Math 1210 § 4. **Third Midterm Exam** Name: Practice Problems
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1. *For the functions $f(x)$ and $g(x)$ defined on the closed interval $[-1,2]$, decide whether the Mean Value Theorem for Derivatives applies. If it does, find all possible values of c; if not, state the reason. For each function, sketch the graph on the given interval. (Text problems 189[9,21].)*

$$f(x) = x + |x|; \qquad g(x) = \frac{x}{x-3}.$$

The Mean Value Theorem for Derivatives cannot be applied to $f(x)$ because it is not differentiable at $x = 0$ where it has a kink. Indeed, the conclusion fails: $f(-1) = 0$ and $f(2) = 4$ so the secant line from $x = -1$ to $x = 2$ has the slope $\frac{4}{3}$. However, $f'(x) = 0$ if $x < 0$ and $f'(x) = 2$ if $x > 0$. Thus there is no intermediate point $c$ where the derivative exists and equals $\frac{4}{3}$.

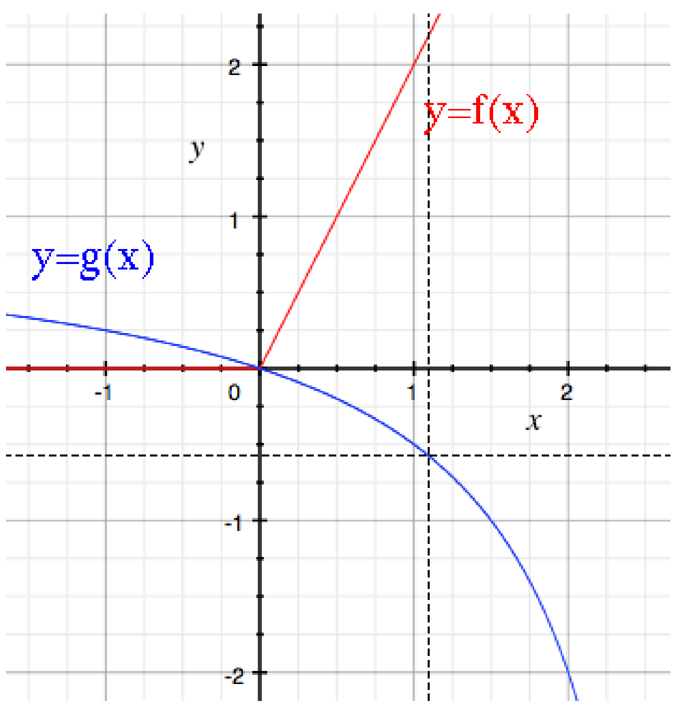
$g(x)$ is a rational function, thus is differentiable wherever it doesn't blow up, which occurs at $x = 3$. So the Mean Value Theorem for Derivatives applies to $g(x)$ because it is continuous on $[-1,2]$ and differentiable on $(-1,2)$. We wish to solve the equation

$$\frac{g(2) - g(-1)}{2 - (-1)} = g'(c)$$

for $c \in (-1,2)$. Substituting, $g'(x) = (x-3)^{-1} - x(x-3)^{-2} = -3(x-3)^{-2}$

$$-\frac{3}{4} = \frac{-2 - \frac{1}{4}}{3} = \frac{\frac{2}{2-3} - \frac{-1}{(-1)-3}}{2-(-1)} = -\frac{3}{(c-3)^2}$$

Thus $4 = (c-3)^2$ or $\pm 2 = c - 3$ so $c = 5$ or $c = 1$. Only $1 \in (-1,2)$ so that $\boxed{c = 1}$ is the desired value.



2. *Suppose $F(x) = D$ for all $x \in (a, b)$. Show that then there is a constant $C$ such that $f(x) = Dx + C$ for all $x \in (a, b)$. (Text problem 189[31].)*

This cries for an application of the Mean Value Theorem for Derivatives because information about the derivative is given on an interval and information about the function itself is sought.

We claim that there is a constant $C$ so that $H(x) = F(x) - Dx = C$ for all $x \in (a, b)$. Pick any point $x_1 \in (a, b)$. Let $C = F(x_1) - Dx_1$. We now show that the difference $H(x)$ is this same $C$ for all $x$. Indeed, pick any other $x \in (a, b)$. The function $H$ is differentiable in $(a, b)$ because it is the difference of differentiable functions. Hence it is continuous in the closed interval from $x_1$ to $x$ and differentiable in the same open interval. Hence we may apply the Mean Value Theorem for Derivatives in this interval. That means that there is a point $\xi$ between $x_1$ and $x$ such that

$$\frac{H(x) - H(x_1)}{x - x_1} = H'(\xi).$$

But because

$$H'(\xi) = F'(\xi) - D = 0$$

we conclude that

$$F(x) - Dx = H(x) = H(x_1) = C.$$

In other words, for every $x \in (a, b)$, $F(x) = Dx + C$ as desired.

3. *Using the Bisection Method, find a real root of $f(x)$ in the interval $[0, 1]$ accurate to two decimal places.*

$$f(x) = x^4 - 5x^3 + 1 = 0.$$

First, observe that $f(0) = 1$ and $f(1) = -3$ so that the continuous function $f(x)$ crosses the $x$-axis in between. Second, at each stage, we shall compute the left and right endpoints, as well as the error that the midpoint makes in approximating the root. $f(x)$ will be positive at all left endpoints $\ell_n$ and negative at all right endpoints $r_n$. The midpoint is $m_n = \frac{1}{2}(\ell_n + r_n)$. Because the interval is bisected at each stage starting from $\ell_1 = 0$ and $r_1 = 1$, the total length of the interval is

$$r_n - \ell_n = \frac{1}{2^{n-1}} (r_1 - \ell_1) = \frac{1}{2^{n-1}}$$

The midpoint is at most half of the length of the interval from any point in the interval, so the distance between root $\rho$ and the midpoint is

$$|m_n - \rho| \le \frac{1}{2}(r_n - \ell_n) = \frac{1}{2^n} (r_1 - \ell_1) = \frac{1}{2^n}.$$

For two decimal point accuracy, this number has to be less than .005. Solving $.005 = 2^{-n}$ we find $n = 7.64$ thus we need to repeat the halving eight times ($2^{-8} = 0.00390625$). Let us proceed with the bisection. At each stage $f(\ell_n) > 0$ and $f(r_n) < 0$. We determine $f(m_n)$. If it is positive, set $\ell_{n+1} = m_n$ and $r_{n+1} = r_n$. If it is negative, set $\ell_{n+1} = \ell_n$ and $r_{n+1} = m_n$. If it is zero, stop because $m_n$ is the root.

Looking at the table, the Bisection Method approximation to the root is $\boxed{\rho \approx 0.61}$ which makes an error less than 0.00390625 which means it's good to two decimal places.

| n | l_n | m_n | r_n | f(m_n) |
|---|---|---|---|---|
| 1 | 0 | 0.5 | 1.0 | 0.4375 |
| 2 | 0.5 | 0.75 | 1.0 | -0.7929688 |
| 3 | 0.5 | 0.625 | 0.75 | -0.06811523 |
| 4 | 0.5 | 0.5625 | 0.625 | 0.2102203 |
| 5 | 0.5625 | 0.59375 | 0.625 | 0.07768345 |
| 6 | 0.59375 | 0.609375 | 0.625 | 0.006471694 |
| 7 | 0.609375 | 0.6171875 | 0.625 | -0.03039622 |
| 8 | 0.609375 | 0.61328125 | 0.6171875 | -0.01185633 |

4. *Using Newton's Method, find a real root of* $f(x)$ *in the interval* $[0,1]$.

$$f(x) = x^4 - 5x^3 + 1 = 0.$$

It's the same function as in the previous problem so we may compare. The sequence of approximations to the zero $\rho$ proceeds by improving each successive approximation of $\rho$ as follows. One computes the tangent line to the graph of $f(x)$ at $x_n$ and takes the next point to be the intersection of the tangent line with the $x$-axis. The point slope form for a point $(x,y)$ on the tangent line through $(x_n f(x_n))$ with slope $f'(x_n)$ is

$$y - f(x_n) = f'(x_n)(x - x_n).$$

The next point corresponds to $y = 0$ in this equation or

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

For our function, $f'(x) = 4x^3 - 15x^2$, the formula reads

$$x_{n+1} = x_n - \frac{x_n^4 - 5x_n^3 + 1}{4x_n^3 - 15x_n^2}.$$

Starting at $x_1 = 1$ we get the following sequence.

| n | x_n |
|---|---|
| 1 | 1.0000000 |
| 2 | 0.7272727 |
| 3 | 0.6266341 |
| 4 | 0.6111192 |
| 5 | 0.6107597 |
| 6 | 0.6107595 |
| 7 | 0.6107595 |
| 8 | 0.6107595 |

The error at each step is the square of the previous error. Seven decimal place accuracy is obtained already in the sixths iteration. The root is $\boxed{\rho = 0.6107595}$.

5. *Using the Fixed Point Method, find a real root of $f(x)$ in the interval* $[0, 1]$.

$$f(x) = x^4 - 5x^3 + 1 = 0.$$

To use the fixed point method, the equation $f(x) = 0$ has to be converted to the fixed point form $x = g(x)$. Then the method proceeds by starting at $x_1 = 1$ and then iterating

$$x_{n+1} = g(x_n).$$

The catch is that $|g'(x)| < 1$ must hold in the vicinity of the root $\rho$. Note that $f'(1) = -10$ so that we may try to add $10x$ to both sides to find

$$x^4 - 5x^3 + 1 + 10x = 10x$$

or

$$x = g(x) = \frac{1}{10}\left(x^4 - 5x^3 + 1\right) + x$$

Starting at $x_1 = 1$ we iterate $x_{n+1} = g(x_n)$ to obtain

| n | x_n | n | x_n |
|---|---|---|---|
| 1 | 1.0000000 | 14 | 0.6107951 |
| 2 | 0.7000000 | 15 | 0.6107784 |
| 3 | 0.6525100 | 16 | 0.6107695 |
| 4 | 0.6317286 | 17 | 0.6107648 |
| 5 | 0.6215997 | 18 | 0.6107623 |
| 6 | 0.6164404 | 19 | 0.6107610 |
| 7 | 0.6137570 | 20 | 0.6107603 |
| 8 | 0.6123467 | 21 | 0.6107599 |
| 9 | 0.6116015 | 22 | 0.6107597 |
| 10 | 0.6112066 | 23 | 0.6107596 |
| 11 | 0.6109970 | 24 | 0.6107595 |
| 12 | 0.6108857 | 25 | 0.6107595 |
| 13 | 0.6108266 | 26 | 0.6107595 |

The Fixed Point Method also converges to the zero $\boxed{\rho = 0.6107595}$ but at a geometric rate.

6. *Calculate the following indefinite integrals.*

   (a) $\displaystyle\int \sec^2 x - 5\sin x \, dx$

   (b) $\displaystyle\int \frac{y^3(y^2+1)^2}{\sqrt{y}}\, dy$

   (c) $\displaystyle\int \frac{\sin z}{\sqrt{1+\cos z}}\, dz$

$$\int \sec^2 x - 5\sin x\, dx = \int \sec^2 x\, dx - 5\int \sin x\, dx = \boxed{\tan x + 5\cos x + C}$$

$$\int \frac{y^3(y^2+1)^2}{\sqrt{y}}\, dy = \int \frac{y^3(y^4+2y^2+1)}{\sqrt{y}}\, dy = \int \frac{y^7+2y^5+y^3}{\sqrt{y}}\, dy = \int y^{\frac{13}{2}} + 2y^{\frac{9}{2}} + y^{\frac{5}{2}}\, dy$$

$$= \int y^{\frac{13}{2}}\, dy + 2\int y^{\frac{9}{2}}\, dy + \int y^{\frac{5}{2}}\, dy = \boxed{\frac{2}{15}y^{\frac{15}{2}} + \frac{4}{11}y^{\frac{11}{2}} + \frac{2}{7}y^{\frac{7}{2}} + C}$$

$$\int \frac{\sin z}{\sqrt{1+\cos z}}\, dz = -\int (1+\cos z)^{-\frac{1}{2}}(-\sin z)\, dz = \boxed{-2\,(1+\cos z)^{\frac{1}{2}} + C}$$

7. *Evaluate the integral* $\mathcal{I} = \displaystyle\int \cos^5\left[\left(x^2+3\right)^7\right] \sin\left[\left(x^2+3\right)^7\right] \left(x^2+3\right)^6 x\,dx.$

Substitute

$$u = \cos\left[\left(x^2+3\right)^7\right].$$

Then

$$du = -14 \sin\left[\left(x^2+3\right)^7\right] \left(x^2+3\right)^6 x\,dx$$

Dividing by $-14$ we insert in the integral to find

$$\mathcal{I} = -\frac{1}{14}\int u^5\,du = -\frac{1}{6\cdot 14}u^6 + C.$$

Substituting $u$ back

$$\mathcal{I} = \boxed{-\frac{1}{84}\cos^6\left[\left(x^2+3\right)^7\right] + C.}$$

8. *Find* $f(x)$ *where* $f''(x) = 3\sqrt[3]{x+7}$.

We integrate twice.

$$\begin{aligned} f'(x) &= \int f''(x)\,dx = 3\int (x+7)^{\frac{1}{3}}\,dx = 3\cdot\frac{3}{4}(x+7)^{\frac{4}{3}} + C_1; \\ f(x) &= \int f'(x)\,dx = \int \frac{9}{4}(x+7)^{\frac{4}{3}} + C_1\,dx = \\ &= \frac{9}{4}\cdot\frac{3}{7}(x+7)^{\frac{7}{3}} + C_1 x + C_2 = \boxed{\frac{27}{28}(x+7)^{\frac{7}{3}} + C_1 x + C_2} \end{aligned}$$

where $C_1$ and $C_2$ are two arbitrary constants of integration.

9. *Find each sum.(a.)* $\quad S = \displaystyle\sum_{i=1}^{n}\left(i^2 - 3i + 5\right) \qquad$ *(b.)* $\quad T = \displaystyle\sum_{i=1}^{n} 2\cdot 3^{i-1}$

For the first sum, we recall the formulæ

$$\sum_{i=1}^{n} 1 = n; \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}; \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Using linearity, the first sum becomes

$$\begin{aligned} S &= \sum_{i=1}^{n}\left(i^2 - 3i + 5\right) = \sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + 5\sum_{i=1} 1 \\ &= \frac{n(n+1)(2n+1)}{6} - 3\cdot\frac{n(n+1)}{2} + n = \boxed{\frac{n^3 - 3n^2 - n}{3}} \end{aligned}$$

Because $2 = 3 - 1$, the second is a telescoping sum

$$\begin{aligned} T &= \sum_{i=1}^{n} 2\cdot 3^{i-1} = \sum_{i=1}^{n}\left(3^i - 3^{i-1}\right) \\ &= (3-1) + \left(3^2 - 3\right) + \left(3^3 - 3^2\right) + \cdots + \left(3^n - 3^{n-1}\right) = \boxed{3^n - 1}. \end{aligned}$$

Alternately, we may use the formula for a geometric sum

$$\sum_{j=0}^{k} ar^j = a\frac{1 - r^{k+1}}{1 - r}$$

with $a = 2$, $j = i - 1$, $r = 3$ and $k = n - 1$ so that

$$T = \sum_{i=1}^{n} 2 \cdot 3^{i-1} = \sum_{j=0}^{n-1} 2 \cdot 3^{j} = 2\frac{1 - 3^n}{1 - 3} = \boxed{3^n - 1.}$$

10. *Using geometry, find a lower and upper estimate for the integral. Then find the value of the definite integral by approximating with a Riemann Sum and taking the limit. Check your answer by computing the integral using antiderivatives.*

$$\int_{-3}^{5} 7x + 2\,dx$$

The increasing function $f(x) = 7x + 2$ ranges between $f(-3) = -19$ to $f(5) = 37$. Thus the integral $\mathcal{I}$ lies between

$$\boxed{-152} = -19[5 - (-3)] \le \mathcal{I} \le 37[5 - (-3)] = \boxed{296}.$$

Since the function $f(x) = 7x + 2$ is continuous on the interval $[-3, 5]$ we know from the Integrability Theorem that the function is integrable and the limit of any sequence of Riemann sums as partitions get finer will converge to the integral. Let's use the partition of the interval $[-3, 5]$ of length $5 - (-3) = 8$ into $n$ equal parts. Thus for $i = 0, \ldots, n$, the partition points, width and sample points are

$$x_i = -3 + \frac{8i}{n}; \qquad \Delta x_i = x_i - x_{i-1} = \frac{8}{n}; \qquad \bar{x}_i = x_i = -3 + \frac{8i}{n}.$$

The Riemann sum with this data is

$$\begin{aligned}
\sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i &= \sum_{i=1}^{n} \left[7\left(-3 + \frac{8i}{n}\right) + 2\right]\frac{8}{n} \\
&= \frac{7 \cdot 8^2}{n^2}\sum_{i=1}^{n} i - \frac{8 \cdot 19}{n}\sum_{i=1}^{n} 1 \\
&= \frac{448}{n^2} \cdot \frac{n(n+1)}{2} - \frac{152}{n} \cdot n \\
&= \frac{224(n+1)}{n} - 152
\end{aligned}$$

Taking the limit, the integral is

$$\int_{-3}^{5} 7x + 2\,dx = \lim_{n\to\infty}\left(\frac{224(n+1)}{n} - 152\right) = 224 - 152 = \boxed{72}.$$

Using the antiderivatives we see

$$\begin{aligned}
\int_{-3}^{5} 7x + 2\,dx = \left[\frac{7}{2}x^2 + 2x\right]_{-3}^{5} &= \left[\frac{7}{2}5^2 + 2 \cdot 5\right] - \left[\frac{7}{2}(-3)^2 - 2 \cdot 3\right] \\
&= \frac{7}{2}(25 - 9) + 2[5 - (-3)] = \boxed{72}
\end{aligned}$$

11. *Find* $\int_{-2}^{1} x^2\,[[x]]\,dx$. *Recall:* $[[x]]$ *is the greatest integer function. You may use* $\int_0^b x^2 = \frac{b^3}{3}$.

Observe that

$$[[x]] = \begin{cases} -2, & \text{if } -2 \le x < -1; \\ -1, & \text{if } -1 \le x < 0; \\ 0, & \text{if } 0 \le x < 1; \\ 1, & \text{if } 1 \le x < 2; \end{cases}$$

Note that in the interval $[-2, 1]$ the function $[[x]]$ jumps at the integers, so that the function $[[x]]$ differs from $i$ only at one point in the interval $[i, i+1]$ so both functions have the same integral over this interval. Thus we may split the integration into intervals $[i, i+1]$ and replace $[[x]]$ by $i$. Thus

$$\begin{aligned}
\int_{-2}^{1} x^2\,[[x]]\,dx &= \int_{-2}^{-1} x^2\,[[x]]\,dx + \int_{-1}^{0} x^2\,[[x]]\,dx + \int_{0}^{1} x^2\,[[x]]\,dx \\
&= -2\int_{-2}^{-1} x^2\,dx - \int_{-1}^{0} x^2\,dx + 0\cdot\int_{0}^{1} x^2\,dx \\
&= -2\left(\int_{0}^{-1} x^2\,dx - \int_{0}^{-2} x^2\,dx\right) + \int_{0}^{-1} x^2\,dx \\
&= -2\left(\frac{(-1)^3}{3} - \frac{(-2)^3}{3}\right) + \frac{(-1)^3}{3} \\
&= -2\left(-\frac{1}{3} + \frac{8}{3}\right) - \frac{1}{3} = \boxed{-5.}
\end{aligned}$$

12. *Find* $\frac{d}{dx}$ *of the functions*

$$G(x) = \int_0^x 3t^4 + \sqrt[3]{t}\,dt; \quad H(x) = \int_1^x xt\,dt; \qquad I(x) = \int_{-x^2}^{x} \frac{t\,dt}{\sqrt{1+t^2}}.$$

Applications of the First Fundamental Theorem of Calculus.

$$G'(x) = \frac{d}{dx}\int_0^x 3t^4 + \sqrt[3]{t}\,dt = \boxed{3x^4 + \sqrt[3]{x}}.$$

$x$ is a constant as far as the integration with respect to $t$ is concerned, so may be factored out of the integral first. Then the product rule is used.

$$H'(x) = \frac{d}{dx}\left(x\int_1^x t\,dt\right) = \int_1^x t\,dt + x\frac{d}{dx}\int_1^x t\,dt = \left[\frac{x^2}{2}\right]_1^x + x\cdot x = \boxed{\frac{3}{2}x^2 - \frac{1}{2}}.$$

To use the Chain Rule, we put

$$F(u) = \int_0^u \frac{t\,dt}{\sqrt{1+t^2}} \qquad \text{so} \qquad F'(u) = \frac{u}{\sqrt{1+u^2}}.$$

Hence

$$\begin{aligned}
I'(x) &= \frac{d}{dx}\int_{-x^2}^{x} \frac{t\,dt}{\sqrt{1+t^2}} = \frac{d}{dx}\big(F(x) - F(-x^2)\big) \\
&= \frac{x}{\sqrt{1+x^2}} - \frac{(-x^2)}{\sqrt{1+(-x^2)^2}}\cdot(-2x) = \boxed{\frac{x}{\sqrt{1+x^2}} - \frac{2x^3}{\sqrt{1+x^4}}}
\end{aligned}$$

13. *Find the following definite integrals.*

(a) $\displaystyle\int_1^7 \frac{(1+\sqrt{t})^3}{\sqrt[4]{t}}\,dt$

(b) $\displaystyle\int_2^4 \frac{1+\tau^2}{(1+3\tau+\tau^3)^4}\,d\tau$

(c) $\displaystyle\int_0^{\pi/2} \cos\theta\cos(\pi\sin\theta)\,d\theta$

$$\begin{aligned}
\int_1^7 \frac{(1+\sqrt{t})^3}{\sqrt[4]{t}}\,dt &= \int_1^7 \frac{1+3\sqrt{t}+3t+t^{3/2}}{\sqrt[4]{t}}\,dt \\
&= \int_1^7 t^{-1/4}+3t^{1/4}+3t^{3/4}+t^{5/4}\,dt \\
&= \left[\frac{4}{3}t^{3/4}+\frac{12}{5}t^{5/4}+\frac{12}{7}t^{7/4}+\frac{4}{9}t^{9/4}\right]_1^7 \\
&= \boxed{\frac{4}{3}\cdot 7^{3/4}+\frac{12}{5}\cdot 7^{5/4}+\frac{12}{7}\cdot 7^{7/4}+\frac{4}{9}\cdot 7^{9/4}-\frac{4}{3}-\frac{12}{5}-\frac{12}{7}-\frac{4}{9}.}
\end{aligned}$$

Let $u = 1+3\tau+\tau^3$. Then $du = (3+3\tau^2)\,d\tau = 3(1+\tau^2)\,d\tau$, $u = 15$ when $\tau = 2$ and $u = 77$ when $\tau = 4$. Hence

$$\begin{aligned}
\int_{\tau=2}^4 \frac{1+\tau^2}{(1+3\tau+\tau^3)^4}\,d\tau &= \frac{1}{3}\int_{u=15}^{77} u^{-4}\,du \\
&= \frac{1}{3}\left[-\frac{1}{3}u^{-3}\right]_{u=15}^{77} \\
&= \boxed{\frac{1}{9\cdot 15^3}-\frac{1}{9\cdot 77^3}}
\end{aligned}$$

Let $\upsilon = \pi\sin\theta$ so $d\upsilon = \pi\cos\theta\,d\theta$. Also $\upsilon = 0$ when $\theta = 0$ and $\upsilon = \pi$ when $\theta = \pi/2$. Hence

$$\begin{aligned}
\int_{\theta=0}^{\pi/2} \cos\theta\cos(\pi\sin\theta)\,d\theta &= \frac{1}{\pi}\int_{\upsilon=0}^{\pi} \cos\upsilon\,d\upsilon \\
&= \frac{1}{\pi}\Big[\sin\upsilon\Big]_{\upsilon=0}^{\pi} \\
&= \frac{1}{\pi}\Big[\sin\pi-\sin 0\Big] = \boxed{0.}
\end{aligned}$$

14. *Using symmetry and periodicity, evaluate the following integral.*

$$\int_0^{7\pi} |\sin 2x|\,dx$$

The function is $\pi/2$-periodic. To see it, use the trig identity $\sin(\theta+\pi) = -\sin\theta$ so

$$f\left(x+\frac{\pi}{2}\right) = \left|\sin\left[2\left(x+\frac{\pi}{2}\right)\right]\right| = |\sin[2x+\pi]| = |-\sin[2x]| = |\sin[2x]| = f(x)$$

Using the next problem, we see that integrating over any interval of length equal to a period gives the same value. Also the interval $[0, 7\pi]$ contains 14 periods. Thus

$$\mathcal{I} = \int_0^{7\pi} |\sin 2x|\,dx = 14\int_0^{\pi/2} |\sin 2x|\,dx = 14\int_{-\pi/4}^{\pi/4} |\sin 2x|\,dx = 28\int_0^{\pi/4} |\sin 2x|\,dx$$

because $f(x)$ is even:

$$f(-x) = |\sin[2(-x)]| = |-\sin 2x| = |\sin 2x| = f(x).$$

Now for $0 \le x \le \pi/4$, $\sin 2x \ge 0$ so $f(x) = |\sin 2x| = \sin 2x$. It follows that the integral is

$$\begin{aligned}\mathcal{I} = 28\int_0^{\pi/4} |\sin 2x|\,dx = 28\int_0^{\pi/4} \sin 2x\,dx &= 28\left[-\frac{1}{2}\cos 2x\right]_{x=0}^{\pi/4} \\ = 14\left[-\cos\frac{\pi}{2} + \cos 0\right] &= 14\left[-0+1\right] = \boxed{14.}\end{aligned}$$

15. *Let $f(x)$ be a continuous $p$-periodic function on $\mathbf{R}$. Show that the following holds any $a$. (Text problem 258[49].)*

$$\int_a^{a+p} f(t)\,dt = \int_0^p f(t)\,dt.$$

The numbers $kp$ are uniformly spaced on the real line as $k$ runs through the integers. The number $a$ falls between two of these. Let $k$ be an integer such that $(k-1)p \le a < kp$. Then $kp \le a + p < (k+1)p$. By interval additivity,

$$\int_{t=a}^{a+p} f(t)\,dt = \int_{t=a}^{kp} f(t)\,dt + \int_{\tau=kp}^{a+p} f(\tau)\,d\tau$$

Now change variables in each integral. Let $t = (k-1)p + u$. Then $du = dt$, $u = a - (k-1)p$ when $t = a$ and $u = p$ when $t = kp$. Let $\tau = kp + v$. Then $dv = d\tau$, $v = 0$ when $t = kp$ and $v = a - (k-1)p$ when $\tau = a + p$. It follows that

$$\int_{t=a}^{kp} f(t)\,dt + \int_{\tau=kp}^{a+p} f(\tau)\,d\tau = \int_{u=a-(k-1)p}^{p} f[(k-1)p+u]\,du + \int_{v=0}^{a-(k-1)p} f[kp+v]\,dv$$

Now let us use periodicity. $f(x) = f(p+x) = f(2p+x)$ and so on. Thus $f[(k-1)p+u] = f[u]$ and $f[kp+v] = f[v]$. Substituting, the integrals become

$$\begin{aligned}&\int_{u=a-(k-1)p}^{p} f[(k-1)p+u]\,du + \int_{v=0}^{a-(k-1)p} f[kp+v]\,dv \\ &\qquad = \int_{u=a-(k-1)p}^{p} f[u]\,du + \int_{v=0}^{a-(k-1)p} f[v]\,dv \\ &\qquad = \int_{v=0}^{a-(k-1)p} f[v]\,dv + \int_{u=a-(k-1)p}^{p} f[u]\,du \\ &\qquad = \int_{v=0}^{p} f[v]\,dv\end{aligned}$$

by interval additivity, as desired.

16. *Use symmetry or periodicity to help you evaluate the following integrals.*

(a) $\displaystyle\int_{-2}^{2} |x^5| + x^7\,dx$

(b) $\displaystyle\int_{-\pi/2}^{\pi/2} (\sin y + \cos y)^2\,dy$

(c) $\displaystyle\int_{-3}^{3} \frac{z^2 \sin^3 z\, \cos^5 z}{(1+\sin^4 z)^3}\,dz$

Observe that $x^7$ is odd and $|x|$ is even so $|x^5|$ is even. Then use $x = |x|$ if $x \ge 0$.

$$\begin{aligned}\int_{-2}^{2} |x^5| + x^7\,dx &= \int_{-2}^{2} |x^5|\,dx + \int_{-2}^{2} x^7\,dx = 2\int_{0}^{2} |x^5|\,dx + 0\\ &= 2\int_0^2 x^5\,dx = 2\left[\frac{x^6}{6}\right]_0^2 = \frac{2^6}{3} - 0 = \boxed{\frac{64}{3}.}\end{aligned}$$

Observe that $\sin y$ is odd and $\cos y$ is even so $\sin y\, \cos y$ is odd. Then notice that $\sin^2 y + \cos^2 y = 1$. Finally, the area under a constant is length times height.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} (\sin y + \cos y)^2\,dy &= \int_{-\pi/2}^{\pi/2} \sin^2 y + 2\sin y\, \cos y + \cos^2 y\,dy\\ &= \int_{-\pi/2}^{\pi/2} \sin^2 y + \cos^2 y\,dy + \int_{-\pi/2}^{\pi/2} 2\sin y\, \cos y\,dy\\ &= \int_{-\pi/2}^{\pi/2} 1\,dy + 0 = \boxed{\pi.}\end{aligned}$$

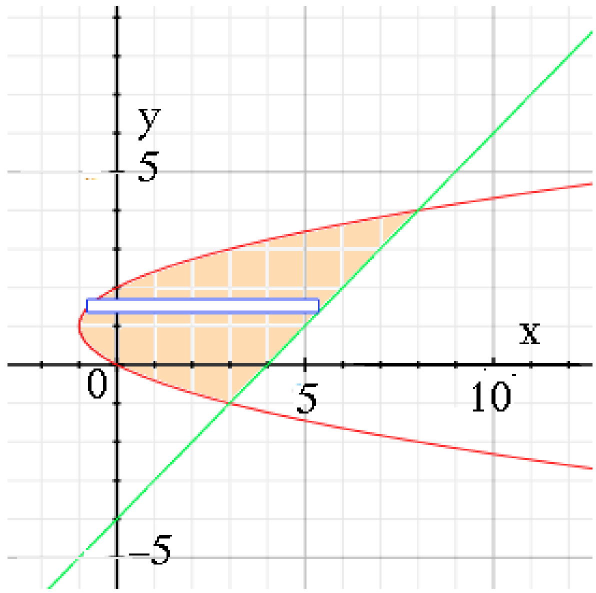
Observe that $z$, $\sin z$ are odd and $\cos z$ is even. Thus $\sin^2 z$, $\sin^4 z$, $(1+\sin^4 z)^{-3}$ and $\cos^5 z$ are all even because odd times odd is even and a function of even is even. Also $\sin^3 z$ is odd. The integrand is a product of even functions times a single odd function, thus odd. Hence

$$\int_{-3}^{3} \frac{x^2 \sin^3 z\, \cos^5 z}{(1+\sin^4 z)^3}\,dz = \boxed{0.}$$

17. *Sketch the region bounded by the graphs of the equations. Show a typical slice. Make an estimate of the area. Set up an integral for the area and evaluate it.*

$$x = y^2 - 2y, \qquad x - y - 4 = 0.$$

Both equations express $x$ as a function of $y$, therefore it is natural to integrate wth respect to $y$, adding together infinitesimally narrow horizontal strips. The first $x = y^2 - 2y$ is a parabola opening to the right which is cut by the second $x = y + 4$, a line.




One can estimate the area by counting the number of unit squares in the graph. That gives an estimate of $A \approx 21$. They intersect when

$$y^2 - 2y = x = y + 4$$

which is equivalent to

$$0 = y^2 - 3y - 4 = (y - 4)(y + 1)$$

whose roots are $y = 4, -1$. Thus the intercepts are the points $(8, 4)$ and $(3, -1)$. The area is then the sum of strips from the parabola to the line. Thus the total area is

$$\begin{aligned}
A &= \int_{y=-1}^{4} \left[y + 4\right] - \left[y^2 - 2y\right] dy \\
&= \int_{y=-1}^{4} 3y + 4 - y^2 \, dy \\
&= \left[\frac{3}{2}y^2 + 4y - \frac{1}{3}y^3\right]_{y=-1}^{4} \\
&= \frac{3}{2}\left[4^2 - (-1)^2\right] + 4\left[4 - (-1)\right] - \frac{1}{3}\left[4^3 - (-1)^3\right] \\
&= \frac{3}{2} \cdot 15 + 4 \cdot 5 - \frac{1}{3} \cdot 65 = \boxed{\frac{125}{6}} \approx 20.833.
\end{aligned}$$

18. *Find the area of the triangle with vertices at* $(-1, 4)$, $(2, -2)$ *and* $(5, 1)$ *by integration. (Text problem 280[30].)*

The equation of a line through the points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$$

The equation of the line from $(-1, 4)$ to $(2, -2)$ is

$$y - 4 = \frac{-2 - 4}{2 - (-1)}(x + 1) = -2(x + 1) \qquad \text{or} \qquad y = -2x + 2.$$

The equation of the line from $(2,-2)$ to $(5,1)$ is

$$y-(-2)=\frac{1-(-2)}{5-2}(x-2)=x-2 \qquad \text{or} \qquad y=x-4.$$

The equation of the line from $(5,1)$ to $(-1,4)$ is

$$y-1=\frac{4-1}{-1-5}(x-5)=-\frac{1}{2}(x-5) \qquad \text{or} \qquad y=-\frac{1}{2}x+\frac{7}{2}.$$

The $x$-coordinate of the second vertex falls between the other two. Its $y$-cooodinate is below the line from $(-1,4)$ to $(5,1)$. Thus the area of the triangle is the sum of the areas of the sub-triangles to the left and right of the line $x=2$. Thus

$$\begin{aligned}
A &= \int_{-1}^{2}\left[-\frac{1}{2}x+\frac{7}{2}\right]-\left[-2x+2\right]dx+\int_{2}^{5}\left[-\frac{1}{2}x+\frac{7}{2}\right]-\left[x-4\right]dx\\
&= \int_{-1}^{2}\frac{3}{2}x+\frac{3}{2}\,dx+\int_{2}^{5}-\frac{3}{2}x+\frac{15}{2}\,dx\\
&= \left[\frac{3}{4}x^2+\frac{3}{2}x\right]_{-1}^{2}+\left[-\frac{3}{4}x^2+\frac{15}{2}x\right]_{2}^{5}\\
&= \left[\frac{3}{4}\left(2^2-(-1)^2\right)+\frac{3}{2}\left(2-(-1)\right)\right]+\left[-\frac{3}{4}\left(5^2-2^2\right)+\frac{15}{2}\left(5-2\right)\right]\\
&= \frac{3}{4}\cdot 3+\frac{3}{2}\cdot 3-\frac{3}{4}\cdot 21+\frac{15}{2}\cdot 3=\boxed{\frac{27}{2}}=13.5.
\end{aligned}$$

To check using geometry, the line $x=2$ that cuts the corner at $y=-2$ and the opposite line at $y=5/2$ has length $9/2$. Thinking of this line as the base of the triangles to the left and to the right and adding their areas we get same answer.

$$A=\frac{1}{2}b_1h_1+\frac{1}{2}b_2h_2=\frac{1}{2}\cdot\frac{9}{2}(2-(-1))+\frac{1}{2}\cdot\frac{9}{2}(5-2)=\frac{27}{2}.$$