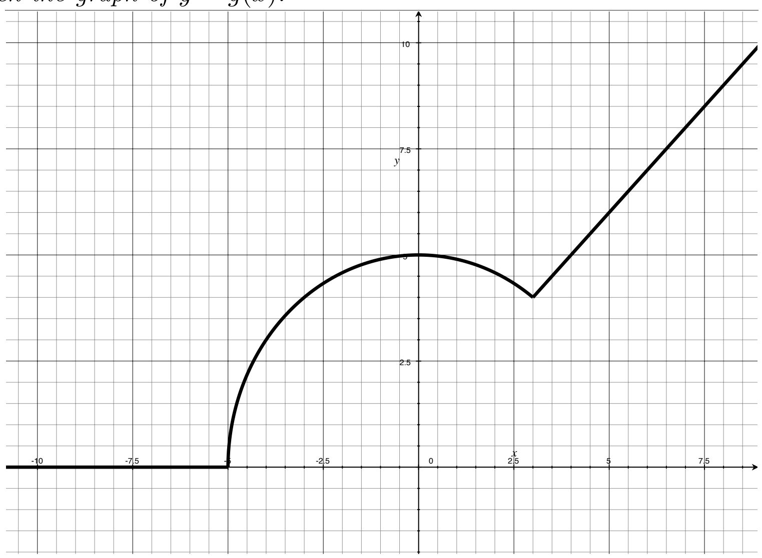
$$\lim_{x\to a} \sqrt{f(x)} = \sqrt{\lim_{x\to a} f(x)} = \sqrt{L}.$$

By the product and cube root laws,

$$\lim_{x\to a} \sqrt{f(x)} \cdot \sqrt[3]{g(x)} = \left(\lim_{x\to a} \sqrt{f(x)}\right)\left(\lim_{x\to a} \sqrt[3]{g(x)}\right) = \sqrt{L}\sqrt[3]{\lim_{x\to a} g(x)} = \sqrt{L}\cdot\sqrt[3]{M}.$$

4. *Let the function $g(x)$ be defined piecewise by* $g(x) = \begin{cases} 0, & \text{if } x < -5; \\ \sqrt{25 - x^2}, & \text{if } -1 \le x < 3; \\ x + 1, & \text{if } 3 \le x. \end{cases}$

(a) *Sketch the graph of $y = g(x)$.*



(b) *Find the limit if it exists and explain:* $\lim_{x \to 3} g(x)$

The limit exists if both left and right limits exist at $x = 3$ and they are equal. Using the left definition, by the difference and square laws,

$$\lim_{x \to 3-} (25 - x^2) = \lim_{x \to 3-} 25 - \lim_{x \to 3-} x^2 = 25 - \left(\lim_{x \to 3-} x \right)^2 = 25 - 3^2 = 16$$

is positive. Hence we may apply the square root law

$$\lim_{x \to 3-} g(x) = \lim_{x \to 3-} \sqrt{25 - x^2} = \sqrt{\lim_{x \to 3-} (25 - x^2)} = \sqrt{16} = 4.$$

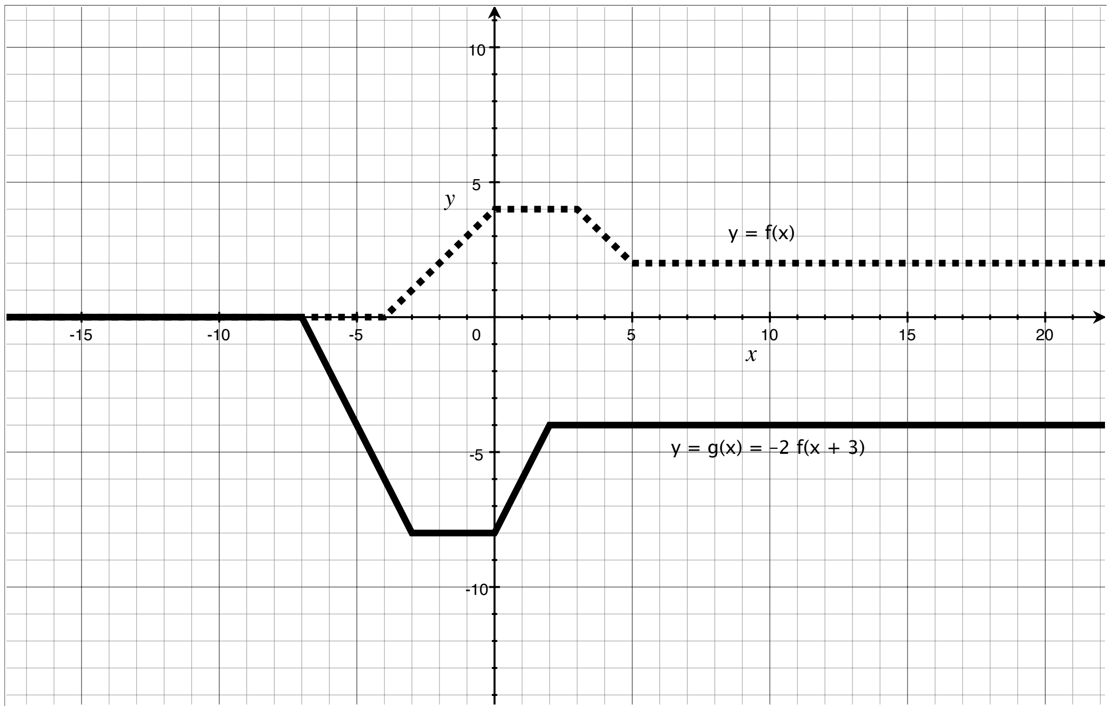
Using the right definition, the right limit follows from the sum law

$$\lim_{x \to 3+} g(x) = \lim_{x \to 3+} (x + 1) = \lim_{x \to 3+} x + \lim_{x \to 3+} 1 = 3 + 1 = 4.$$

Since both left and right limits exist and are equal, the limit exists and equals 4.

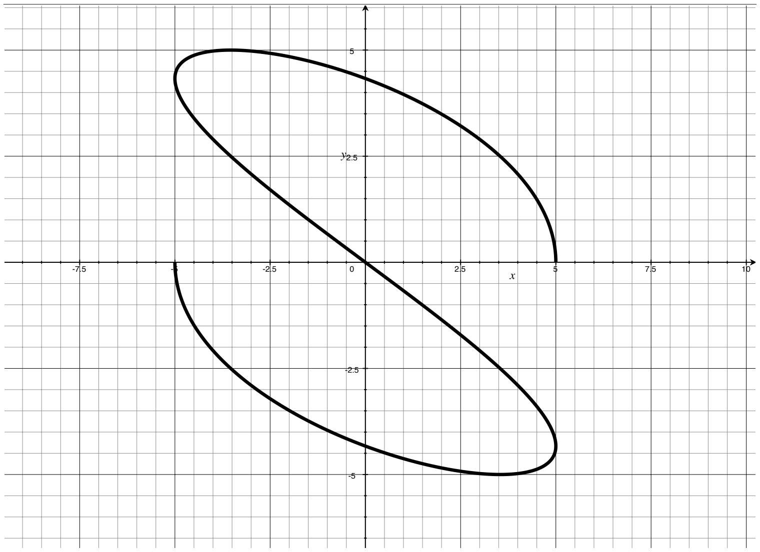
5. (a) *Consider the function $f(x)$ depicted in the graph. Draw the graph of the transformed function $g(x)$ on the same graph, where*

$$g(x) = -2f(x-1)$$



(b) *Which of the following parametric curves is depicted in the diagram? Why?*

$$(A)\begin{cases} x = 5\cos 3t \\ y = 5\sin 2t \\ 0 \le t \le 2\pi \end{cases} \qquad (B)\begin{cases} x = 5\cos 3t \\ y = 5\sin 2t \\ 0 \le t \le \pi \end{cases} \qquad (C)\begin{cases} x = 5\cos 2t \\ y = 5\sin 3t \\ 0 \le t \le \pi \end{cases}$$



The parametric curve giving this figure is (B).

In (A), because $0 \le t \le 2\pi$, both $x(t) = 5\cos 3t$ and $y(t) = 5\sin 2t$ run through respectively three and two complete cycles and the curve returns to their starting point at $t = 2\pi$. Thus the figure is a complete Lissajous Figure which continues whithout start and end.

In (C), as t runs through $0 \le t \le \pi$, the $x(t) = 5\cos 2t$ runs through one cycle from 5 to -5 to 5. This is not the case in the diagram, where x runs from 5 to -5 to 5 to -5.

For comparison, here are the figures (A) and (C).