Math 1210 § 4. **First Midterm Exam** Name: Solutions

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1. *Let $f(x)$ be defined for all real numbers x. Give the limit expression for the slope of the tangent line to $y = f(x)$ at $x = c$. Using the expression, compute m_{tan} of $f(x) = \dfrac{x}{x+3}$ at $x = c$.*

$$\boxed{m_{\tan} = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}}$$

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{c+h}{c+h+3} - \dfrac{c}{c+3}}{h} \\ &= \lim_{h\to 0} \frac{(c+h)(c+3) - c(c+h+3)}{h(c+h+3)(c+3)} \\ &= \lim_{h\to 0} \frac{c^2 + 3c + ch + 3h - c^2 - ch - 3c}{h(c+h+3)(c+3)} \\ &= \lim_{h\to 0} \frac{3h}{h(c+h+3)(c+3)} \\ &= \lim_{h\to 0} \frac{3}{(c+h+3)(c+3)} = \frac{3}{(c+0+3)(c+3)} = \boxed{\frac{3}{(c+3)^2}}. \end{aligned}$$

2. *Find the limit if it exists.*

(a) $\displaystyle \lim_{x\to -4} \frac{x^2 + 2x - 8}{x+4}$

$$\begin{aligned} \lim_{x\to -4} \frac{x^2 + 2x - 8}{x+4} &= \lim_{x\to -4} \frac{(x+4)(x-2)}{x+4} \\ &= \lim_{x\to -4} x - 2 \\ &= \lim_{x\to -4} x - \lim_{x\to -4} 2 = -4 - 2 = \boxed{-6}. \end{aligned}$$

(b) $\displaystyle \lim_{x\to 3} \frac{x(x+1)}{\sqrt{x^2+16}}$

$$\lim_{x\to 3} \frac{x(x+1)}{\sqrt{x^2+16}} = \frac{\lim_{x\to 3} x(x+1)}{\lim_{x\to 3} \sqrt{x^2+16}} \qquad \text{since the denominator limit is nonzero as we shall see}$$

$$\begin{aligned} &= \frac{\lim_{x\to 3} x \cdot \lim_{x\to 3} (x+1)}{\sqrt{\lim_{x\to 3} x^2 + 16}} \\ &= \frac{\lim_{x\to 3} x \cdot \left(\lim_{x\to 3} x + \lim_{x\to 3} 1\right)}{\sqrt{\lim_{x\to 3} x^2 + \lim_{x\to 3} 16}} \\ &= \frac{3 \cdot (3+1)}{\sqrt{\left(\lim_{x\to 3} x\right)^2 + 16}} = \frac{3 \cdot (3+1)}{\sqrt{3^3 + 16}} = \boxed{\frac{12}{5}}. \end{aligned}$$

(c) $\displaystyle\lim_{x\to 5} \frac{\left[\left[\frac{x}{2}\right]\right]}{|x-5|}$

Note that for x near 5, $\dfrac{x}{2}$ is near 2.5 so that the greatest integer part $\left[\left[\dfrac{x}{2}\right]\right] = 2$. Also $|x-5| > 0$ for all $x \neq 5$. Thus the ratio is positive for both $x > 5$ and $x < 5$, thus the left and right limits at 5 both tend to infinity. Thus the two sided limit is

$$\lim_{x\to 5} \frac{\left[\left[\frac{x}{2}\right]\right]}{|x-5|} = \boxed{\infty}.$$

3. *Find the limit if it exists.*

(a) $\displaystyle\lim_{x\to 0} \frac{1-\cos(2x)}{3x}$

$$\lim_{x\to 0} \frac{1-\cos(2x)}{3x} = \lim_{x\to 0} \frac{2}{3}\left(\frac{1-\cos(2x)}{2x}\right) \qquad \text{Let } u = 2x.\ u \to 0 \text{ as } x \to 0.$$
$$= \frac{2}{3}\lim_{u\to 0} \frac{1-\cos(u)}{u} = \frac{2}{3}\cdot 0 = \boxed{0}.$$

(b) $\displaystyle\lim_{x\to\infty} \frac{x+1}{\sqrt{4x^2+1}}$

$$\lim_{x\to\infty} \frac{x+1}{\sqrt{4x^2+1}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{4+\frac{1}{x^2}}}$$
$$= \frac{\lim_{x\to\infty} 1+\frac{1}{x}}{\lim_{x\to\infty} \sqrt{4+\frac{1}{x^2}}}$$
$$= \frac{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{1}{x}}{\sqrt{\lim_{x\to\infty} 4 + \frac{1}{x^2}}}$$
$$= \frac{1+0}{\sqrt{\lim_{x\to\infty} 4 + \lim_{x\to\infty} \frac{1}{x^2}}}$$
$$= \frac{1}{\sqrt{4+0}} = \boxed{\frac{1}{2}}$$

(c) $\displaystyle\lim_{x\to -\infty} x^{-\frac{1}{3}} \sin\left(x^3\right)$

Note that the function $\sin(x^3)$ oscillates as $x \to -\infty$ between

$$-1 \leq \sin(x^3) \leq 1.$$

The cube root $\sqrt[3]{x} \to -\infty$ as $x \to -\infty$. Thus dividing by $|\sqrt[3]{x}|$ we get

$$-\frac{1}{|\sqrt[3]{x}|} \leq x^{-\frac{1}{3}} \sin\left(x^3\right) \leq \frac{1}{|\sqrt[3]{x}|}.$$

The first and last terms in this inequality tend to zero as $x \to -\infty$. By the Squeeze Theorem, it follows that the middle term tends to zero too.

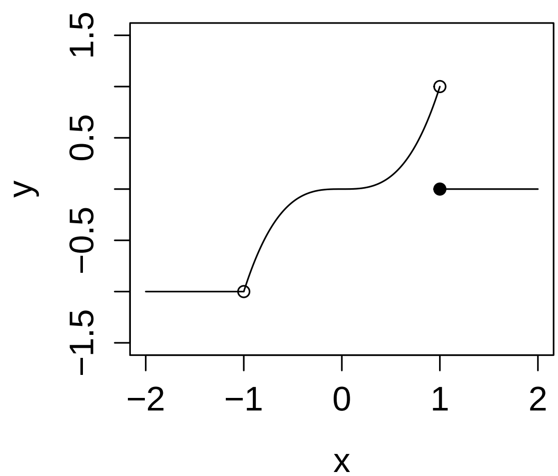
$$\lim_{x\to-\infty} x^{-\frac{1}{3}} \sin\left(x^3\right) = \boxed{0}.$$

4. *Let the function $g(x)$ be defined piecewise by* $g(x) = \begin{cases} -1, & \text{if } x < -1; \\ x^3, & \text{if } -1 < x < 1; \\ 0, & \text{if } 1 \le x. \end{cases}$

Sketch the graph of $y = g(x)$. What is the domain of $g(x)$? Find the limits if they exist:

$$\lim_{x\to1} g(x), \qquad \lim_{x\to1-} g(x)$$

What are the values of x where $g(x)$ is discontinuous? How should $g(x)$ be defined at $x = -1$ to make it continuous there? Explain.



The domain of the function is all points except $x = -1$ so $\mathcal{D} = (-\infty - 1) \cup (-1, \infty)$. The function has a jump at $x = 1$ so the two sided limit does not exist

$$\lim_{x\to1} g(x) \quad \boxed{\text{Does not exist.}}$$

On the other hand, for x near 1 and $x < 1$ we have $g(x) = x^3$ so

$$\lim_{x\to1-} g(x) = \lim_{x\to1-} x^3 = \boxed{1}.$$

Because $g(x)$ has a jump, it is discontinuous at $\boxed{x = 1}$. It is continuous at all other points of its domain.

There is a hole in the graph at $x = -1$. By extending the definition of g to include $\boxed{g(-1) = -1}$, the function becomes continuous there because it has a two sided limit at -1. From the left $x < -1$, $g(x) = -1$ so left limit is -1. From the right $-1 < x < 1$ the function is $g(x) = x^3$ so its limit at -1 is also -1. Since the limits from both sides are consistent, the limit exists at $x = -1$ and it equals the new $g(-1)$ making it continuous:

$$g(-1) = -1 = \lim_{x\to-1} g(x) \qquad \text{so } g(x) \text{ is continuous at } x = -1.$$

5. *Let* $f(x) = \dfrac{(x-2)^4}{x^4-1}$.

Find the horizontal asymptotes of $y = f(x)$*, if any. Find the vertical asymptotes of* $y = f(x)$*, if any. Determine the signs of* $f(x)$ *in the regions* $-\infty < -1 < 1 < 2 < \infty$*. Sketch the graph of* $y = f(x)$ *using this information. Be sure to indicate any horizontal and vertical asymptotes and zeros.*

Horizontal asymptotes are found from taking limits. For both limits to $+\infty$ and $-\infty$ we have

$$\lim_{x\to\pm\infty} \frac{(x-2)^4}{x^4-1} = \lim_{x\to\pm\infty} \frac{(1-\frac{2}{x})^4}{1-\frac{1}{x^4}} = \frac{(1-0)^4}{1-0} = 1.$$

Thus both left tail and right tail have horizontal asymptote $\boxed{y=1}$. The function $f(x)$ blows up when $x = \pm 1$ so there are two vertical asymptotes at $\boxed{x=-1}$ and $\boxed{x=1}$.

We observe that the numerator $(x-2)^4$ is always positive. The denominator factors

$$x^4 - 1 = (x^2+1)(x^2-1).$$

The first factor is positive. The second is negative if $-1 < x < 1$ and is otherwise positive. The signs of f are thus

Region	$x < -1$	$-1 < x < 1$	$1 < x < 2$	$2 < x$
Sign of $f(x)$	$\frac{(+)}{(+)} > 0$	$\frac{(+)}{(-)} < 0$	$\frac{(+)}{(-)} < 0$	$\frac{(+)}{(+)} > 0$

The graph has a zero at $\boxed{x=2}$. This is just a schematic sketch of the function (*e.g.*, the actual value $f(0) = -16$). The actual plot is much more dramatic!