Math 1310-1, Fall 2022

FINAL EXAM

To receive partial credit you must justify your work (intermediate steps). Write clearly and highlight your final answer. If you run out of room use another page and clearly indicate where. No textbooks, student-made formula sheets, calculators, computers, cell phones or mobile devices are allowed.
By signing below you acknowledged having read the above instructions and that you will abide by the University Honor Code.

NAME: ____________________

Unid: ____________________

SIGNATURE: ____________________

Common geometric and trigonometric Identities

Area of triangle $bh/2$
Area of circle πr^2
Circumference of circle: $2\pi r$
Surface of sphere $4\pi r^2$
Volume of sphere $\frac{4}{3}\pi r^3$
Volume of cone $\frac{1}{3}\pi r^2 h$

Derivative identities

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

Some integral identities

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$

Pythagorean Identities

$$\cos^2(x) + \sin^2(x) = 1$$
$$\tan^2(x) + 1 = \sec^2(x)$$

Double Angle Identities:

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$
$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

Trig Functions at Certain Angles.

Trig. Function	Angle in Radians	Value
cos	0	1
cos	$\frac{\pi}{6}$	$\sqrt{3}/2$
cos	$\frac{\pi}{4}$	$1/\sqrt{2}$
cos	$\frac{\pi}{3}$	$1/2$
cos	$\frac{\pi}{2}$	0
sin	0	0
sin	$\frac{\pi}{6}$	$1/2$
sin	$\frac{\pi}{4}$	$1/\sqrt{2}$
sin	$\frac{\pi}{3}$	$\sqrt{3}/2$
sin	$\frac{\pi}{2}$	1
tan	0	0
tan	$\frac{\pi}{6}$	$1/\sqrt{3}$
tan	$\frac{\pi}{4}$	1
tan	$\frac{\pi}{3}$	$\sqrt{3}$
tan	$\frac{\pi}{2}$	∞

(25 pts) 1. Consider the function $f(x) = \frac{1}{x+3}$. Use the limit-based-definition of the derivative to find the tangent line $y = mx + b$ to the curve at $x = -1$.

Solution:

$$\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h+3} - \frac{1}{x+3}}{h} \\
= \lim_{h\to 0} \frac{\frac{x+3}{x+h+3} - \frac{x+h+3}{x+3}}{h} \\
= \lim_{h\to 0} \lim_{h\to 0} \frac{\frac{-h}{(x+h+3)(x+3)}}{h} \\
= \lim_{h\to 0} \lim_{h\to 0} -\frac{1}{(x+h+3)(x+3)} \\
= -\frac{1}{(x+3)^2} \\
f'(-1) = -\frac{1}{4} \\
y = -\frac{1}{4}(x+1) + \frac{1}{2}.
\end{aligned}$$

(25 pts) 2. Compute the following limts:

(a) $\lim_{x\to 0} \frac{\sin(x)}{x}$

(b) $\lim_{x\to\infty} (x)^{\frac{1}{x}}$

Solution:

$$
\begin{aligned}
(a) \lim_{x\to 0} \frac{\sin(x)}{x} &= \lim_{x\to 0} \frac{\cos(x)}{1} = 1 \\
(b) \ln\left(\lim_{x\to\infty} (x)^{\frac{1}{x}}\right) \\
&= \lim_{x\to\infty} \frac{\ln(x)}{x} \\
&= \lim_{x\to\infty} \frac{1}{x} = 0 \\
\implies \lim_{x\to\infty} (x)^{\frac{1}{x}} &= e^0 = 1.
\end{aligned}
$$

(40 pts) 3. Compute the following derivatives using the appropriate derivative rules (do not use the limit-based differentiation procedure). Clearly delineate your work for each subpart.

(a) $\frac{d}{dx}[x^2 + x^{\frac{2}{3}} + 3]$.

(b) $\frac{d}{dx}[x \ln(x)]$.

(c) $\frac{d}{dx}[\ln(\sin^{-1}(x))]$. Note -1 is not an exponent.

(d) $\frac{d}{dx}\left(\frac{xe^x}{2+\sqrt{x}}\right)$.

(25 pts) 4. A perfectly spherical balloon is inflating at a rate of 2 m^3 per minute (i.e. $V'(t) = 2$). At what rate is the diameter increasing $d'(t)$ when the diameter is 4 meters: $d(t) = 4$? Hint: find an expression for $V(t)$ in terms of $d(t)$ using basic geometry facts.

Solution:

$$
\begin{aligned}
V(t) &= \frac{4}{3}\pi\left(\frac{d(t)}{2}\right)^3 \\
V' &= 4\pi\left(\frac{d(t)}{2}\right)^2 \frac{d'(t)}{2} \\
4 &= 4\pi\left(2\right)^2 d' \\
d' &= \frac{1}{4\pi}.
\end{aligned}
$$

(25 pts) 5. You are designing a half-cylinder trough that will be designed to hold a volume of water for animals to feed from. The trough must hold 1m^3 of water in the half-cylinder and has an open top plane $ABCD$ (so animals can access the water). To minimize cost of manufacture, you must choose the diameter $d = AD$ (same as BC) and the width $w = AB$ (same as DC) so that the surface area of the tank is minimum subject to the constraint that the trough holds 1m^3 of volume.

In the optimized tank, is the diameter larger than the width, or vice versa, or are they equal?

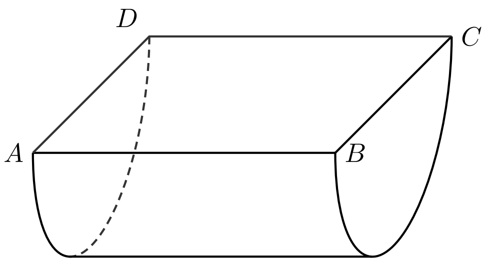

Figure 3

Solution:

$$
\begin{aligned}
V &= \tfrac{1}{2}\pi r^2 w = 1 \\
w = \frac{2}{\pi r^2} A &= \pi r^2 + \pi r w \\
A(r) &= \pi r^2 + \pi r \frac{2}{\pi r^2} \\
&= \pi r^2 + \frac{2}{r} \\
A' &= 2\pi r - \frac{2}{r^2} = 0 \\
\pi r^3 &= 1 \\
r &= \pi^{-1/3}. \\
w &= 2\pi^{2/3}/\pi = 2r = d.
\end{aligned}
$$

(25 pts) 6. Consider the Riemann sum

$$\lim_{n\to\infty}\sum_{i=1}^{n}(\tfrac{2i}{n})e^{-4\frac{i}{n}}$$

that represents a specific integral with a specific limits of integration $\int_?^? ?dx$.

(a) Suppose there are N Δx that are evenly spaced across the interval $[a,b]$. Determine an interval of integration $[a,b]$ and the Δx for any positive integer n.

(b) Using your result in (a), determine the integral represented by the sum above: $\int_?^? ?dx$?

Solution: There are many integrals this could equal but this is but one:

$$\begin{aligned}\Delta x &= 2/N\\ x &\in [0,2]\\ x_n &= 2n/N\\ f(x) &= e^{-2x}\\ &\int_0^2 e^{-2x}dx\end{aligned}$$

(25 pts) 7. Consider the function

$$F(x) = \int_0^x \frac{s^2 - 1}{s^2 + e^s} ds.$$

(a) Determine the interval(s) where F is an increasing function, and a decreasing function.

(b) Find any local maximum points.

(c) Find any local minimum points.

Hint: use the FTC, but do not try to find an antiderivative of the integrand.

Solution:

$$\begin{aligned} \frac{dF}{dx} &= \frac{s^2 - 1}{s^2 + e^s} \\ &= \frac{(s+1)(s-1)}{s^2 + e^s} = 0 \\ &\implies s = -1, 1 \end{aligned}$$

increasing: $(\infty, -1)$, and $(1, \infty)$

decreasing: $(-1, 1)$.

max: $x = -1$

min $x = 1$

(40 pts) 8. Compute the following integrals (a-c)—one more is on the next page.

(a) $\int_1^3 x + \frac{1}{x} + x^{\frac{1}{3}}\ dx$

(b) $\int xe^{-x} dx$

(c) $\int_{-2}^{3} \frac{1}{(x+1)(x-2)}\ dx$

(25 pts) 9. Suppose the function

$$v(t) = \frac{t}{1+t^2}$$

represents the velocity (measured in meters/second) of an object at time $t \geq 0$. Compute the distance that the object travels if starting at time $t = 0$.

(a) Setup the integral that computes the distance traveled—don't forget the dummy variable.

(b) How far does the object go at time $t = 100$? Leave your answer as an algebraic expression.

(c) How far as $t \to \infty$?

Solution:

$$\begin{aligned} A &= \int_0^{100} \frac{t}{1+t^2}\, dt \\ &= [\frac{1}{2} \ln(1+t^2)]_0^{100} \end{aligned}$$

(25 pts) 10. Consider the solid of revolution constructed by revolving an area around the x-axis. The area in the x-y-plane is enclosed by $y = 1 - \frac{1}{2}x$ and the horizontal line $y = 0$ (the y-axis), and $x = 0$ (the x-axis).

(a) Draw and describe the solid of revolution in simple terms (it is a simple solid).

(b) Set up the integral that computes the volume of the solid of revolution using the method of disks/washers, but do not compute the integral.

(c) Compute the integral and confirm it agrees with the volume defined via the standard formula.

Solution: It's a cone.

$$\int_0^2 \pi(1 - \tfrac{1}{2}x)^2 dx = \frac{1}{3}\pi 2$$