Math 1310

Midterm Exam 2

1. (20 points) Compute the derivative of $f(x) = ax - bx^2$ using the limit-based definition.

> **Solution:**
>
> $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{a(x+h) - b(x+h)^2 - ax + bx^2}{h} \\ &= \lim_{h\to 0} \frac{ax + ah - bx^2 - b2xh - bh^2 - ax + bx^2}{h} \\ &= \lim_{h\to 0} \frac{ah - b2xh - bh^2}{h} = \lim_{h\to 0} \frac{h}{h} \frac{a - b2x - bh}{1} \\ &= a - b2x. \end{aligned}$$

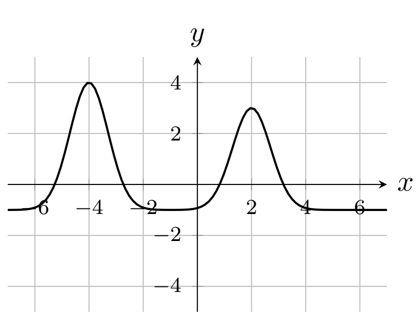
2. (20 points) Determine if the derivative of

$$f(x) = \begin{cases} 2x - 1, & x < 2 \\ x^2 - 1, & x \geq 2 \end{cases}$$

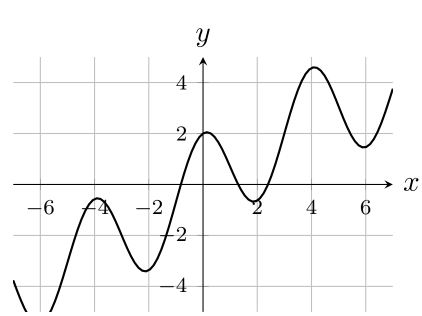
exists at $x = 2$.

> **Solution:**
>
> $$f(2^-) = 3, \quad f(2^+) = 3$$
> $$f'(x) = \begin{cases} 2, & x < 2 \\ 2x, & x \geq 2 \end{cases}$$
> $$f'(2^-) = 2 \neq f'(2^+) = 4.$$
>
> The left and right limits of the derivative do not agree so the limit does not exists at $x = 2$.

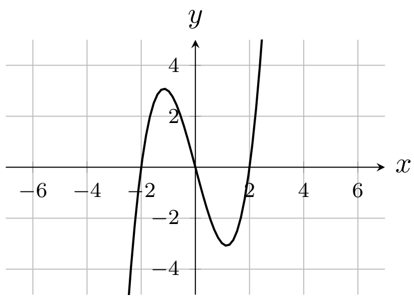
3. (30 points) Match the graphs of the four functions ((A)–(D)) given with the graphs of their derivatives ((a)–(d)). Explain your answer. For example, identify some distinct feature(s) of the graphs.

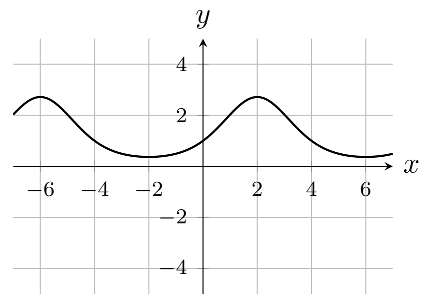


(A)



(B)


y
x
4
2
−2
−4
−6 −4 −2 2 4 6
(C)


y
x
4
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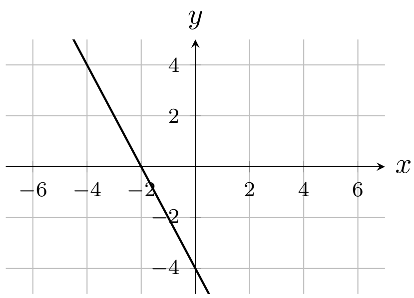
−6 −4 −2 2 4 6
(E)


y
x
4
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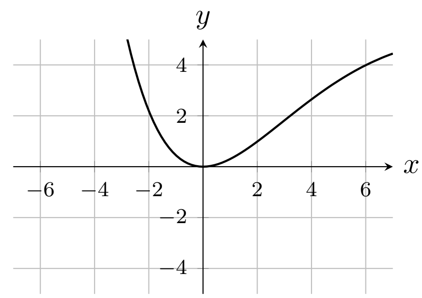
−6 −4 −2 2 4 6
(D)


y
x
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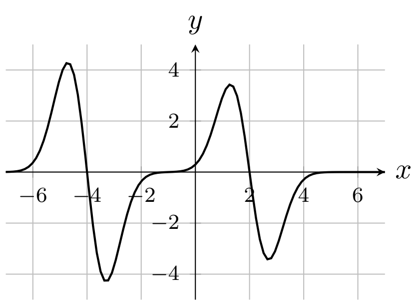
−6 −4 −2 2 4 6
(F)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x
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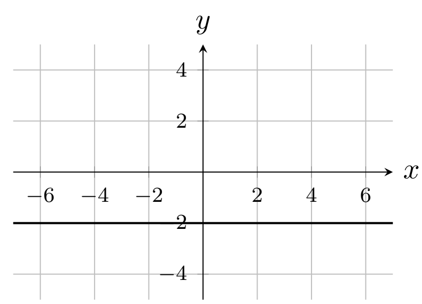
(a)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x
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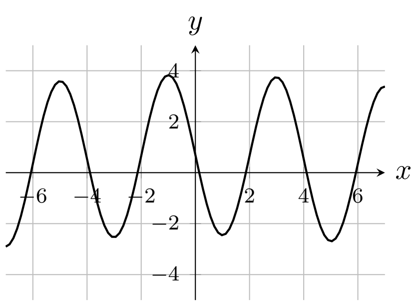
(d)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x
4
2
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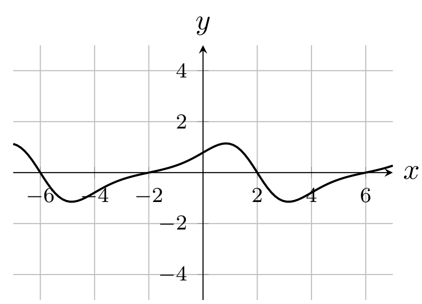
−4
−6 −4 −2 2 4 6
(b)


y
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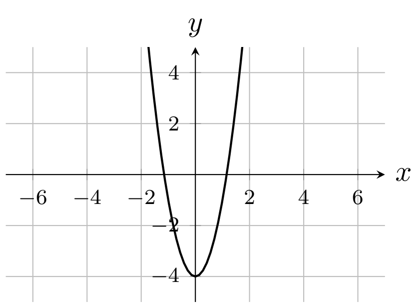
(e)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x
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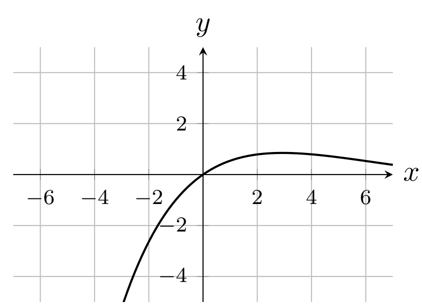
−6 −4 −2 2 4 6
(c)


y
x
4
2
−2
−4
−6 −4 −2 2 4 6
(f)

Solution:

4. (50 points) Compute the following derivatives using the derivative rules

(a) $\frac{d}{dx}\left(4x^4 - 2x^3 + 1\right)$

(b) $\frac{d}{dx}\big(\arctan(2x)\ln(x)\big)$

(c) $\frac{d}{dx}\left(\frac{e^{4x}}{\arccos(x)}\right)$

(d) $\frac{d}{dx}\ln\left(\frac{1}{x}\right)$

(e) $\frac{d}{dx}\cos\left(x^2 + \sin(x)\right)$

Solution:

(a) $\frac{d}{dx}\left(4x^4 - 2x^3 + 1\right) = 16x^3 - 6x^2$

(b) $\frac{d}{dx}\big(\arctan(2x)\ln(x)\big) = \frac{1}{x}\arctan(2x) + 2\ln(x)\frac{1}{1+4x^2}$

(c)

$$\frac{d}{dx}\left(\frac{e^{4x}}{\arccos(x)}\right) = \frac{4\arccos(x)e^{4x} - e^{4x}\frac{-1}{\sqrt{1-x^2}}}{\big[\arccos(x)\big]^2}$$
$$= e^{4x}\frac{4\arccos(x) + (1-x^2)^{-\frac{1}{2}}}{\big[\arccos(x)\big]^2}.$$

(d)

$$\frac{d}{dx}\ln\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}}\frac{d}{dx}\frac{1}{x}$$
$$= -x\cdot\frac{1}{x^2} = -\frac{1}{x}$$

(e)

$$\frac{d}{dx}\cos\left(x^2 + \sin(x)\right) = -\sin(x^2 + \sin(x))\frac{d}{dx}(x^2 + \sin(x))$$
$$= \sin\left(x^2 + \sin(x)\right)\left(-(2x + \cos(x))\right)$$

5. (20 points) Consider the curve defined by the equation $\sqrt{xy+1} = x - 1$

(a) Find an expression for $\frac{dy}{dx}$

(b) Demonstrate that the point $(6, 4)$ lies on this curve. (pick $n = 1, 5, 9, ..$)

(c) Find the slope dy/dx of the tangent line to the curve at the point in (b).

Solution:

(a)

$$\frac{1}{2\sqrt{xy+1}}(y + xy') = 1$$

Rearranging,

$$y' = \frac{2\sqrt{xy+1} - y}{x}$$

(b) We substitute $x = 6$, $y = 4$ into the equation, and find that it works. (c)

$$y' = \frac{2 \times 5 - 4}{6} = 1.$$

6. (20 points) Use the linear approximation of $f(x) = \sqrt[3]{x}$ to estimate the value of $\sqrt[3]{1001}$. Leave your answer as an expression involving sums and fractions of whole numbers. Hint: a easy-to-compute nearby point is $\sqrt[3]{1000} = 10$. Show your work.

Solution: We know that $f(1000) = \sqrt[3]{1000} = 10$ because $10^3 = 1000$, is a known point nearby $x = 1001$. Also,

$$f'(x) = (x^{1/3})' = \tfrac{1}{3}x^{-2/3},$$

so

$$f'(1000) = \tfrac{1}{3}(100)^{-2/3} = \tfrac{1}{3}(10)^{-2} = \tfrac{1}{300}$$

so,

$$f(1001) \approx 10 + \tfrac{1}{300}(1001 - 1000) = 10 - \tfrac{1}{300}$$

note

$$\tfrac{1}{300} = 0.3333333 \times 0.001 = 0.00333333.$$

so,

$$f(1001) \approx 10 - \tfrac{1}{300} = 10.00333333.$$

The ten-decimal-point value is $\sqrt[3]{1001} = 10.0033322228$. The linear approximation is accurate to the first five decimal places.