Math 1310-1, Fall 2021

FINAL EXAM

To receive partial credit you must justify your work (intermediate steps). Write clearly and highlight your final answer. If you run out of room use another page and clearly indicate where.

No textbooks, student-made formula sheets, calculators, computers, cell phones or mobile devices are allowed.

By signing below you acknowledged having read the above instructions and that you will abide by the University Honor Code.

NAME: ________________________________

Unid: ________________________________

SIGNATURE: ________________________________

THINK WELL

Trigonometric Identities

Derivative identities

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

Some integral identities

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + C$$

Pythagorean Identities

$$\cos^2(x) + \sin^2(x) = 1$$
$$\tan^2(x) + 1 = \sec^2(x)$$

Double Angle Identities:

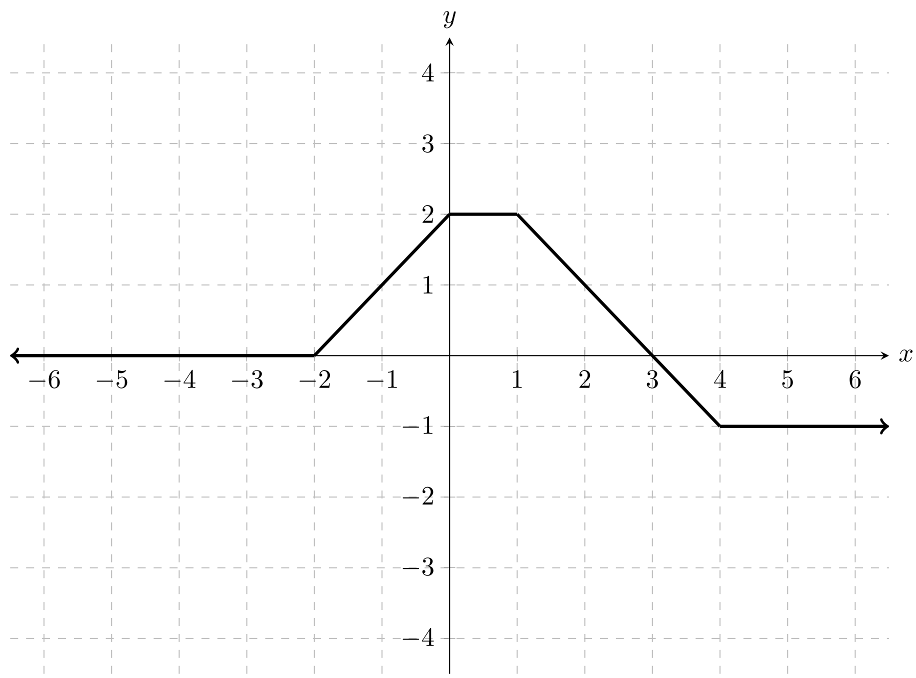
$$\sin\big(A+B\big) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin\big(A-B\big) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos\big(A+B\big) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos\big(A-B\big) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\tan\big(A+B\big) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$
$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

Trig Functions at Certain Angles.

Trig. Function	Angle in Radians	Value
cos	0	1
cos	$\frac{\pi}{6}$	$\sqrt{3}/2$
cos	$\frac{\pi}{4}$	$1/\sqrt{2}$
cos	$\frac{\pi}{3}$	$1/2$
cos	$\frac{\pi}{2}$	0
sin	0	0
sin	$\frac{\pi}{6}$	$1/2$
sin	$\frac{\pi}{4}$	$1/\sqrt{2}$
sin	$\frac{\pi}{3}$	$\sqrt{3}/2$
sin	$\frac{\pi}{2}$	1
tan	0	0
tan	$\frac{\pi}{6}$	$1/\sqrt{3}$
tan	$\frac{\pi}{4}$	1
tan	$\frac{\pi}{3}$	$\sqrt{3}$
tan	$\frac{\pi}{2}$	∞

(20 pts) 1. Consider the function $f(x)$ depicted in the graph. Draw a graph of the transformed function on the same axes.

$$-2f(x-2).$$



(20 pts) 2. Consider the function $f(x) = \frac{x+2}{x+1}$.

(a) Find the secant line $y = mx + b$ through the points $x = 0$ and $x = 2$.

(b) Use the limit-based-definition of the derivative to find the line $y = mx + b$ tangent to the curve at $x = 0$.

Solution:

$$
\begin{aligned}
(a)\ y &= mx+b \\
f(2) = 2,\ f(0) &= 4/3 \\
m = \tfrac{6/3-4/3}{2} &= 1/3. \\
y &= (1/3)x+2. \\
(b)\ f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{x+h+2}{x+h+1}-\frac{x+2}{x+1}}{h} \\
= \lim_{h\to 0} &\frac{(x+h+2)(x+1)-(x+2)(x+h+1)}{h(x+h+1)(x+1)} \\
= \lim_{h\to 0}\lim_{h\to 0}\frac{-h}{h(x+h+1)(x+1)} &= \frac{-1}{(x+1)(x+1)} \\
y &= x+2.
\end{aligned}
$$

(40 pts) 3. Compute the following derivatives using the appropriate derivative rules (do not use the limit-based differentiation procedure). Clearly delineate your work for each subpart.

(a) $\frac{d}{dx}[e^{-x^2}]$.

(b) $\frac{d}{dx}[x^2\cos(x)]$.

(c) $\frac{d}{dx}[\ln(\tan^{-1}(x))]$.

(d) $\frac{d}{dx}\left(\frac{xe^x}{2+\sqrt{x}}\right)$.

(20 pts) 4. Consider the equation for an ellipse

$$4x^2 + \frac{y^2}{9} = 5.$$

(a) Find the slope m of the tangent line y' to the ellipse at the point $(x, y) = (1, 3)$.

(b) Specify the equation for the tangent line $y = mx + b$ at the point in (a).

Solution:

$$\begin{aligned} 8x + \frac{2}{9}yy' &= 5 \\ y' &= 9\frac{5 - 8x}{2y} \\ y'(1) = 3\frac{5 - 8}{2} &= -9/2. \\ y &= -\frac{9}{2}(x - 1) + 3. \end{aligned}$$

(20 pts) 5. Consider the function $f(x)$ defined on the positive real line $[0, \infty)$

$$f(x) = \frac{x}{x^2+1}$$

(a) Find the interval(s) on the positive real line where $f(x)$ is increasing and decreasing.

(b) Find the x-value(s), if any, where $f(x)$ has zero slope.

(c) Find the point(s) of inflection, if any, for $f(x)$ and regions of the positive real line where the function is concave up and down.

(d) Using the fact that $f(0) = 0$, and the results you found from above, sketch a graph of $f(x)$, indicating the intervals and points in (a)-(c), correctly represent increasing and decreasing regions, and concavity.

Solution: For this question, instructors should choose an explicit value for $a > 0$

$$\begin{aligned}
f'(x) &= \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \\
f''(x) &= \frac{-2x(x^2+1)^2 - 4x(1-x^2)(x^2+1)}{(x^2+1)^4} \\
&= \frac{-2x(x^2+1)[(x^2+1)+2(1-x^2)]}{(x^2+1)^4} \\
&= \frac{-2x(x^2+1)[-x^2+3]}{(x^2+1)^4} \\
f'(x) = 0 &\implies 1-x^2 = 0, \quad x = 1. \\
\textit{increasing}: &\quad [0,1) \\
\textit{decreasing}: &\quad (1,\infty) \\
f''(x) = 0 &\implies x^2 = 3 \\
x = \sqrt{3} \textit{concave down}: &\quad [0,\sqrt{3}) \\
\textit{concave up}: &\quad (\sqrt{3},\infty)
\end{aligned}$$

(30 pts) 6. Evaluate the following limits. Use limit laws, and identify the indeterminate form in order to use L'Hospital's rule if necessary.

(a) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right)$.

(b) $\lim_{x\to -1}\frac{\ln(x+2)}{x+1}$.

(c) $\lim_{x\to 0}[\cos(x)]^{\frac{1}{x}}$.

Solution: It's a $\infty - \infty$ indeterminate form

$$\begin{aligned}
&\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right)\\
&=\lim_{x\to 0^+}\left(\frac{\sin(x)-x}{x\sin(x)}\right)\\
&=\lim_{x\to 0^+}\left(\frac{\cos(x)-1}{\sin(x)+x\cos(x)}\right)\\
&=\lim_{x\to 0^+}\left(\frac{-\sin(x)}{2\cos(x)-x\sin(x)}\right)=0
\end{aligned}$$

(20 pts) 7. Consider the Riemann sum

$$\lim_{N\to\infty}\sum_{n=1}^{N}(\tfrac{2n}{N})e^{-4\frac{n}{N}}$$

that represents a specific integral with a specific limits of integration $\int_?^? ?dx$.

(a) Suppose there are N Δx that are evenly spaced across the interval $[a,b]$. Determine the interval of integration $[a,b]$ and the Δx for any positive integer N.

(b) Using your result in (a), determine the integral represented by the sum above: $\int_?^? ?dx$?

Solution: There are many integrals this could equal but this is but one:

$$\begin{aligned}
\Delta x &= 2/N\\
x &\in [0,2]\\
x_n &= 2n/N\\
f(x) &= e^{-2x}\\
\int_0^2 & e^{-2x}dx
\end{aligned}$$

(20 pts) 8. Compute the derivative $\frac{dF}{dx}$ given

$$F(x) = \int_2^x \frac{e^{-s}}{s^2 + \sin(s)} ds.$$

Solution:

$$\frac{dF}{dx} = \frac{e^{-x}}{x^2 + \sin(x)}$$

(20 pts) 9. Suppose the function

$$v(t) = te^{-2t},$$

represents the velocity (measured in meters/second) of an object at time $t \geq 0$. Compute the distance that the object travels starting at time $t = 0$ and ending at time $t = 100$ by computing the correct integral. Leave your answer as an algebraic expression.

Solution:

$$A = \int_0^{100} te^{-at}\ dt$$

$$= [\frac{te^{-at}}{a} + \frac{e^{-at}}{a^2}]_0^{100}$$

(40 pts) 10. Compute the following four integrals (a-d)—two more are on the next page.

(a) $\int_1^3 x + \frac{1}{x} + \sqrt{x}\ dx$

(b) $\int_0^1 5x\sqrt{3x^2+2}\ dx$

(c) $\int x\cos(x)dx$

(d) $\int \frac{x}{(x+1)(x-2)}\ dx$

(20 pts) 11. Consider the solid of revolution constructed by revolving an area around the vertical y-axis. The area in the x-y-plane is defined as the area enclosed by $y = -x^3 + x$ and the horizontal line $y = 0$, for $x \geq 0$ (the x-axis). Set up the integral that computes the volume of the solid of revolution using the method of disks/washers, but do not compute the integral.

Solution:

$$\int_0^1 2\pi(x - x^3)dx.$$