1. *Find the derivatives of the following functions.*

   (a) $f(x) = x^3 \sec x.$
   $$f'(x) = \boxed{3x^2 \sec x + x^3 \sec x \tan x}$$

   (b) $g(x) = \dfrac{\sin x}{1 + x^2}.$
   $$g'(x) = \boxed{\dfrac{\cos(x)\left(1 + x^2\right) - \sin(x)\,(2x)}{\left(1 + x^2\right)^2}}$$

   (c) $h(x) = \cos\left(\sqrt{5 + 6x}\right).$
   $$h'(x) = \boxed{-\sin\left(\sqrt{5 + 6x}\right) \cdot \dfrac{6}{2\sqrt{5 + 6x}}}$$

   (d) $y(x)$ *defined by the equation* $x^2 + xy + y^2 = 19.$
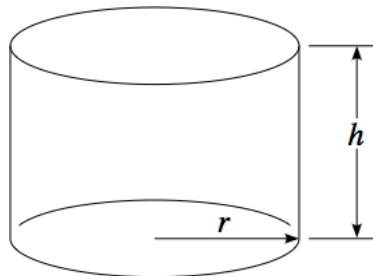   Differentiating implicitly
   $$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
   Solving for the derivative, for (x, y) on the curve,
   $$\frac{dy}{dx} = \boxed{-\frac{2x + y}{x + 2y}}$$

2. *Find the dimensions h and r of the right circular cylinder with least surface area that encloses a volume of 16π cubic inches. (The surface area is the sum of the areas of the top, bottom and curved side.)*

   The surface area is sum to top and bottom areas plus the area of the side
   $$A = 2\pi r^2 + 2\pi rh.$$

   The total volume enclosed is
   $$V = \pi r^2 h = 16\pi.$$

   Solving for h
   $$h = \frac{16}{r^2}.$$

   Inserting into the surface area, we find the objective function
   $$A(r) = 2\pi r^2 + \frac{32\pi}{r}.$$

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The objective function is to be minimized over all feasible radii $0 < r$. Differentiating,

$$\frac{dA}{dr} = 4\pi r - \frac{32\pi}{r^2} = 4\pi r \left\{ 1 - \frac{8}{r^3} \right\}$$

Setting the derivative equal to zero we find that the roots are $r = 0$ or $1 - \frac{8}{r^3} = 0$ which means $r = 2$. The derivative $\frac{dA}{dr}$ is negative if $0 < r < 2$ and positive if $2 < r$ which means that $A(r)$ has a global minimum $A(2) = 32\pi$ when $\boxed{r = 2}$. At the minimum, the corresponding $h = \frac{16}{2^2} = \boxed{4}$.

3. *About how much will the radius of a sphere whose volume is 30 cubic feet increase if it is uniformly coated by .02 cubic feet of paint?*

[Hint: The volume of a sphere of radius r is $\quad V = \frac{4}{3}\pi r^3.$]

We can either differentiate first or solve for r first. The resulting derivative gives a relation between dV and dr which is approximately Δr, the desired quantity. We are given that V increases from $V = 30$ cubic feet to $V + \Delta V = 30.02$ cubic feet after painting, so $\Delta V = .02$. Solving for r and differentiating yields

$$r = \left(\frac{3}{4\pi} \right)^{\frac{1}{3}} V^{\frac{1}{3}}$$

$$dr = \left(\frac{3}{4\pi} \right)^{\frac{1}{3}} \frac{1}{3} V^{-\frac{2}{3}} dV$$

At $V = 30$ cubic feet and $dV = \Delta V = .02$ cubic feet, we find that the change in radius is approximately

$$\Delta r \approx dr = \left(\frac{1}{4\pi} \right)^{\frac{1}{3}} 3^{-\frac{2}{3}} \cdot 30^{-\frac{2}{3}} (.02) = \boxed{\left(\frac{1}{4\pi} \right)^{\frac{1}{3}} \cdot \frac{.02}{90^{\frac{2}{3}}}} = 0.000428.$$

4. *Sand is being unloaded from a conveyor belt at a rate of 5 cubic feet per minute. It falls into a pile shaped like an inverted right cone which has an angle of $30°$ from the horizontal. How fast is the height increasing when the pile is 10 feet high?*

[Hint: the volume of a right circular cone radius r and height h is $\quad V = \frac{1}{3}\pi r^2 h.$]
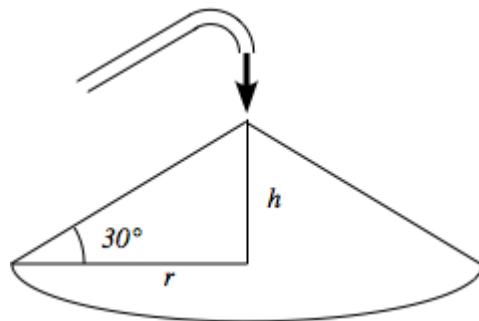
The variable r and h in feet, V in cubic feet are functions of t in minutes. We are given $\frac{dV}{dt} = 5$ cubic feet per minute and seek $\frac{dh}{dt}$ in feet per minute when $h = 10$ feet. Since the ratio of height to base is constant, we have

$$\frac{h}{r} = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

so that $r = \sqrt{3}h$. Hence, the equation relating height to volume is

$$V = \frac{1}{3}\pi \left(\sqrt{3}h\right)^2 h = \pi h^3.$$

Differentiating,

$$\frac{dV}{dt} = 3\pi h^2 \frac{dh}{dt}.$$

The desired rate,

$$\frac{dh}{dt} = \frac{1}{3\pi h^2}\frac{dV}{dt}$$

Finally we may substitute values at the wanted instant: $h = 10$ feet and volume growth is $\frac{dV}{dt} = 5$ cubic feet per minute.

$$\frac{dh}{dt} = \frac{1}{3\pi \cdot 10^2} \cdot 5 = \boxed{\frac{1}{60\pi}} \text{ feet per minute.}$$

5. *Consider the function $f(x) = x^4 - x^3$ on the interval $[-1, 2]$. Find the zeros. Determine where $f(x)$ is positive or negative. Find the critical points of $f(x)$ on the interval $[-1, 2]$. Determine where f is increasing or decreasing. For the interval $[-1, 2]$, find local minimum, local maximum, global minimum, global maximum points or determine that they don't exist. Explain the tests that helped you decide which is which. Determine where $f(x)$ is concave up or concave down and find the inflection points. Sketch the graph of $f(x)$.*

We may factor

$$f(x) = x^4 - x^3 - x^3(x - 1)$$

to see that $f(x)$ has zeros $\boxed{x = 0}$ or $\boxed{x = 1}$. Also $\boxed{f(x) > 0 \text{ if } x > 1 \text{ or } x < 0.}$ On the other hand, $\boxed{f(x) < 0 \text{ if } 0 < x < 1.}$

Differentiating and factoring

$$f'(x) = 4x^3 - 3x^2 = x^2(4x - 3)$$

to see that $f(x)$ has stationary points where $f'(x) = 0$ or at $\boxed{x = 0}$ or $\boxed{x = \frac{3}{4}}$. We also see that $\boxed{f'(x) < 0 \text{ if } x < 0 \text{ or } 0 < x < \frac{3}{4}.}$ On the other hand, $\boxed{f'(x) > 0 \text{ if } \frac{3}{4} < x.}$ Thus $f(x)$ is $\boxed{\text{decreasing if } x < \frac{3}{4}}$ and $\boxed{\text{increasing if } \frac{3}{4} < x.}$ There are no singular points. Thus the only the stationary and endpoints $\boxed{-1, 0, \frac{3}{4}, 2 \text{ are critical points.}}$ for the interval $[-1, 2]$.

The function is decreasing to the right of the left endpoint, thus $\boxed{-1 \text{ is a relative maximum.}}$ $x = 0$ is not a relative min nor max since the function is decreasing on both sides. The function is decreasing to the left and increasing on the right of $x = \frac{3}{4}$, thus $\boxed{\frac{3}{4} \text{ is a relative minimum.}}$

The function is increasing to the left of the right endpoint, thus $\boxed{2 \text{ is a relative maximum.}}$ Evaluating the function at the critical points, $f(-1) = 2$, $f(0) = 0$, $f(\frac{3}{4}) = -\frac{27}{256}$ and $f(2) = 8$. Thus the smallest and largest are the global extrema: $\boxed{f(\frac{3}{4}) = -\frac{27}{256} \text{ is a global minimum}}$ and $\boxed{f(2) = 8 \text{ is a global maximum}}$ on the interval $[-1, 2]$.

Differentiating and factoring again,

$$f''(x) = 12x^2 - 6x = 6x(2x - 1)$$

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we see that $f''(x) = 0$ when $x = 0$ or $x = \frac{1}{2}$. Thus $\boxed{f''(x) < 0 \text{ if } 0 < x < \frac{1}{2}.}$ and

$\boxed{f''(x) > 0 \text{ if } x < 0 \text{ or } \frac{3}{4} < x.}$. It follows that $\boxed{f \text{ is concave down if } 0 < x < \frac{1}{2}.}$ and

$\boxed{f \text{ is concave up if } x < 0 \text{ or } \frac{3}{4} < x.}$. the concavity changes at both

$\boxed{\text{inflection points } (0,0) \text{ and } (\frac{1}{2}, -\frac{1}{16}).}$