Math 1210 § 4. **First Midterm Exam** Name: Practice Problems
Treibergs September 4, 2015

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**Practice Questions.** The upcoming exam questions may be similar to these, but not identical. Understand the concepts so that you handle minor variations.

1. Determine whether the indicated limits exist. If they do, find them. Be sure to show all work.

   (a) $\lim\limits_{x\to 3} \dfrac{x^2 - x - 6}{x^2 - 2x - 3}$

   (b) $\lim\limits_{x\to 4} \dfrac{x^2 + x}{x^2 + 2x + 3}$

   (c) Recall that $[[x]]$ denotes the greatest integer part of $x$.
   $\lim\limits_{x\to 5} [[x^2 + 1]]$

2. Sketch the graph of

$$f(x) = \begin{cases} x+1, & \text{if } x < 0; \\ x, & \text{if } 0 \le x < 1; \\ \dfrac{1}{x}, & \text{if } 1 \le x. \end{cases}$$

   Find each of the following limits or state that it does not exist. Explain.

   (a) $\lim\limits_{x\to -\infty} f(x)$.

   (b) $\lim\limits_{x\to 0} f(x)$.

   (c) $\lim\limits_{x\to 0-} f(x)$.

   (d) $\lim\limits_{x\to 1} f(x)$.

   (e) $\lim\limits_{x\to \infty} f(x)$.

3. Determine whether the indicated limits exist. If they do, find them. Be sure to show all work.

   (a) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+3} - \sqrt{3}}{x}$

   (b) $\lim\limits_{x\to 4} \dfrac{x-4}{|x-4|}$

   (c) $\lim\limits_{x\to 0} x^2 \left[\left[\dfrac{1}{x^2}\right]\right]$

4. Recall the Main Limit Theorem

   **Theorem A. Main Limit Theorem.** Let $n$ be a positive integer, $k$ be a constant and $f$ and $g$ functions that have a limit at $c$. Then

   1. $\lim\limits_{x\to c} k = k$;
   2. $\lim\limits_{x\to c} x = c$;
   3. $\lim\limits_{x\to c} kf(x) = k \lim\limits_{x\to c} f(x)$;
   4. $\lim\limits_{x\to c} \big(f(x) + g(x)\big) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$;

5. $\lim_{x\to c}\big(f(x) - g(x)\big) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$;
6. $\lim_{x\to c}\big(f(x)\cdot g(x)\big) = \Big(\lim_{x\to c} f(x)\Big)\cdot\Big(\lim_{x\to c} g(x)\Big)$;
7. $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$;
8. $\lim_{x\to c}(f(x))^n = \Big(\lim_{x\to c} f(x)\Big)^n$;
9. $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$, provided $\lim_{x\to c} f(x) > 0$ when $n$ is even.

Use Theorem A to find the limit. Justify each step by appealing to a numbered statement.

$\lim_{x\to 2}\sqrt{-2w^3 + 7w^2}$.

5. Determine whether the indicated limits exist. If they do, find them. Be sure to show all work.
   (a) $\lim_{x\to 0}\dfrac{\tan 2x}{\tan 3x}$
   (b) $\lim_{\theta\to 0}\dfrac{\theta\cot\theta}{\sec\theta}$
   (c) $\lim_{t\to 0}\dfrac{1-\cos t}{\sin^2 t}$

6. Determine whether the indicated limit exists. If it does, find it. Be sure to show all work.

   $\lim_{x\to 0}(-1)^{[\![\frac{1}{x}]\!]}\sin(x)$

7. Determine whether the indicated limits exist. if they do, find them. Be sure to show all work.
   (a) $\lim_{x\to\infty}\dfrac{(2x+3)^4}{x^4+1}$
   (b) $\lim_{x\to 1}\dfrac{x+2}{x^3-1}$
   (c) $\lim_{x\to 2}\dfrac{x^2+3x}{(x-2)^4}$

8. Let $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$

   Determine whether $f(x)$ is continuous on the interval $[0, 2]$.

9. A particle moves along a coordinate line and $s$, its directed distance in centimeters from the origin after $t$ seconds is given by $s = f(t) = \sqrt{7t+1}$. Using just the limit definition, find the instantaneous velocity of the particle after 5 seconds.

10. Using just the limit definition, for each $x$, determine whether $F(x)$ is differentiable at $x$, and if it is, find $F'(x)$, where $F(x) = \dfrac{1}{x+2}$.

**Solutions.**

1. Determine whether the indicated limits exist. If they do, find them. Be sure to show all work.

   (a)

$$\lim_{x\to 3}\frac{x^2-x-6}{x^2-2x-3}=\lim_{x\to 3}\frac{(x-3)(x+2)}{(x-3)(x+1)}=\lim_{x\to 3}\frac{(x+2)}{(x+1)}=\frac{3+2}{3+1}=\boxed{\frac{5}{4}}.$$

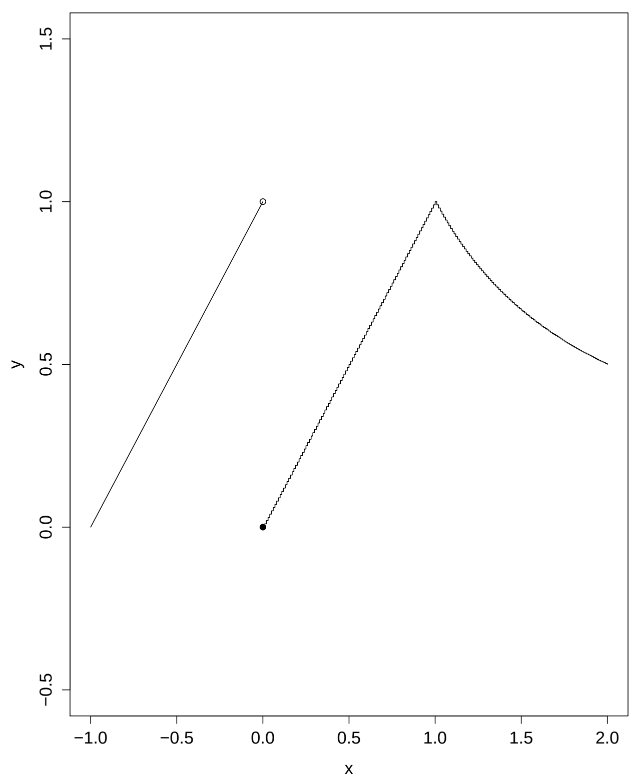
   (b)

$$\lim_{x\to 4}\frac{x^2+x}{x^2+2x+3}=\frac{4^2+4}{4^2+2\cdot 4+3}=\boxed{\frac{20}{27}}$$

   (c) $\lim_{x\to 5}[[x^2+1]]$ Note that $[[y]]$ jumps whenever $y$ is an integer. Also we see that $\lim_{x\to 5} x^2+1 = 5^2+1=26$. Since $x^2+1$ is an increaseing function, we see that the function $f(x)=[[x^2+1]]$ equals 26 for $x$'s close to but greater than 5, and $f(x)=24$ for $x$'s close to but less than 5. Because $f(x)$ jumps at $x=5$, the limit $\boxed{\text{does not exist}}$.

2. Sketch the graph of

$$f(x)=\begin{cases} x+1, & \text{if } x<0;\\ x, & \text{if } 0\le x<1;\\ \dfrac{1}{x}, & \text{if } 1\le x.\end{cases}$$



Find each of the following limits or state that it does not exist. Explain.

(a) $\lim_{x \to -\infty} f(x) = \boxed{-\infty}$ . $f(x)$ decreases to $-\infty$ without bound as $x \to -\infty$.

(b) $\lim_{x \to 0} f(x)$ $\boxed{\text{Does not exist.}}$. The function jumps at $x = 0$. The function approaches different values for $x > 0$ and $x < 0$ so there is no consistent limiting value.

(c) $\lim_{x \to 0-} f(x) = \boxed{1}$. The function approaches one as $x \to 0$ from the left.

(d) $\lim_{x \to 1} f(x) = \boxed{1}$. The left and right limits have the same limiting value, so the two-sided limit exists at $x = 1$.

(e) $\lim_{x \to \infty} f(x) = 0$. The quantity $1/x$ decreases to zero as $x \to \infty$.

3. Determine whether the indicated limits exist. if they do, find them. Be sure to show all work.

(a)

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{x-3} - \sqrt{3}}{x} &= \lim_{x \to 0} \frac{\left(\sqrt{x+3} - \sqrt{3}\right)\left(\sqrt{x+3} + \sqrt{3}\right)}{x\left(\sqrt{x+3} + \sqrt{3}\right)} \\
&= \lim_{x \to 0} \frac{(x+3) - 3}{x\left(\sqrt{x+3} + \sqrt{3}\right)} \\
&= \lim_{x \to 0} \frac{x}{x\left(\sqrt{x+3} + \sqrt{3}\right)} \\
&= \lim_{x \to 0} \frac{1}{\sqrt{x+3} + \sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}}
\end{aligned}$$

(b) $\lim_{x \to 4} \frac{x-4}{|x-4|}$ $\boxed{\text{Does not exist.}}$ The function $f(x) = \frac{x-4}{|x-4|}$ equals $+1$ for $x > 4$ and $-1$ for $x < 4$, thus has a jump at $x = 4$. The two sided limit does not exist because $f(x)$ has differing limits upon left and right approaches $x \to 4$.

(c) $\lim_{x \to 0} x^2 \left[\!\left[ \frac{1}{x^2} \right]\!\right] = \boxed{1}$.

This can be seen from the Squeeze Theorem. Note that as $x \to 0$ the fraction $\frac{1}{x^2}$ tends ti infinity without bound. The greatest integer part drops the fractional part of this value, thus is at most one smaller, and so satisfies the inequality

$$\frac{1}{x^2} - 1 \leq \left[\!\left[ \frac{1}{x^2} \right]\!\right] \leq \frac{1}{x^2}$$

Multiplying by $x^2$, we see that the function $f(x) = x^2 \left[\!\left[ \frac{1}{x^2} \right]\!\right]$ satisfies

$$1 - x^2 \leq f(x) \leq 1$$

so that in the limit, because the first and last expressions tend to one, the middle quantity does also by the Squeeze Theorem.

4. Recall the Main Limit Theorem

**Theorem A. Main Limit Theorem.** Let $n$ be a positive integer, $k$ be a constant and $f$ and $g$ functions that have a limit at $c$. Then

1. $\lim_{x\to c} k = k$;
2. $\lim_{x\to c} x = c$;
3. $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$;
4. $\lim_{x\to c} \big(f(x)+g(x)\big) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$;
5. $\lim_{x\to c} \big(f(x)-g(x)\big) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$;
6. $\lim_{x\to c} \big(f(x)\cdot g(x)\big) = \left(\lim_{x\to c} f(x)\right)\cdot\left(\lim_{x\to c} g(x)\right)$;
7. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$;
8. $\lim_{x\to c} (f(x))^n = \left(\lim_{x\to c} f(x)\right)^n$;
9. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$, provided $\lim_{x\to c} f(x) > 0$ when $n$ is even.

Use theorem A. to find the limit. Justify each step by appealing to a numbered statement.

$$\lim_{x\to 2} \sqrt{-2w^3+7w^2} = \sqrt{\lim_{x\to 2} -2w^3+7w^2}$$ by (9). We're taking the $n = 2$nd root, which is even, but inner limit is positive, as we shall see.

$$= \sqrt{\lim_{x\to 2} -2w^3 + \lim_{x\to 2} 7w^2}$$ by (4), the limit of a sum is the sum of a limit.

$$= \sqrt{-2\lim_{x\to 2} w^3 + 7\lim_{x\to 2} w^2}$$ by (3), a constant multiple may be pulled out.

$$= \sqrt{-2\left(\lim_{x\to 2} w\right)^3 + 7\left(\lim_{x\to 2} w\right)^2}$$ by (8), limit of power is the power of limit.

$$= \sqrt{-2\cdot 2^3 + 7\cdot 2^2} = \sqrt{12} = \boxed{2\sqrt{3}}.$$ by (2), limit of $x$ itself is $c$.

5. Determine whether the indicated limits exist. If they do, find them. Be sure to show all work.

(a)

$$\lim_{x\to 0} \frac{\tan 2x}{\tan 3x} = \lim_{x\to 0} \frac{\left(\dfrac{2}{\cos 2x}\right)\left(\dfrac{\sin 2x}{2x}\right)}{\left(\dfrac{3}{\cos 3x}\right)\left(\dfrac{\sin 3x}{3x}\right)}$$

$$= \frac{\left(\lim_{x\to 0}\dfrac{2}{\cos 2x}\right)\left(\lim_{x\to 0}\dfrac{\sin 2x}{2x}\right)}{\lim_{x\to 0}\left(\dfrac{3}{\cos 3x}\right)\left(\lim_{x\to 0}\dfrac{\sin 3x}{3x}\right)}$$

$$= \frac{\left(\lim_{y\to 0}\dfrac{2}{\cos y}\right)\left(\lim_{y\to 0}\dfrac{\sin y}{y}\right)}{\lim_{z\to 0}\left(\dfrac{3}{\cos z}\right)\left(\lim_{z\to 0}\dfrac{\sin z}{z}\right)}$$ where we let $y = 2x$ and $z = 3x$.

So

$$\lim_{x\to 0} \frac{\tan 2x}{\tan 3x} = \frac{\left(\frac{2}{1}\right)(1)}{\left(\frac{3}{1}\right)(1)} = \boxed{\frac{2}{3}}.$$

(b)

$$\begin{aligned}\lim_{\theta\to 0} \frac{\theta \cot\theta}{\sec\theta} &= \lim_{\theta\to 0} \frac{\theta\,\cos^2\theta}{\sin\theta} \\ &= \lim_{\theta\to 0} \frac{\cos^2\theta}{\left(\frac{\sin\theta}{\theta}\right)} \\ &= \frac{\left(\lim_{\theta\to 0}\cos\theta\right)^2}{\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)} = \frac{1^2}{1} = \boxed{1}.\end{aligned}$$

(c)

$$\begin{aligned}\lim_{t\to 0} \frac{1-\cos t}{\sin^2 t} &= \lim_{t\to 0} \frac{1-\cos t}{1-\cos^2 t} \\ &= \lim_{t\to 0} \frac{1-\cos t}{(1-\cos t)\,(1+\cos t)} \\ &= \lim_{t\to 0} \frac{1}{1+\cos t} \\ &= \frac{\lim_{t\to 0} 1}{\lim_{t\to 0} 1 + \lim_{t\to 0}\cos t} = \frac{1}{1+1} = \boxed{\frac{1}{2}}.\end{aligned}$$

6. Determine whether the indicated limit exists. If it does, find it. Be sure to show all work.

$$\lim_{x\to 0} (-1)^{\left[\!\left[\frac{1}{x}\right]\!\right]}\ \sin(x) = \boxed{0}.$$

This can be seen from the Squeeze Theorem. Note that as $x \to 0$ the fraction $\frac{1}{x}$ tends to plus or minus infinity without bound. The greatest integer part drops the fractional part of this value, and so also tends to plus or minus infinity. Raising $-1$ to this power oscillates between $\pm 1$. Thus we have a bound

$$\left|(-1)^{\left[\!\left[\frac{1}{x}\right]\!\right]}\right| \leq 1$$

Multiplying by $|\sin x|$, we see that

$$\left|\sin(x)\,(-1)^{\left[\!\left[\frac{1}{x}\right]\!\right]}\right| \leq |\sin x|$$

which is the same as saying

$$-|\sin x| < \sin x\,(-1)^{\left[\!\left[\frac{1}{x}\right]\!\right]} \leq |\sin x|.$$

Since the absolute value function and the sine function are both continuous, the composite $|\sin x|$ is continuous also. Its limit as $x \to 0$ is $|\sin(0)| = 0$. Because the first and last expressions tend to zero, the middle quantity does also by the Squeeze Theorem.

7. Determine whether the indicated limits exist. if they do, find them. Be sure to show all work.

(a)

$$\lim_{x\to\infty} \frac{(2x+3)^4}{x^4+1} = \lim_{x\to\infty} \frac{\left(2+\frac{3}{x}\right)^4}{1+\frac{1}{x^4}}$$
$$= \frac{\lim_{x\to\infty}\left(2+\frac{3}{x}\right)^4}{\lim_{x\to\infty}\left(1+\frac{1}{x^4}\right)}$$
$$= \frac{\left(\lim_{x\to\infty} 2+\lim_{x\to\infty}\frac{3}{x}\right)^4}{\lim_{x\to\infty} 1+\lim_{x\to\infty}\frac{1}{x^4}}$$
$$= \frac{\left(\lim_{x\to\infty} 2+3\lim_{x\to\infty}\frac{1}{x}\right)^4}{\lim_{x\to\infty} 1+\lim_{x\to\infty}\frac{1}{x^4}} = \frac{(2+3\cdot 0)^4}{1+0} = \boxed{16}.$$

(b)

$$\lim_{x\to 1}\frac{x+2}{x^3-1} = \lim_{x\to 1}\frac{x+2}{(x-1)(x^2+x+1)} \quad \boxed{\text{Does not exist}}.$$

When $x$ is close to one the numerator $x+2$ is close to 3 and $x^2+x+1$ is also close to 3. Thus the ratio is approximately $\frac{1}{x-1}$ which is close to $+\infty$ or $-\infty$ depending on whether $x > 1$ or $x < 1$. Since there is a jump, the two sided limit does not exist.

(c) $\lim_{x\to 2}\frac{x^2+3x}{(x-2)^4} = \boxed{\infty}$

Note that $(x-2)^4$ stays positive and tends to zero as $x \to 2$. However, for $x$ near 2, the numerator $x^2+3x$ is close to 10, therefore, the ratio becomes unboundedly large for both left and right approaches of $x$ to 2.

8. Let $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$

Determine whether $f(x)$ is continuous on the interval $[0, 2]$.

$f$ $\boxed{\text{is continuous on } [0,2]}$. We need to check that for each $c$ such that $0 \le c \le 2$ we have

$$\lim_{x\to c} f(x) = f(c).$$

For $c = 0$ the limit

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+}\frac{\sin x}{x} = 1 = f(0).$$

For $0 < c$ we know that both functions $x$ and $\sin x$ are continuous at $c$ and that $x$ is nonzero at $c$. Therefore the limit for $0 < c < 2$ is

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{\sin x}{x} = \frac{\lim\limits_{x\to c} \sin x}{\lim\limits_{x\to c} x} = \frac{\sin c}{c} = f(c).$$

Similarly, at the right endpoint $c = 2$,

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} \frac{\sin x}{x} = \frac{\lim\limits_{x\to 2-} \sin x}{\lim\limits_{x\to 2-} x} = \frac{\sin 2}{2} = f(2).$$

9. A particle moves along a coordinate line and $s$, its directed distance in centimeters from the origin after $t$ seconds is given by $s = f(t) = \sqrt{7t+1}$. Using just the limit definition, find the instantaneous velocity of the particle after 5 seconds.

   Inserting into the formula for the instantaneous velocity at $t$ we compute

$$\begin{aligned}
V &= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{7(t+h)+1} - \sqrt{7t+1}}{h} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{7(t+h)+1} - \sqrt{7t+1}\right)\left(\sqrt{7(t+h)+1} + \sqrt{7t+1}\right)}{h\left(\sqrt{7(t+h)+1} + \sqrt{7t+1}\right)} \\
&= \lim_{h\to 0} \frac{[7(t+h)+1] - [7t+1]}{h\left(\sqrt{7(t+h)+1} + \sqrt{7t+1}\right)} \\
&= \lim_{h\to 0} \frac{7h}{h\left(\sqrt{7(t+h)+1} + \sqrt{7t+1}\right)} \\
&= \lim_{h\to 0} \frac{7}{\sqrt{7(t+h)+1} + \sqrt{7t+1}} = \boxed{\frac{7}{2\sqrt{7t+1}} \text{ centimeters per second.}}
\end{aligned}$$

10. Using just the limit definition, for each $x$, determine whether $F(x)$ is differentiable at $x$, and if it is, find $F'(x)$, where $F(x) = \dfrac{1}{x+2}$. The function is differentiable if the limit of the difference quotient exists. Indeed,

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(t)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{(x+h)+2} - \dfrac{1}{x+2}}{h} \\
&= \lim_{h\to 0} \frac{[x+2] - [(x+h)+2}{h[(x+h)+2]\,[x+2]} \\
&= \lim_{h\to 0} \frac{-h}{h[(x+h)+2]\,[x+2]} \\
&= \lim_{h\to 0} \frac{-1}{[(x+h)+2]\,[x+2]} = \boxed{-\frac{1}{(x+2)^2}}.
\end{aligned}$$

The derivative is defined when division by zero does not occur, namely when $x \neq -2$.