Math 1210 § 4. **Second Midterm Exam** Name: Practice Problems
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1. *Use the limit definition of derivative to compute the derivative of* $f(x) = \dfrac{1}{1+x^2}$ *at* $x = a$.

   Inserting the function into the limit of the difference quotient yields

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{1}{1+(a+h)^2} - \dfrac{1}{1+a^2}}{h} \\ &= \lim_{h\to 0} \frac{a^2-(a+h)^2}{h[1+(a+h)^2][1+a^2]} \\ &= \lim_{h\to 0} \frac{a^2-(a^2+2ah+h^2)}{h[1+(a+h)^2][1+a^2]} \\ &= \lim_{h\to 0} \frac{-2ah-h^2}{h[1+(a+h)^2][1+a^2]} \\ &= \lim_{h\to 0} \frac{-2a-h}{[1+(a+h)^2][1+a^2]} \\ &= \boxed{-\frac{2a}{[1+a^2]^2}.} \end{aligned}$$

2. *Find the limits*

   (a) $\displaystyle\lim_{h\to 0} \frac{\sin(\pi+h)-\sin(\pi)}{h}$

   This is the limit of a difference quotient with $f(x) = \sin x$ and $a = \pi$. The limit is the derivative. Thus

$$\begin{aligned} \lim_{h\to 0} \frac{\sin(\pi+h)-\sin(\pi)}{h} &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= f'(a) = \cos\pi = \boxed{-1.} \end{aligned}$$

   (b) *Let* $g(x)$ *and* $k(x)$ *be differentiable at* a. *Find* $\displaystyle\lim_{h\to 0} \frac{g(a+h)k(a+h)^2 - g(a)k(a)^2}{h}$

   This is the limit of a difference quotient with $f(x) = g(x)k(x)^2$ and $x = a$.

$$\begin{aligned} \lim_{h\to 0} \frac{g(a+h)k(a+h)^2 - g(a)k(a)^2}{h} &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= f'(a) = \boxed{g'(a)k(a)^2 + 2g(a)k(a)k'(a).} \end{aligned}$$

3. *Suppose that $f(x)$ is differentiable at $x = a$ and that $f(a) \neq 0$. Use the limit of difference quotients (and not the quaotient rule) to find $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right)$ at $x = a$.*

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{f}\right)(a) &= \lim_{h\to 0} \frac{\dfrac{1}{f(a+h)} - \dfrac{1}{f(a)}}{h} \\
&= \lim_{h\to 0} \frac{f(a) - f(a+h)}{hf(a+h)f(a)} \\
&= \lim_{h\to 0} \left(-\frac{1}{f(a+h)f(a)}\right)\left(\frac{f(a+h) - f(a)}{h}\right) \\
&= \lim_{h\to 0} \left(-\frac{1}{f(a+h)f(a)}\right) \cdot \lim_{h\to 0} \left(\frac{f(a+h) - f(a)}{h}\right) \\
&= -\left(\frac{1}{f(a)^2}\right) \cdot f'(a) = \boxed{-\frac{f'(a)}{f(a)^2}.}
\end{aligned}$$

We have used the fact that the differentiable function is continuous at a so that

$$\lim_{h\to 0} f(a+h) = f(a).$$

4. *Find the derivatives.*

(a) $f(x) = (1 + 2x^3)^4(5 + 6x^7)^8$.
Using the product and chain rules,

$$f'(x) = \boxed{4(1 + 2x^3)^3 \cdot 6x^2 \cdot (5 + 6x^7)^8 + (1 + 2x^3)^4 \cdot 8(5 + 6x^7)^7 \cdot 42x^6.}$$

(b) $g(x) = \dfrac{(1 + 2x^3)^4}{(5 + 6x^7)^8}$.
Using the quotient and chain rules,

$$g'(x) = \boxed{\frac{4(1 + 2x^3)^3 \cdot 6x^2 \cdot (5 + 6x^7)^8 - (1 + 2x^3)^4 \cdot 8(5 + 6x^7)^7 \cdot 42x^6}{(5 + 6x^7)^{16}}.}$$

(c) $k(x) = \dfrac{\sin x + \cos x}{\tan x}$.
Using the quotient and chain rules,

$$k'(x) = \boxed{\frac{(\cos x - \sin x) \cdot \tan x - (\sin x + \cos x) \cdot \sec^2 x}{\tan^2 x}.}$$

(d) $\ell(x) = \left(\dfrac{\cos x}{\sin 2x}\right)^5$.
Using the chain and quotient rules,

$$\ell'(x) = \boxed{5\left(\frac{\cos x}{\sin 2x}\right)^4 \cdot \left(\frac{-\sin x \sin 2x - 2\cos x \cos 2x}{\sin^2 2x}\right).}$$

(e) $m(x) = \sin^2[\sin(\cos x)]$.
Using the chain rule,

$$m'(x) = \boxed{2\sin[\sin(\cos x)]\cos[\sin(\cos x)] \cdot \cos(\cos x) \cdot (-\sin x).}$$

5. *The dial of a standard clock has a* 10*-centimeter radius. One end of an elastic string is attached to the rim at 12 and the other to the tip of the* 10*-centimeter minute hand. At what rate is the string stretching at 12:15 (assuming that the clock is not slowed by this stretching)? If the hour hand is* 7*-centimeters long, how fast are the tips of the hands separating at 12:20? (Text problems 124[72,73].)*

   Let us locate the tip of the minute hand t minutes after high noon. One revolution takes 60 minutes so that the coordinates are

$$(x(t), y(t)) = \left(10 \sin \frac{2\pi}{60} t, 10 \cos \frac{2\pi}{60} t\right)$$

   12:00 has coordinates $(0, 10)$. Then the length of the string $\ell(t)$ is from Pythagorean Theorem

$$\begin{aligned}
\ell(t) &= \sqrt{[0 - x(t)]^2 + [10 - y(t)]^2} = \sqrt{100 \sin^2 \frac{2\pi}{60} t + \left[10 - 10 \cos \frac{2\pi}{60} t\right]^2} \\
&= \sqrt{100 \sin^2 \frac{2\pi}{60} t + 100 - 200 \cos \frac{2\pi}{60} t + 100 \cos \frac{2\pi}{60} t} \\
&= 10 \sqrt{2 - 2 \cos \frac{2\pi}{60} t}
\end{aligned}$$

   The rate of stretching is

$$\ell'(t) = \frac{10 \cdot 2 \sin \frac{2\pi}{60} t \cdot \frac{2\pi}{60}}{2\sqrt{2 - 2 \cos \frac{2\pi}{60} t}} = \frac{\pi \sin \frac{\pi}{30} t}{3\sqrt{2 - 2 \cos \frac{\pi}{30} t}}$$

   At $t = 15$ minutes,

$$\ell'(15) = \frac{\pi \sin \frac{\pi}{2}}{3\sqrt{2 - 2 \cos \frac{\pi}{2}}} = \boxed{\frac{\pi}{3\sqrt{2}} \text{ centimeters per minute}}$$

   Let us locate the tip of the hour hand t minutes after high noon. One revolution takes $12 \cdot 60 = 720$ minutes so that its coordinates are

$$(x_h(t), y_h(t)) = \left(7 \sin \frac{2\pi}{720} t, 7 \cos \frac{2\pi}{720} t\right)$$

   The angle α in radians between the minute and hour hand for $0 \le t \le 30$ (when both hands are in the right half face) is

$$\alpha(t) = \frac{2\pi t}{60} - \frac{2\pi t}{720} = \frac{11\pi t}{360}.$$

   Let $c(t)$ denote the distance between the minute and hour hand. By the law of cosines applied to the triangle with sides minute hand, hour hand and the segment tip to tip we have

$$c(t) = \sqrt{10^2 + 7^2 - 2 \cdot 10 \cdot 7 \cos \alpha(t)} = \sqrt{149 - 140 \cos \alpha(t)}$$

Its derivative is

$$c'(t) = \frac{140 \sin \alpha(t) \cdot \alpha'(t)}{2\sqrt{149 - 140 \cos \alpha(t)}}$$

At $t = 20$ minutes, this is

$$c'(20) = \frac{70 \sin \alpha(20) \cdot \alpha'(20)}{\sqrt{149 - 140 \cos \alpha(20)}} = \frac{70 \sin \frac{11\pi}{18} \cdot \frac{11\pi}{360}}{\sqrt{149 - 140 \cos \frac{11\pi}{18}}}$$

whch is approximately $\boxed{0.4500}$ centimeters per minute.

6. *Find the third derivatives.*

(a) $f(x) = (4 - 7x)^5$

$$\begin{aligned} f'(x) &= 5(4-7x)^4 \cdot (-7) = -35(4-7x)^4 \\ f''(x) &= -35 \cdot 4(4-7x)^3 \cdot (-7x) = 980(4-7x)^3 \\ f'''(x) &= 980 \cdot 3(4-7x)^2 \cdot (-7) = \boxed{-20580(4-7x)^2.} \end{aligned}$$

(b) $g(x) = \dfrac{2x^2}{6-x}$.

It's easier to first rewrite $g(x) = 2x^2(6-x)^{-1}$ and use product, power and chain rules.

$$\begin{aligned} g'(x) &= 2 \cdot 2x(6-x)^{-1} + 2x^2 \cdot (-1)(6-x)^{-2} \cdot (-1) \\ &= 4x(6-x)^{-1} + 2x^2(6-x)^{-2} \\ g''(x) &= 4(6-x)^{-1} + 4x \cdot (-1)(6-x)^{-2} \cdot (-1) + \\ &\quad 2 \cdot 2x(6-x)^{-2} + 2x^2 \cdot (-2)(6-x)^{-3} \cdot (-1) \\ &= 4(6-x)^{-1} + 8x(6-x)^{-2} + 4x^2(6-x)^{-3} \\ g'''(x) &= 4 \cdot (-1)(6-x)^{-2} \cdot (-1) + 8(6-x)^{-2} + 8x \cdot (-2)(6-x)^{-3} \cdot (-1) \\ &\quad + 4 \cdot 2x(6-x)^{-3} + 4x^2 \cdot (-3)(6-x)^{-4} \cdot (-1) \\ &= \boxed{12(6-x)^{-2} + 24x(6-x)^{-3} + 12x^2(6-x)^{-4}.} \end{aligned}$$

(c) $h(x) = x \cos\left(\dfrac{\pi}{x}\right)$.

It's easier to first rewrite $h(x) = x\cos(\pi x^{-1})$ and use product, power and chain rules.

$$\begin{aligned} h'(x) &= \cos(\pi x^{-1}) - x \sin(\pi x^{-1}) \cdot (-\pi x^{-2}) \\ &= \cos(\pi x^{-1}) + \pi x^{-1} \sin(\pi x^{-1}) \\ h''(x) &= -\sin(\pi x^{-1}) \cdot (-\pi x^{-2}) - \pi x^{-2} \sin(\pi x^{-1}) + \pi x^{-1} \cos(\pi x^{-1}) \cdot (-\pi x^{-2}) \\ &= \pi x^{-2} \sin(\pi x^{-1}) - \pi x^{-2} \sin(\pi x^{-1}) - \pi^2 x^{-3} \cos(\pi x^{-1}) \\ &= -\pi^2 x^{-3} \cos(\pi x^{-1}) \\ h'''(x) &= -\pi^2 \cdot (-3)x^{-4} \cos(\pi x^{-1}) + \pi^2 x^{-3} \sin(\pi x^{-1}) \cdot (-\pi x^{-2}) \\ &= \boxed{3\pi^{-2} x^{-4} \cos(\pi x^{-1}) - \pi^3 x^{-5} \sin(\pi x^{-1}).} \end{aligned}$$

7. *A projectile is fired directly upward from the ground with an initial velocity of v_0 feet per second. Its height in t seconds is given by $s = v_0 t - 16t^2$ feet. What must the initial velocity be for the projectile to reach a maximum height of 1 mile? (Text problem 129[35].)*

   The maximum occurs when the derivative vanishes. The derivative

$$\frac{ds}{dt} = v_0 - 32t$$

   is zero at $t = v_0/32$. There the height is

$$s\left(\frac{v_0}{32}\right) = v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 = \frac{v_0^2}{64}.$$

   Since there are 5280 feet in a mile, we solve for v_0 at the maximum height

$$5280 = \frac{v_0^2}{64}$$

   so $v_0 = \boxed{581.3089}$ feet per second.

8. *An object moves along a horizontal coordinate line in such a way that the position is $s = t^3 - 3t^2 - 24t - 6$. Here s is measured in centimeters and t in seconds. When is the object slowing down; that is, when is the* speed *decreasing? (Text problem 130[37].)*

   This is a little tricky because the speed is the *absolute value* of the velocity. The velocity is the derivative of the position and may be either positive or negative.

$$v(t) = \frac{ds}{dt} = 3t^2 - 6t - 24 = 3(t^2 - 2t - 8) = 3(t-4)(t+2)$$

   The change in the velocity is given by the acceleration

$$a(t) = \frac{dv}{dt} = 6t - 6$$

   If the velocity is positive, namely when $t < -2$ or when $t > 4$, the speed is the velocity, and it is decreasing when the acceleration is negative, namely when $t < 1$.

   If the velocity is negative, namely when $-2 < t < 4$, then the speed is negative of the velocity and it is decreasing when the velocity is increasing, namely when the acceleration is positive which is when $t > 1$.

   Thus the speed is decreasing for both of the times $\boxed{t < -2 \text{ or } 1 < t < 4}$. During $t < -2$ the velocity is positive, which is decreasing because the acceleration is negative. During $1 < t < 4$ the velocity is negative and getting less negative (increasing) because the acceleration is positive.

9. *Assuming that the equation defines a differentiable function of x, find $D_x y$ by implicit differentiation.*

$$x^3\sqrt{y-1} = xy + 3$$

   Viewing $y = y(x)$ as a function of x we take derivatives with respect to x.

$$3x^2\sqrt{y-1} + \frac{x^3}{2\sqrt{y-1}} \cdot \frac{dy}{dx} = y + x\frac{dy}{dx}.$$

   Solving for the derivative

$$\left(\frac{x^3}{2\sqrt{y-1}} - x\right)\frac{dy}{dx} = y - 3x^2\sqrt{y-1}$$

so

$$\frac{dy}{dx} = \frac{y - 3x^2\sqrt{y-1}}{\dfrac{x^3}{2\sqrt{y-1}} - x} = \boxed{\frac{2y\sqrt{y-1} - 6x^2(y-1)}{x^3 - 2x\sqrt{y-1}}}.$$

10. Show that the normal line to $x^3 + y^3 = 3xy$ at $\left(\frac{3}{2}, \frac{3}{2}\right)$ passes through the origin. (Text problem 134[41].)

Viewing $y = y(x)$ as a function of x we take derivatives with respect to x

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}.$$

Solving for the derivative yields

$$(y^2 - x)\frac{dy}{dx} = y - x^2$$

or

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}.$$

The slope of the tangent line is found by substituting the point in question $(x, y) = \left(\frac{3}{2}, \frac{3}{2}\right)$ which yields

$$m_{\tan} = \frac{\left(\frac{3}{2}\right) - \left(\frac{3}{2}\right)^2}{\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)} = -1.$$

The slope of the normal line, which is perpendicular to the tangent line is then

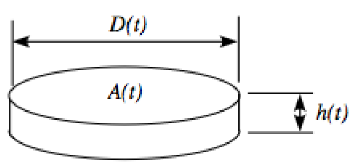
$$m_{\text{norm}} = -\frac{1}{m_{\tan}} = -\frac{1}{-1} = 1.$$

The point-slope form of the normal line through the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ is

$$y - \frac{3}{2} = 1 \cdot \left(x - \frac{3}{2}\right).$$

The point $(x, y) = (0, 0)$ satisfies this equation, so lies on the normal line.

11. *Assume that an oil spill is being cleaned up by deploying bacteria that consume the oil at* 4 *cubic feet per hour. The oil itself is modeled in the form of a thin cylinder, whose height is the thickness of the oil slick. When the thickness is* .001 *foot, the cylinder is* 500 *feet in diameter. If the height is decreasing at* .0005 *foot per hour, at what rate is the area of the slick changing? (Text problem 141[8].)*



(1.) Diagram and variables. Let t denote time in hours, $h(t)$ the thickness in feet, $D(t)$ the diameter in feet, $V(t)$ the volume in cubic feet and $A(t)$ the area in square feet. (2.) Givens and wanted values. We are given that

$$\frac{dV}{dt} = -4, \quad \frac{dh}{dt} = -.0005, \text{ when } D = 500 \text{ and } h = .001. \text{ We seek } \frac{dA}{dt} \text{ at that instant.}$$

(3.) Equation relating variables. Let us find the rate of diameter decrease and use it to find the rate of area change. The volume of a cylinder is

$$V = \frac{\pi}{4} D^2 h.$$

(4.) Differentiate. Remembering that all variables change in time, we must use the product rule to differentiate with respect to time

$$\frac{dV}{dt} = \frac{\pi}{2} Dh \frac{dD}{dt} + \frac{\pi}{4} D^2 \frac{dh}{dt}.$$

(5.) Substitute known quantities and solve for wanted quantity. Solving for the rate of diameter increase,

$$\frac{dD}{dt} = \frac{2}{\pi D h} \frac{dV}{dt} - \frac{D}{2h} \frac{dh}{dt}.$$

At the instant in question

$$\frac{dD}{dt} = \frac{2}{\pi(500)(.001)}(-4) - \frac{500}{2(.001)}(-.0005) = 119.907 \text{ feet per hour}$$

The thinning is faster than the volume decrease resulting in the growth of the diameter. Now the area is related to the diameter by

$$A = \frac{\pi}{4} D^2.$$

Differentiating

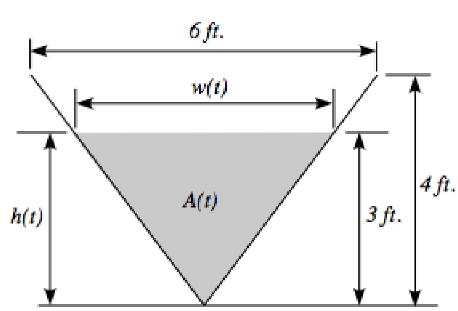
$$\frac{dA}{dt} = \frac{\pi}{2} D \frac{dD}{dt}.$$

At the instant in question, the rate of area increase is

$$\frac{dA}{dt} = \frac{\pi}{2}(500)(119.907) = \boxed{94174.74 \text{ square feet per hour.}}$$

Alternately, we could have eliminated D and expressed volume in terms of A before differentiating.

12. *A trough* 12 *feet long has the cross section in the form of an isosceles triangle (with base at the top)* 4 *feet deep and* 6 *feet across. If water is filling the trough at a rate of* 9 *cubic feet per minute, how fast is the water level rising when the water is* 3 *feet deep? (Text problem 149[37].)*



(1.) Diagram and variables. The diagram shows a cross section of the trough. Let t be time in minutes, $V(t)$ volume of water in cubic feet, $h(t)$ height of the water, $w(t)$ width of the water and ℓ the length of the trough all in feet and $A(t)$ the area of a section of the water in square feet. (2.) Givens and wanted values. We are given that

$$\frac{dV}{dt} = 9, \text{ when } h = 3 \text{ feet. We seek } \frac{dh}{dt} \text{ at that instant.}$$

(3.) Equation relating variables. The water cross section is a triangle whose base is $w(t)$ feet and whose height is $h(t)$ feet. The water and trough sections are similar triangles so

$$\frac{w(t)}{h(t)} = \frac{6}{4} \qquad \text{so } w(t) = \frac{3}{2}h(t).$$

The volume of the water is the volume of a triangular cylinder of length $\ell = 12$ feet so

$$V(t) = A(t)\ell = \frac{1}{2}w(t)h(t)\ell = \frac{1}{2}\cdot\frac{3}{2}h(t)\cdot h(t)\cdot 12 = 9h(t)^2.$$

(4.) Differentiate. We find

$$\frac{dV}{dt} = 18h(t)\frac{dh}{dt}.$$

Solving for the rate of change in height,

$$\frac{dh}{dt} = \frac{1}{18h(t)}\frac{dV}{dt}.$$

(5.) Substitute known quantities and solve for wanted quantity. At the desired instant,

$$\frac{dh}{dt} = \frac{1}{18\cdot 3}\cdot 9 = \boxed{\frac{1}{6}\text{ feet per minute.}}$$

13. *Use differentials to approximate* $\sqrt[3]{63.91}$.

The number is close to $64 = 4^3$. Writing the differential of the function $y = \sqrt[3]{x}$, we have

$$dy = \frac{1}{3}x^{-\frac{2}{3}}\,dx$$

Then if $x = 64$ and $dx = \Delta x = 63.91 - 64 = -.09$ then the differential approximation is

$$y(x+\Delta x) - y(x) = \Delta y \approx dy = \frac{1}{3}x^{-\frac{2}{3}}\,dx.$$

Substituting values,

$$\sqrt[3]{63.91} - 4 \approx \frac{1}{3}\cdot 64^{-\frac{2}{3}}\cdot(-.09) = -\frac{1}{16}\cdot .03 = -0.001875$$

Thus

$$\sqrt[3]{63.91} \approx 4 - 0.001875 = \boxed{3.998125.}$$

14. *A tank has the shape of a cylinder with hemispherical ends. If the cylindrical part is* 100 *centimeters long and has an outside diameter of* 20 *centimeters, about how much paint is required to coat the outside of the tank to a thickness of 1 millimeter? (Text problem 147[35].)*

We express the volume of the tank (including the tank itself and the space enclosed) V in cubic centimeters in terms of the radius of the cylindrical part r in centimeters. Then the volume of paint is $\Delta V = V(r+\Delta r) - V(r) = V(10.1) - V(10)$, where $r = 10$ centimeters, half of the diameter, and $\Delta r = .1$ centimeter (one millimeter). The volume is the sum of the volumes of the cylinder plus the two hemispherical ends

$$V(r) = \pi r^2\ell + \frac{4}{3}\pi r^3 = 100\pi r^2 + \frac{4}{3}\pi r^3.$$

Its differential is

$$dV = 200\pi r\,dr + 4\pi r^2\,dr.$$

Then the volume of paint is

$$\Delta V \approx dV = 200\pi\cdot 10\cdot 0.1 + 4\pi 10^2\cdot 0.1 = \boxed{753.9822\text{ cubic centimeters.}}$$

15. *Identify the critical points and find the maximum value and the minimum value on the interval $I = [-1, 4]$.*

$$f(x) = \frac{x^3}{1 + x^4}$$

The function is everywhere differentiable, so the critical points are the endpoints $x = -1$ or $x = 4$ or the stationary points. To find these, differentiate and set to zero.

$$f'(x) = \frac{3x^2(1 + x^4) - x^3(4x^3)}{(1 + x^4)^2} = \frac{x^2(3 - x^4)}{(1 + x^4)^2}$$

This is zero if $x = 0$ or if $x = \pm\sqrt[4]{3} = \pm 1.316074$. This leaves two stationary points in the interval, $x = 0$ or $x = \sqrt[4]{3}$. By the Critical Point Theorem, the maximum and minimum occur at a critical point. Thus we need to compute the values of the function at these four points and select the smallest and largest.

$$f(-1) = \frac{(-1)^3}{1 + (-1)^4} = -\frac{1}{2}, \qquad f(0) = \frac{(0)^3}{1 + (0)^4} = 0,$$

$$f(\sqrt[4]{3}) = \frac{(\sqrt[4]{3})^3}{1 + (\sqrt[4]{3})^4} = 0.5698768, \qquad f(4) = \frac{(4)^3}{1 + (4)^4} = \frac{64}{257} = 0.2490272.$$

Thus the maximum is $\boxed{0.5698768}$ which occurs at the stationary point $x = \sqrt[4]{3}$ and the minimum is $\boxed{-.5}$ which occurs at the left endpoint $x = -1$.

16. *Determine where the given function is increasing, decreasing, concave up, concave down. Locate the zeros, critical points, horizontal and vertical asymptotes, y-intercepts, extrema, relative extrema and inflection points. Use th first and second derivative tests to decide if the which of the stationary points are local minima and maxima. Sketch the graph.*

$$f(x) = \frac{3x + 1}{x^2 + 1}$$

From $f(0) = 1$ we see that the $\boxed{y\text{-intercept is the point } E = (0, 1).}$. Also, the denominator is bounded away from zero so that $\boxed{\text{there is no vertical asymptote.}}$ Also the limit of $f(x)$ as $x \to \pm\infty$ is zero so $\boxed{y = 0 \text{ is the horizontal asymptote.}}$ Solving $f(x) = 0$ implies $3x + 1 = 0$ of $x = -\frac{1}{3}$. Thus $\boxed{\text{the only zero point is } C = \left(-\frac{1}{3}, 0\right).}$ Since the sign is that of the numerator of $f(x)$, we see that $f(x) < 0$ on $(-\infty - \frac{1}{3})$ and $f(x) > 0$ on $(-\frac{1}{3}, \infty)$. Finally, because $x^2 + 1 \approx \frac{10}{9}$ when x is near $-\frac{1}{3}$, we have $f(x) \approx \frac{27}{10}(x + \frac{1}{3})$ near $x = -\frac{1}{3}$ so the graph crosses the zero from negaive to positive with slope $\frac{27}{10}$ near $x = -\frac{1}{3}$.

Taking the first derivative gives the next batch of clues.

$$f'(x) = \frac{3(x^2 + 1) - (3x + 1) \cdot 2x}{(x^2 + 1)^2} = \frac{-3x^2 - 2x + 3}{(x^2 + 1)^2}$$

The denominator is bounded away from zero so the signs are determined by the numerator $-3x^2 - 2x + 3$ which is a parabola that crosses the y-axis in two points. Solving $-3x^2 - 2x + 3 = 0$ using the quadratic formula, the two roots ω_1 and ω_2 give the stationary points

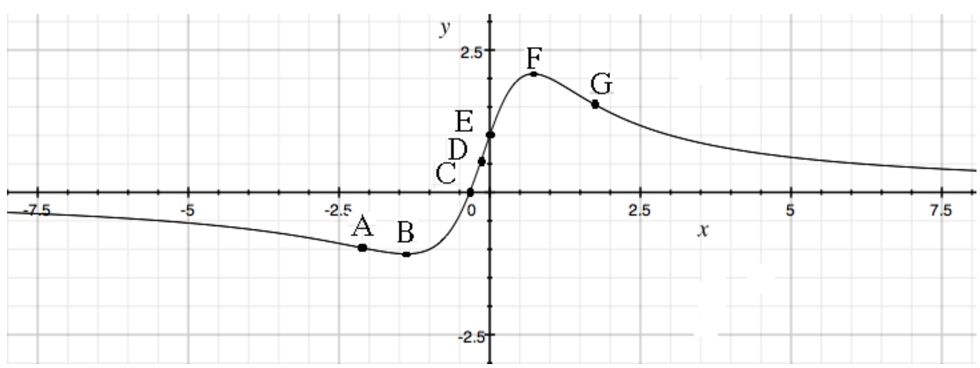
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2)2 \pm \sqrt{(-2)^2 - 4(-3)3}}{2(-3)} = \frac{2 \pm \sqrt{40}}{-6} = -\frac{1}{3} \pm \frac{\sqrt{10}}{3} \approx -1.387, .721$$

Since the function is differentiable on $\mathbf{R}$, there are no singular points and no endpoints. Only stationary points are the $\boxed{\text{critical points: } B = (\omega_1, f(\omega_1)) \text{ and } F = (\omega_2, f(\omega_2)).}$. Since $-3x^2 - 2x + 3$ is a downward parabola, we find that $f'(x) < 0$ on $(-\infty, \omega_1)$, $f'(x) > 0$ on (ω_1, ω_2) and $f'(x) < 0$ on (ω_2, ∞). Hence the function $f(x)$ is $\boxed{\text{increasing on } (\omega_1, \omega_2)}$ and $\boxed{\text{decreasing on } (-\infty, \omega_1) \cup (\omega_2, \infty).}$ Since the function is decreasing to the left and increasing to the right, the first derivative test tells us that $\boxed{B = (\omega_1, f(\omega_1)) \approx (-1.387, -1.081) \text{ is a relative minimum point.}}$ Also the function is increasing to the left and decreasing to the right, $\boxed{F = (\omega_2, f(\omega_2)) \approx (.721, 2.081) \text{ is a relative maximum point.}}$ Now because over the set $(-\frac{1}{3}, \infty)$ where $f(x)$ is positive it increases until it reaches ω_2 and decreases from that point on, $\boxed{f(\omega_2) \approx 2.081 \text{ is the gobal maximum.}}$. Similarly because $f(x)$ is negative exactly on the set $(-\infty, -\frac{1}{3})$ where it decreases until it reaches ω_1 and increases until $x = -\frac{1}{3}$, $\boxed{f(\omega_1) \approx -1.081 \text{ is the global minimum.}}$.

The last clues may be deduced from the second derivative

$$f''(x) = \frac{(-6x-2)\left(x^2+1\right)^2 - (-3x^2-2x+3)\cdot 2\left(x^2+1\right)\cdot 2x}{\left(x^2+1\right)^4} = \frac{6x^3+6x^2-18x-2}{\left(x^2+1\right)^3}$$

The denominator doesn't vanish, its signs depend on the numerator $c(x) = 6x^3 + 6x^2 - 18x - 2$. Because the cubic changes signs $c(-3) = -56$, $c(-2) = 10$, $c(0) = -2$ and $c(2) = 34$ there are three zeros of $c(x)$ (of $f''(x)$), call them $\alpha_1 \approx -2.261$, $\alpha_2 \approx -.108$ and $\alpha_3 \approx 1.369$. The signs of the second derivative are $f''(x) < 0$ if $x \in (-\infty, \alpha_1) \cup (\alpha_2, \alpha_3)$ and $f''(x) > 0$ if $x \in (\alpha_1, \alpha_2) \cup (\alpha_3, \infty)$. Thus f is $\boxed{\text{concave down in } (-\infty, \alpha_1) \cup (\alpha_2, \alpha_3).}$ and $\boxed{\text{concave up in } (\alpha_1, \alpha_2) \cup (\alpha_3, \infty).}$ Observe that at the stationary points $f''(\omega_1) > 0$ and $f''(\omega_2) < 0$. Thus by the second derivative test we confirm that $f(\omega_1)$ is a local minimum and $f(\omega_2)$ is a local maximum. Finally, since the signs of $f''(x)$ change at the α_i's, $\boxed{A = (\alpha_1, f(\alpha_1)) \approx (-2.261, -.946),\ D = (\alpha_2, f(\alpha_2)) \approx (-.108, .668)}$ and $\boxed{G = (\alpha_3, f(\alpha_3)) \approx (1.369, 1.777) \text{ are inflection points.}}$



17. *Find, if possible, the (global) maximum and minimum values of* $f(x) = 3\sqrt{x} - 2x$ *on the interval* $[0, 4]$.

The function is continuous on a closed bounded interval, thus has a global minimum and a global maximum. Let us find the critical points. The derivative is

$$f'(x) = \frac{3}{2\sqrt{x}} - 2,$$

which is defined on $(0, 4]$, and not defined at $x = 0$, thus f is singular at $x = 0$. $f'(x) = 0$ implies $3 = 4\sqrt{x}$ which vanishes only ast the stationary point $x = \frac{9}{16}$. The endpoints are $x = 0$ and $x = 4$. Thus the set of critical points is $\{0, \frac{9}{16}, 4\}$. By the Critical Point Theorem, the maxima and minima may be selected from

$$f(0) = 0, \qquad f\left(\frac{9}{16}\right) = \frac{9}{8}, \qquad f(4) = -2.$$

thus the global maximum is $\boxed{f\left(\frac{9}{16}\right) = \frac{9}{8}}$ and the global minimum is $\boxed{f(4) = -2}$.

18. *Consider* $f(x) = Ax^3 + Bx^2 + Cx + D$ *with* $A > 0$. *Show that* f *has one local minimum and one local maximum if and only if* $B^3 - 3AC > 0$.

A cubic function is smooth on $\mathbf{R}$, so has relative extrema only at stationary points. Thus we seek conditions that guarantee that f has exactly two stationary points. The fact that the stationary points are one local minimum and one local maximum will follow from the cubic nature.

Let us assume that $B^3 - 3AC > 0$ to show that f has one local minimum and one local maximum. Since f is cubic, it is differentiable everywhere so the only critical points are stationary points. The derivative is

$$f'(x) = 3Ax^2 + 2Bx + C = ax^2 + bx + c.$$

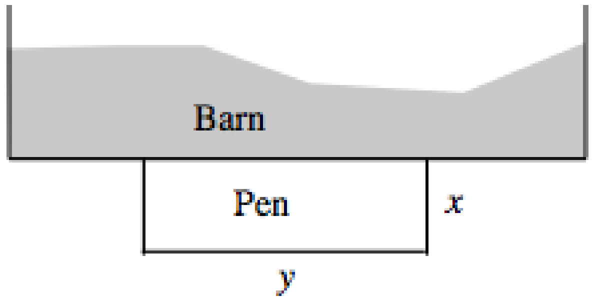
Since $A > 0$, the only roots $\omega_1 < \omega_2$ of f' are given by the quadratic formula

$$\omega_1, \omega_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2B \pm \sqrt{4B^2 - 12AC}}{6A}.$$

Because we assume $4B^2 - 12AC = 4(B^2 - 3AC) > 0$, it shows that f has two distinct stationary points at $x = \omega_1$ and $x = \omega_2$. Furthermore, since roots of a polynomial correspond to factors, $f'(x) = 3A(x - \omega_1)(x - \omega_2)$. It follows that $f'(x) > 0$ if $x < \omega_1$ or $x > \omega_2$ and $f'(x) < 0$ if $\omega_1 < x < \omega_2$. This says that $f' > 0$ to the left of ω_1 and $f' < 0$ to the right, so $f(\omega_1)$ is a local maximum by the First Derivative Test for local maxima. Similarly, $f' < 0$ to the left of ω_2 and $f' > 0$ to the right, so $f(\omega_2)$ is a local minimum. Since these are the only critical points, we have shown that f has exactly one local minimum and exactly one local maximum.

Let us now assume that f has one local minimum and one local maximum to show $B^3 - 3AC > 0$. Since f is cubic, it is everywhere differentiable so has no singular points and no endpoints. The critical points must be stationary points which are the roots of $f'(x) = 3Ax^2 + 2Bx + C$. Since its zeros are given by the quadratic formula above, that there are two roots says that the discriminant is positive, or $4B^2 - 12AC = 4(B^2 - 3AC) > 0$, as desired.

19. *A farmer has 70 feet of fence which he plans to enclose a rectangular pen along one side of his 100 foot barn, as shown in the figure (the side along the barn needs no fence). What are the dimensions of the pen with maximum area?*



(1.) Diagram and variables. Let x be the length of two sides the fence and y the length parallel to the barn wall in feet. (2.) Objective function. We wish to maximize $A = xy$. (3.) Use the conditions to express the objective in terms of one variable. We are given the total length $2x + y = 70$. Solving for y in terms of x

$$y = 70 - 2x.$$

Substituting into the objective function

$$A(x) = xy = x(70 - 2x)$$

(4.) Critical points. We wish to find the maximum of A on the interval of feasible fence lengths $x \in [0, 35]$. Differentiating, we find

$$\frac{dA}{dx} = 70 - 4x.$$

The derivative vanishes at the stationary point $x = \frac{35}{2}$. Being quadratic, the function is differentiable everywhere, thus the set of critical x's is $\{0, 17.5, 35\}$. (5.) Determine the maximum. The corresponding areas are $A(0) = A(35) = 0$ and $A(17.5) = 17.5 \cdot 35 = 612.5$ square feet. Thus the maximal area is $\boxed{\text{612.5 square feet}}$ which occurs when dimensions are $\boxed{x = 17.5 \text{ feet and } y = 35 \text{ feet.}}$

20. *The ZootUte company makes suits. The fixed monthly cost is \$7000 while the cost of manufaturing each unit is \$100. Write an expression for the cost $C(x)$, the total cost of making x suits in a month. ZootUte estimates that 100 units a month can be sold at a unit price of \$250 and that the sales will increase by 10 units for each \$5 decrease in price. Write an expression for the prince $p(x)$ and the revenue $R(x)$ if x units are sold in one month, $x \geq 100$. Write an expression for the profit $P(x)$ if $x \geq 100$. Sketch the graph of $P(x)$. From it estimate x where profit is maximum. Find the exact value using calculus. (Text problem 177[54–57].)*

The cost is the sum of the fixed costs and variable costs

$$C(x) = \boxed{7000 + 100x.}$$

The unit price that can be charged for $x \geq 100$ will decrease \$5 for each additional 10 units made or $\frac{5}{10}$ dollars per unit resulting in demand in dollars per month

$$p(x) = 250 - \frac{1}{2}(x - 100) = \boxed{300 - \frac{1}{2}x} \qquad \text{for } x \geq 100.$$

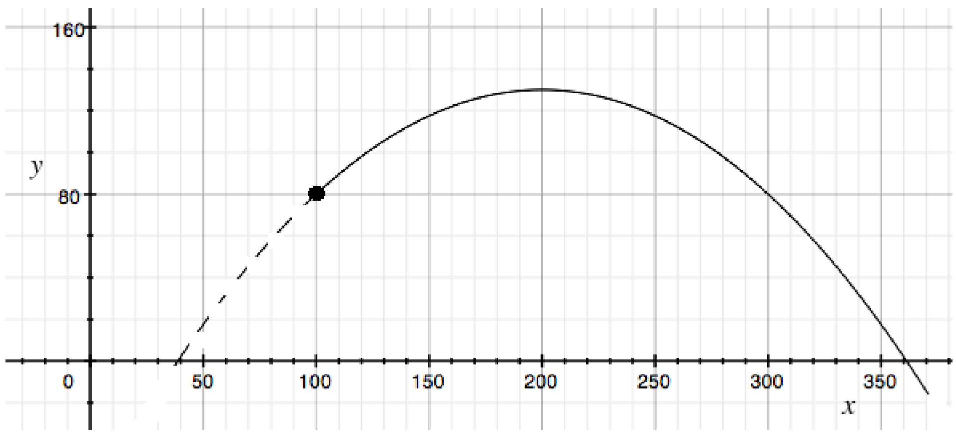
The revenue is number sold times price per unit, or

$$R(x) = x\left(300 - \frac{1}{2}x\right) = \boxed{300x - \frac{1}{2}x^2} \qquad \text{for } x \geq 100.$$

The profit is revenue minus cost,

$$P(x) = R(x) = C(x) = 300x - \frac{1}{2}x^2 - (7000 + 100x) = \boxed{-\frac{1}{2}x^2 + 200x - 7000} \qquad \text{for } x \geq 100.$$

Plotting the parabolic $P(x)$,

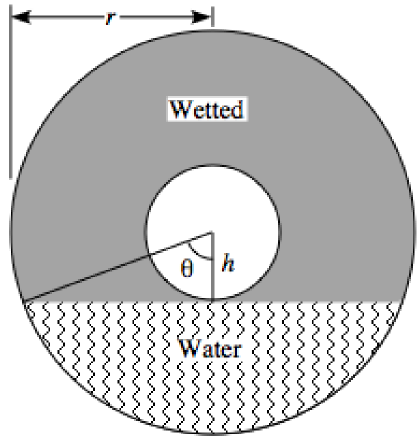


where vertical y-axis is in hundreds of dollars and x-axis units sold. The profit is maximum at about $\boxed{P = 13000}$ dollars at $x = 200$. Taking derivative

$$\frac{dP}{dx} = -x + 200$$

which is zero exactly at $x = 200$. Since $P(x)$ is increasing for $x < 200$ and decreasing for $x > 200$, the stationary point is a global maximum. The exact value of the maximum is $P(200) = \boxed{13000}$ dollars.

21. *A humidifier uses a rotating disk of radius r, which is partially submerged in water. The most evaporation occurs when the exposed wetted region (shown as the shaded region) is maximized. Find the height of the water relative to the center, h that maximizes the wetted region. (Text problem 176[40].)*



(1.) Diagram and variables. Let h be the signed height of the water relative to the center of the disk in the same units as r. Let A be the area of the wetted region and W the area of the water in square units. Let θ denote half the angle of the water from the center.
(2.) Objective function. If $h < 0$ (water level below the center of the disk) then the wetted region is outside the disk of radius $-h$ and above the water. The area of the water is the area of a sector of angle 2θ which is a $\frac{\theta}{\pi}$ part of the whole circle minus the area of the triangle from the corners of the water to the center. Thus for $-r \le h < 0$,

$$W = \pi r^2 \cdot \frac{\theta}{\pi} - 2 \cdot \frac{1}{2}(-h) r \sin\theta = \theta r^2 + hr\sin\theta.$$

Thus the wetted area (the objective function) is

$$A = \pi r^2 - \pi(-h)^2 - W = \pi r^2 - \pi h^2 - \theta r^2 - hr\sin\theta.$$

(3.) Use the conditions to express the objective in terms of one variable. Using the triangle from water to center

$$-h = r\cos\theta$$

The objective may be expressed in terms of $\theta \in \left[0, \frac{\pi}{2}\right)$ by

$$A = \pi r^2 - \pi r^2 \cos^2\theta - \theta r^2 + r^2 \cos\theta\, \sin\theta$$

For $h \ge 0$ the center of the wheel is in the water and the exposed area is just the part of the disk above the water, namely

$$A = (\pi - \theta) r^2 - 2 \cdot \frac{1}{2} hr \sin\theta$$

Substituting $-h = r\cos\theta$, for $\frac{\pi}{2} \le \theta \le \pi$,

$$A = (\pi - \theta) r^2 + r^2 \cos\theta\, \sin\theta.$$

Putting these together, for $0 \le \theta \le \pi$,

$$A(\theta) = \begin{cases} \pi r^2 - \pi r^2 \cos^2\theta - \theta r^2 + r^2\cos\theta\,\sin\theta, & \text{if } \theta < \frac{\pi}{2}; \\ (\pi - \theta) r^2 + r^2 \cos\theta\, \sin\theta, & \text{if } \frac{\pi}{2} \le \theta. \end{cases}$$

(4.) Critical points. Note that $A(\theta)$ is continuous. The derivative is

$$\frac{dA}{d\theta} = \begin{cases} 2\pi r^2 \cos\theta\,\sin\theta - r^2 + r^2\left(-\sin^2\theta + \cos^2\theta\right), & \text{if } \theta < \frac{\pi}{2}; \\ -r^2 + r^2\left(-\sin^2\theta + \cos^2\theta\right), & \text{if } \frac{\pi}{2} \le \theta. \end{cases}$$

thus A is differentiable because the two definitions have the same slope at $\theta = \frac{\pi}{2}$. Replacing with double angle formulas

$$\frac{dA}{d\theta} = \begin{cases} \pi r^2 \sin 2\theta - r^2 + r^2\cos 2\theta, & \text{if } \theta < \frac{\pi}{2}; \\ -r^2 + r^2 \cos 2\theta, & \text{if } \frac{\pi}{2} \le \theta. \end{cases}$$

(5.) Determine the maximum. We notice that $\frac{dA}{d\theta} < 0$ for $\frac{\pi}{2} \le \theta < \pi$ and $\frac{dA}{d\theta}(\pi) = 0$ so $\theta = \pi$ is the only stationary point there, which is also the endpoint. Also $\frac{dA}{d\theta}(0) = 0$ so the other endpoint is also stationary. Looking at $0 < x < \frac{\pi}{2}$ we equate the derivative to zero

$$2\pi r^2 \cos\theta\,\sin\theta - r^2 + r^2\left(-\sin^2\theta + \cos^2\theta\right) = 0$$

or

$$2\pi r^2 \cos\theta\,\sin\theta - r^2\left(\sin^2\theta + [1 - \cos^2\theta]\right) = 0.$$

Hence

$$0 = 2\pi r^2 \cos\theta\,\sin\theta - 2r^2\sin^2\theta = 2r^2\left(\pi\cos\theta - \sin\theta\right)\sin\theta$$

so either $\sin\theta = 0$ corresponding to $\theta = 0$ in the interval or $0 < \theta < \frac{\pi}{2}$ and

$$\pi = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

which happens at $\theta_0 = \tan^{-1}\pi = 1.263$. The critical points occur for θ equal to 0, $\tan^{-1}\pi$ or π. At the critical points, $A(0) = 0$, $A(\theta_0) = 1.879$ and $A(\pi) = 0$. Thus the maximum occurs at θ_0 where $A = 1.879$. We observe that if $\tan(\theta_0) = \pi$ then $\sec^2\theta_0 = 1 + \pi^2$ so

$$\cos\theta_0 = \frac{1}{\sqrt{1+\pi^2}}$$

It follows that the optimal distance below the center is

$$-h = r\cos\theta_0 = \boxed{\frac{r}{\sqrt{1+\pi^2}}}$$