Math 1210 § 4. **Third Midterm Exam** Name: Solutions

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1. *Find the following integrals*

$$(a.)\quad I = \int_0^4 \frac{(3+x)^2}{\sqrt{x}}\,dx = \int_0^4 \frac{9+6x+x^2}{\sqrt{x}}\,dx = \int_0^4 9x^{-1/2} + 6x^{1/2} + x^{3/2}\,dx$$

$$= \left[18x^{1/2} + 4x^{3/2} + \frac{2}{5}x^{5/2}\right]_0^4 = \boxed{18\cdot 4^{1/2} + 4\cdot 4^{3/2} + \frac{2}{5}\cdot 4^{5/2} - 0}$$

$$= 18\cdot 2 + 4\cdot 8 + \frac{2}{5}\cdot 32 = \frac{404}{5}.$$

$$(b.)\quad J = \int_{t=0}^{\pi} \cos(\cos t)\,\sin t\,dt = -\int_{u=1}^{-1} \cos(u)\,du$$

Let $u = \cos t$. Then $du = -\sin t\,dt$, $u = 1$ when $t = 0$ and $u = -1$ when $t = \pi$.

$$J = -\left[\sin u\right]_1^{-1} = -[\sin(-1) - \sin(1)] = \boxed{2\sin 1}.$$

$$(c.)\quad K = \int_{z=0}^{5} \frac{z+1}{(z^2+2z+3)^4}\,dz = \frac{1}{2}\int_{v=3}^{38} v^{-4}\,dv = -\frac{1}{6}\left[v^{-3}\right]_3^{38} = \boxed{\frac{1}{6}\left(\frac{1}{3} - \frac{1}{38}\right)}$$

Let $v = z^2 + 2z + 3$. Then $dv = 2(z+1)\,dz$, $v = 3$ when $z = 0$ and $v = 38$ when $z = 5$.

2. (a) *Write a Riemann Sum that approximates* $\int_0^2 3t^2 + 4\,dt$. *Use a partition with* n *equally spaced intervals.*

$$\boxed{x_i = \frac{2i}{n}}\ \boxed{\bar{x}_i = \frac{2i}{n}}\ \boxed{\Delta x_i = \frac{2}{n}}\ \boxed{\text{Riemann Sum} = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \sum_{i=1}^{n}\left[3\left(\frac{2i}{n}\right)^2 + 4\right]\frac{2}{n}}$$

(b) *Evaluate your Riemann Sum and take its limit to obtain* $\int_0^2 3t^2 + 4\,dt$.

$$\textit{Hint:}\qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2};\qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$R_n = \sum_{i=1}^{n}\left[3\left(\frac{2i}{n}\right)^2 + 4\right]\frac{2}{n} = \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{8}{n}\sum_{i=1}^{n} 1 = \frac{24}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{8}{n}\cdot n$$

We find

$$\int_0^2 3t^2 + 4\,dt = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\frac{4(n+1)(2n+1)}{n^2} + 8\right) = \boxed{16}$$

3. *Find:*

(*a*.) $L = \int \frac{1}{(4+\sqrt{x})^6\sqrt{x}}\,dx = 2\int p^{-6}\,dp = -\frac{2}{5}p^{-5} + C = \boxed{-\frac{2}{5}\left(4+\sqrt{x}\right)^{-5} + C}$

Let $p = 4 + \sqrt{x}$. Then $dp = \frac{1}{2\sqrt{x}}\,dx$.

(*b*.) $M = \int_{\theta=-3}^{3} \sec^2\left(\frac{\theta}{2}\right)\,d\theta = 2\int_{\theta=0}^{3} \sec^2\left(\frac{\theta}{2}\right)\,d\theta = 4\int_{q=0}^{3/2} \sec^2 q\,dq$

Even. Let $q = \frac{\theta}{2}$. Then $dq = \frac{1}{2}\,d\theta$. $q = \frac{3}{2}$ when $\theta = 3$ and $q = 0$ when $\theta = 0$.

$M = 4\Big[\tan q\Big]_0^{3/2} = \boxed{4\tan\frac{3}{2}}$.

(*c*.) $N = \int_0^{10} |\sin(\pi z)|\,dz = 10\int_0^1 |\sin(\pi z)|\,dz = 10\int_0^1 \sin(\pi z)\,dz$

$\sin\pi z$ is 2-periodic so $|\sin\pi z|$ is 1-periodic. There are 10 periods, each contributing the same amount to the integral. For $0 \le z \le 1$, $\sin\pi z \ge 0$ so $|\sin\pi z| = \sin\pi z$.

$N = 10\left[-\frac{\cos\pi z}{\pi}\right]_0^1 = \frac{10}{\pi}\left[-\cos\pi - (-\cos 0)\right] = \frac{10}{\pi}\left[-(-1) - (-1)\right] = \boxed{\frac{20}{\pi}}$

4. *Use bisection to approximate the real root of* $f(x) = 2x^2 - 1$ *on the interval* $[0, 2]$. *Your answer should be at most* $\frac{1}{32}$ *from the actual root.*

We start from the interval $[a_1, b_1] = [0, 2]$ and bisect each step. $f(0) = -1$ and $f(2) = 7$. We maintain $f(a_n) < 0$ and $f(b_n) > 0$. $m_n = \frac{1}{2}(a_n + b_n)$ is the midpoint. If the root is in the interval $\rho \in [a_n, b_n]$ then the error made by approximating ρ with m_n is $|\rho - m_n| \le \frac{1}{2}|b_n - a_n|$. Since $b_n - a_n = 2^{2-n}$ in this case, the error is $|m_n - \rho| \le 2^{1-n}$. This is $\frac{1}{32}$ when $n = 6$.

n	a_n	m_n	b_n	$f(m_n)$
1	0	1	2	$2\,(1)^2 - 1 = 1 > 0$
2	0	$\frac{1}{2}$	1	$2\left(\frac{1}{2}\right)^2 - 1 = \frac{1-2}{2} < 0$
3	$\frac{1}{2}$	$\frac{3}{4}$	1	$2\left(\frac{3}{4}\right)^2 - 1 = \frac{9-8}{8} > 0$
4	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$2\left(\frac{5}{8}\right)^2 - 1 = \frac{25-32}{32} < 0$
5	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$	$2\left(\frac{11}{16}\right)^2 - 1 = \frac{121-128}{128} < 0$
6	$\frac{11}{16}$	$\frac{23}{32}$	$\frac{3}{4}$	
7				

Thus the desired approximation is $m_6 = \boxed{\frac{23}{32}} = 0.71875$. The actual root is $\rho = \frac{1}{\sqrt{2}} = 0.7071067814$ with error $m_6 - \rho = 0.0116432186$ which is less than $\frac{1}{32} = 0.03125$.

5. (a) *Flo drives 600 miles to Denver in 7 hours. Show that she exceeded the 75 miles per hour speed limit at some time during the trip. You may assume that the function giving her position $s(t)$ in miles at time t hours is continuous for $0 \le t \le 7$ and differentiable for $0 < t < 7$.*

This is an application of the Mean Value Theorem for Derivatives. There is a time $0 < c < 7$ such that

$$v(c) = s'(c) = \frac{s(7) - s(0)}{7 - 0} = \frac{600 - 0}{7 - 0} > \frac{525}{7} = 75 \text{ mph}.$$

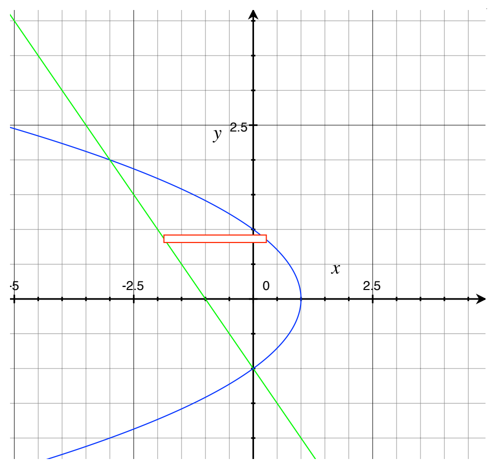
(b) *Sketch the region bounded by the graphs of the two equations. Show a typical slice and approximate its area. Find the limits and set up an integral for the area but DO NOT CALCULATE THE INTEGRAL.*

$$x + y^2 = 1, \qquad x + y = -1.$$

Regard both equations as expressing x in terms of y.

$$x = 1 - y^2, \qquad x = -y - 1.$$

The parabola opens left and passes through $(x, y) = (1, 0)$. It is cut by the line because the line passes through $(-1, 0)$ inside the parabola.



The two curves intersect at solutions of

$$1 - y^2 = x = -y - 1$$

or

$$0 = y^2 - y - 2 = (y + 1)(y - 2).$$

It follows that the intercepts happen when $y = -1$ and $y = 2$.

The better slice is horizontal. Its height is Δy_j and its width is right − left = $(1 - y^2) - (-y - 1)$. Thus the area is

$$\Delta A_i = \Big[(1 - y_i^2) - (-y_i - 1)\Big] \Delta y_i$$

Summing and passing to the limit, we find that the area between the curves is

$$A = \boxed{\int_{y=-1}^{2} \Big[(1 - y^2) - (-y - 1)\Big]\, dy}$$