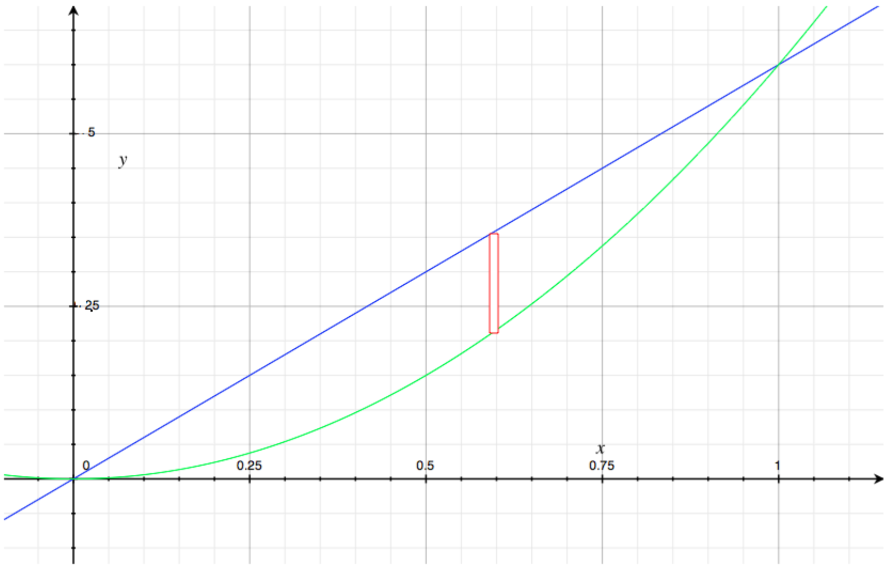
Math 1210 § 4. **Final Exam** Name: Practice Problems

Treibergs November 30, 2015

1. *Find the volume of the solid generated by revolving about the x-axis the region bounded by the line $y = 6x$ and the $y = 6x^2$. (Text problem 286[18].)*



A sketch of the region and the cross section of a washer is indicated. The curves intersect when

$$6x = y = 6x^2$$

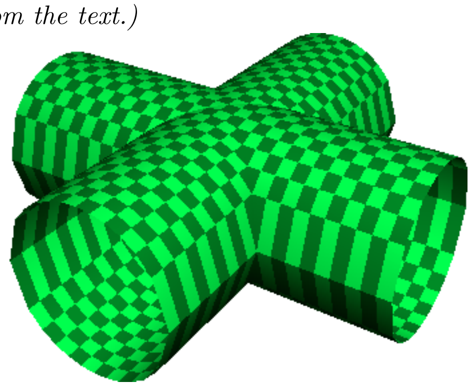
which happens when $0 = x(1 - x)$ or at $x = 0$ and $x = 1$. The volume of a washer centered on the x-axis with outside radius $y = 6x$ and inside radius $y = 6x^2$ is

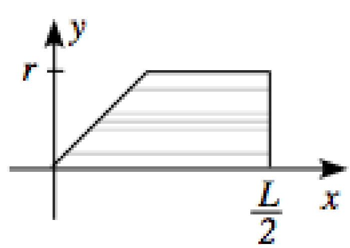
$$dV = \pi\left([6x]^2 - [6x^2]^2\right)\,dx = 36\pi\left(x^2 - x^4\right)\,dx$$

Integrating, we find the volume

$$V = 36\pi\int_0^1 \left(x^2 - x^4\right)\,dx = 36\pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \boxed{36\pi\left(\frac{1}{3} - \frac{1}{5}\right)} = \frac{24\pi}{5}.$$

2. *Find the volume inside the "+" in the figure, assuming that both cylinders have radius r and length $L \geq 2r$ and their axes cross at right angles at the centers. (Problem 286[29] from the text.)*





Center the "+" at the origin and align its axes with the x and y coordinate axes. Taking only the part of the "+" in the first quadrant, in the halfspace $x \ge y$ we get a region D which has $\frac{1}{16}$ of the total volume. The diagrams show the perspective view of the "+" and the top view of D. Then we can compute the volume of vertical slabs for y-coordinates $0 \le y \le r$ and $y \le x \le \frac{L}{2}$ which lie between the surfaces $0 \le z \le \sqrt{r^2 - y^2}$. The volume of such a slab is length times height times width

$$dV = \left(\frac{L}{2} - y\right) \sqrt{r^2 - y^2}\, dy$$

Then the total volume is

$$V = 16 \int_0^r \left(\frac{L}{2} - y\right) \sqrt{r^2 - y^2}\, dy = 8L \int_0^r \sqrt{r^2 - y^2}\, dy - 16 \int_0^r \left(r^2 - y^2\right)^{1/2} y\, dy$$

We do the first integral geometrically. It represents the area of one quarter circle of radius r, thus

$$\int_0^r \sqrt{r^2 - y^2}\, dy = \frac{\pi}{4} r^2.$$

We can write the antiderivative of the second

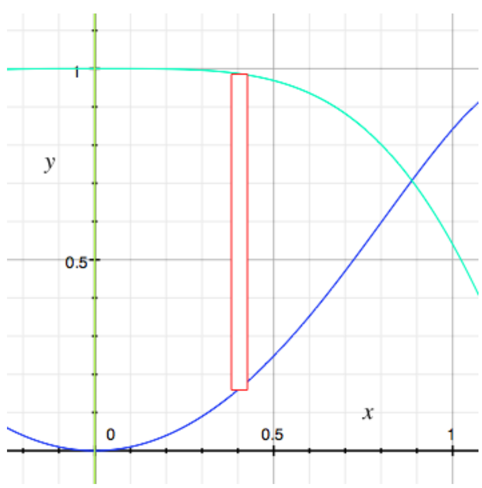
$$\int_0^r \left(r^2 - y^2\right)^{1/2} y\, dy = \left[-\frac{1}{3}\left(r^2 - y^2\right)^{3/2}\right]_0^r = -\frac{1}{3}\left[\left(r^2 - r^2\right)^{3/2} - \left(r^2\right)^{3/2}\right] = \frac{r^3}{3}.$$

the total volume is thus

$$V = \boxed{2\pi\, L\, r^2 - \frac{16}{3} r^3}$$

This makes sense because the sum of the volumes of two cylinders is $2\pi\, L\, r^2$.

3. *The region in the first quadrant bounded by $x = 0$, $y = \sin(x^2)$ and $y = \cos(x^2)$ is revolved about the y-axis. Find the volume of the resulting solid. (Text problem 293[21])*



The three curves bounding the region and a typical thin rectangle inside are indicated in the diagram. The region and rectangle are revolved about the y-axis generating the solid and a shell. The volume of the shell is

$$dV = 2\pi x \left\{\cos(x^2) - \sin(x^2)\right\}\, dx$$

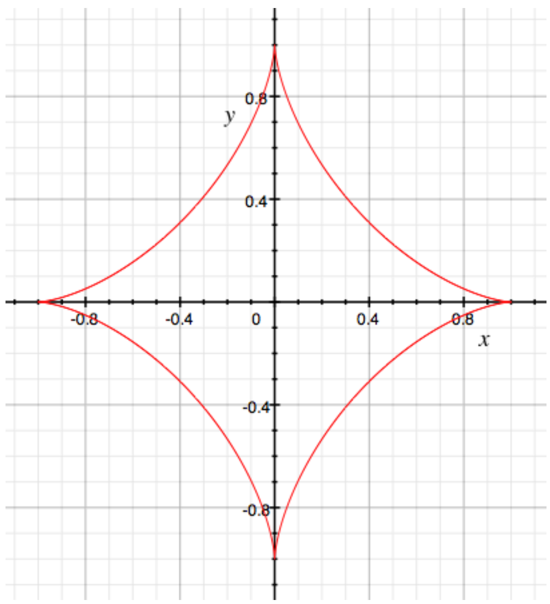
The intersection at $\cos(x^2) = \sin(x^2)$ happens when $x^2 = \frac{\pi}{4}$ or $x = \frac{\sqrt{\pi}}{2}$. The volume is thus

$$\begin{aligned} V &= \pi \int_0^{\frac{\sqrt{\pi}}{2}} \left\{\cos(x^2) - \sin(x^2)\right\} 2x\, dx \\ &= \pi \left[\sin(x^2) + \cos(x^2) \right]_0^{\frac{\sqrt{\pi}}{2}} \\ &= \pi \left[\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - \sin(0) - \cos(0) \right] \\ &= \pi \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 \right] = \boxed{\pi[\sqrt{2} - 1]}. \end{aligned}$$

4. *Sketch the graph of the four-cusped hypocycloid and find its length. (Text problem 300[17].)*

$$x(t) = a \sin^3 t, \qquad y(t) = a\cos^3 t, \qquad 0 \le t \le 2\pi.$$

We observe that $(x(t), y(t))$ move once around the origin but are pulled inward relative to a circle. For $a = 1$ we may write $y = \pm\left(1 - |x|^{\frac{2}{3}}\right)^{\frac{3}{2}}$.



The infinitesimal length for a parametric curve is

$$ds = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt$$

Because

$$\dot{x} = 3a \sin^2 t\, \cos t, \qquad \dot{y} = -3a\cos^2 t\, \sin t$$

we get using $\sin 2t = 2 \sin t\, \cos t$

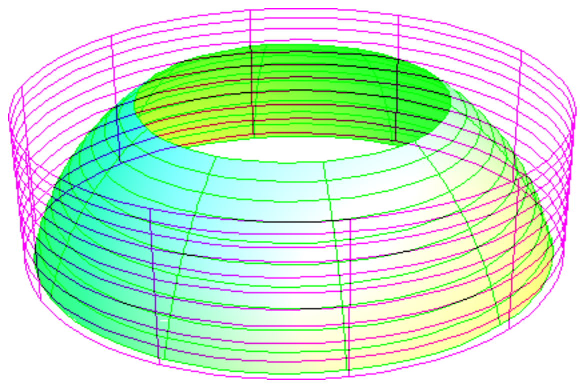
$$\begin{aligned} \dot{x}^2 + \dot{y}^2 &= 9a^2 \left(\sin^4 t\, \cos^2 t + \cos^4 t\, \sin^2 t \right) \\ &= 9a^2 \left(\sin^2 t + \cos^2 t \right) \sin^2 t\, \cos^2 t \\ &= 9a^2 \sin^2 t\, \cos^2 t \\ &= \frac{9}{4} a^2 \sin^2 2t \end{aligned}$$

Each of the four quadrants adds the same length so the total

$$S = 4\int_0^{\frac{\pi}{2}} ds = 4\int_0^{\frac{\pi}{2}} \frac{3a}{2}\sin 2t\,dt = 3a\Big[-\cos 2t\Big]_0^{\frac{\pi}{2}} = 3a\Big[-\cos\pi + \cos 0\Big] = \boxed{6a}.$$

We have used $\sin 2t \ge 0$ for $0 \le t \le \frac{\pi}{2}$. The length is slightly shorter than that of the circle of radius a whose circumference is $2\pi a$.

5. *Show that the area of part of the surface of the sphere of radius a between two parallel planes h units apart ($h < 2a$) is $2\pi ah$. Thus show that if a right cylinder is circumscribed about the sphere, then the two planes parallel to the base of the cylinder bound regions of the same area on the sphere and the cylinder. (Problem 300[32] from the text.)*



Horizontal planes cut a region from the sphere and from the cylinder that girds the sphere. Both regions have the same area. The flattening of the sphere exactly compensates for smaller radius of the ribbon of area.

Let us suppose that the radii of the sphere and cylinder are given by

$$f(x) = \sqrt{a^2 - x^2}; \qquad g(x) = a$$

Note that

$$\dot f(x) = -\frac{x}{\sqrt{a^2 - x^2}}; \qquad \dot g(x) = 0$$

so

$$1 + \dot f^2(x) = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}.$$

Let us cut the sphere and cylinder by the planes $x = b$ and $x = c$ where $-a \le b \le c \le a$, satisfying the condition that $h = c - b \le a - (-a) = 2a$. The element of slant length are respectively

$$ds_{\text{sph}} = \sqrt{1 + \dot f^2(x)}\,dx = \frac{a\,dx}{\sqrt{a^2 - x^2}}; \qquad ds_{\text{cyl}} = \sqrt{1 + \dot g^2(x)}\,dx = dx$$

Computing the area of the spherical surface of revolution between the $x = b$ and $x = c$ planes

$$\begin{aligned} A_{\text{sph}} &= 2\pi \int_b^c f(x)\left(1 + \dot f^2(x)\right)^{\frac{1}{2}}\,dx \\ &= 2\pi a \int_b^c \left(a^2 - x^2\right)^{\frac{1}{2}}\left(a^2 - x^2\right)^{-\frac{1}{2}}\,dx \\ &= 2\pi a \int_b^c dx \\ &= \boxed{2\pi a(c - b)}. \end{aligned}$$

On the other hand, computing the area of the cylindrical surface of revolution between the same planes

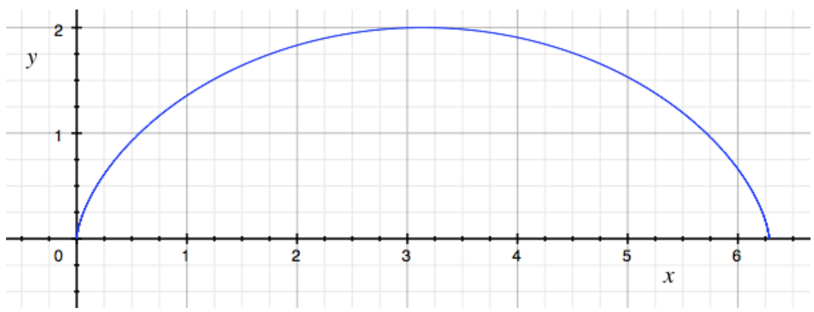
$$\begin{aligned} A_{\text{cyl}} &= 2\pi \int_b^c g(x) \left(1 + \dot{g}^2(x)\right)^{\frac{1}{2}} \, dx \\ &= 2\pi \int_b^c a \cdot 1 \, dx \\ &= \boxed{2\pi a(c-b)} \end{aligned}$$

gives the same area.

6. *One arch of a cycloid generated by a point on the rim of a wheel of radius a rolling along the x-axis has the parametric equations*

$$x(t) = a(t - \sin t), \qquad y(t) = a(1 - \cos t), \qquad \textit{for } 0 \le t \le 2\pi$$

Find the area of the surface generated by revolving this arch about the x-axis. (Text problem 300[33].)



Imagine chopping the surface perpendicular to the x-axis. The infinitesimal circular ribbons have area equal to circumference times the infinitesimal arclength

$$dA = 2\pi y(t) \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} \, dt$$

Now

$$\begin{aligned} \dot{x}(t) &= a(1 - \cos t), \\ \dot{y}(t) &= a \sin t \\ \dot{x}^2(t) + \dot{y}^2(t) &= a^2(1 - \cos t)^2 + a^2 \sin^2 t \\ &= a^2 \left(1 - 2\cos t + \cos^2 t + \sin^2 t\right) \\ &= 2a^2 (1 - \cos t) \\ dA &= 2\sqrt{2}\pi a^2 (1 - \cos t)^{\frac{3}{2}} \, dt \end{aligned}$$

To integrate this, we use the double angle formula. Subtracting

$$\begin{aligned} 1 &= \cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right) \\ \cos t &= \cos^2\left(\frac{t}{2}\right) - \sin^2\left(\frac{t}{2}\right) \\ 1 - \cos t &= 2\sin^2\left(\frac{t}{2}\right) \end{aligned}$$

Because $0 \le t \le 2\pi$ we have $\sin\left(\frac{t}{2}\right) \ge 0$ so the element of area becomes

$$\begin{aligned}
dA &= 2\sqrt{2}\pi a^2 \left(2\sin^2\left(\frac{t}{2}\right)\right)^{\frac{3}{2}} \, dt \\
&= 8\pi a^2 \sin^3\left(\frac{t}{2}\right) \, dt \\
&= 8\pi a^2 \left[\sin^2\left(\frac{t}{2}\right)\right] \sin\left(\frac{t}{2}\right) \, dt \\
&= 8\pi a^2 \left[1 - \cos^2\left(\frac{t}{2}\right)\right] \sin\left(\frac{t}{2}\right) \, dt
\end{aligned}$$

Thus the total area becomes

$$\begin{aligned}
A &= 8\pi a^2 \int_{t=0}^{2\pi} \left[1 - \cos^2\left(\frac{t}{2}\right)\right] \sin\left(\frac{t}{2}\right) \, dt \\
&= -16\pi a^2 \int_{u=1}^{-1} \left[1 - u^2\right] \, du \\
&= -16\pi a^2 \left[u - \frac{1}{3}u^3\right]_{u=1}^{-1} \\
&= -16\pi a^2 \left[-1 - \frac{1}{3}(-1)^3 - 1 + \frac{1}{3}\right] \\
&= \boxed{\frac{64}{3}\pi a^2}
\end{aligned}$$

where we set $u = \cos\left(\frac{t}{2}\right)$ so $du = -\frac{1}{2}\sin\left(\frac{t}{2}\right) \, dt$, $u = 1$ when $t = 0$ and $u = -1$ when $t = 2\pi$.

7. *It requires* 0.05 *joule (newton-meter) of work to stretch a spring from length* 8 *centimeters to* 9 *centimeters and another* 0.10 *joule to stretch it from* 9 *centimeters to* 10 *centimeters. Determine the spring constant and find the natural length of the spring. (Text problem 306[4].)*

Assuming that the spring follows Hooke's Law, the force in newtons required to stretch a spring to x meters if its its natural length ℓ meters is

$$F(x) = k(x - \ell)$$

Then the energy in joules required to stretch this spring from a to b meters is

$$W(a,b) = \int_a^b F(x)\, dx = \int_a^b k(x-\ell)\, dx = \left[\frac{k}{2}x^2 - k\ell x\right]_a^b = \frac{k}{2}(b^2 - a^2) - k\ell(b-a).$$

The given data converted to meters tells us

$$\begin{aligned}
0.05 = W(.08, .09) &= \frac{k}{2}\left((.09)^2 - (.08)^2\right) - k\ell(.09 - .08) = .00085k - .01k\ell \\
0.10 = W(.09, .10) &= \frac{k}{2}\left((.10)^2 - (.09)^2\right) - k\ell(.10 - .09) = .00095k - .01k\ell
\end{aligned}$$

Subtracting

$$0.05 = .00010k$$

yields $\boxed{k = 500}$ joules per meter. Substituting k into the first equation

$$0.05 = .00085(500) - .01(500)\ell = 0.425 - 5\ell$$

so that $\boxed{\ell = .075}$ meters which is 7.5 centimeters.

8. *A tank shaped like the frustum of a cone has circular horizontal cross sections of radius* $4+x$ *feet at the height* x *feet above the base. Find the work done pumping all the oil (density* $\delta = 50$ *pounds per cubic foot) over the edge of the tank. Assume that the tank is* 10 *feet high and is full of oil. (Text problem 306[14].)*

The amount of work to lift the horizontal disk which is x feet above the base of the tank is

$$dW = (\text{distance lifted}) \times (\text{weight of disk}) = (10 - x) \cdot \delta\pi(4 + x)^2\, dx$$

where we put the weight of the disk equal to density times volume. Integrating, we find

$$\begin{aligned}
W &= \int_0^{10} (10 - x) \cdot \delta\pi(4 + x)^2\, dx \\
&= \delta\pi \int_0^{10} (10 - x)(16 + 8x + x^2)\, dx \\
&= \delta\pi \int_0^{10} 160 + 64x + 2x^2 - x^3\, dx \\
&= \delta\pi \left[160x + 32x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^{10} \\
&= \delta\pi \left[160 \cdot 10 + 32(10)^2 + \frac{2}{3}(10)^3 - \frac{1}{4}(10)^4\right] \\
&= 2966.667\delta\pi = \boxed{466003 \text{ foot-pounds}}.
\end{aligned}$$

9. *One cubic foot of gas under a pressure of* 80 *pounds per square inch expands adiabatically to* 4 *cubic feet according to the law* $pv^{1.4} = c$*. Find the work done by the gas. (Problem 306[18] of the text.)*

The initial data lets us compute the pressure constant

$$c = pv^{1.4} = (80)(1^{1.4}) = 80.$$

where c is adjusted for volume measured in cubic feet. Let us assume that the gas is in a piston with cross sectional area A square feet. Then the piston is initially $x_0 = \frac{1}{A}$ feet long. It is $x_1 = \frac{4}{A}$ long at the end. The pressure in the piston of length x is thus

$$p(x) = 80 \cdot (Ax)^{-1.4}$$

pounds per square inch. Then the force on the piston is $144Ap$ pounds (there are 144 square inches in a square foot) and the work done to move the piston from x to $x + dx$ is

$$dW = 144Ap\, dx = 144A \cdot 80 \cdot (Ax)^{-1.4}\, dx$$

Thus the work done by the gas is

$$\begin{aligned}
W &= 144 \cdot 80A^{-.4} \int_{x_0}^{x_1} x^{-1.4}\, dx = 144 \cdot 80A^{-.4} \left[-\frac{1}{.4}x^{-.4}\right]_{\frac{1}{A}}^{\frac{4}{A}} \\
&= -144 \cdot 200A^{-.4} \left[\left(\frac{4}{A}\right)^{-.4} - \left(\frac{1}{A}\right)^{-.4}\right] = \boxed{-144 \cdot 200 \left[4^{-.4} - 1\right]} = 12258.74 \text{ ft-lbs}
\end{aligned}$$

Note that the size of the piston cancelled out.

10. *A 10 pound monkey hangs at the end of a 20 foot chain that weighs $\frac{1}{2}$ pound per foot. How much work does it do climbing the chain to the top? Assume that the end of the chain is attached to the monkey. (Problem 306[20] from the text.)*

Let $w_0 = 10$ pounds be the weight of the monkey and $\delta = 0.5$ in pounds per foot be the density of the chain. We imagine that half of the the chain beneath the monkey droops straight down and the other half returns straight up back to the monkey. The monkey has lifted itself as well as half of the chain beneath to climb up x feet. Thus the downward force (weight) it faces given by

$$f(x) = w_0 + \frac{\delta}{2}x \text{ pounds}$$

Then the total work done is

$$W = \int_a^b f(x)\,dx = \int_0^{20} w_0 + \frac{\delta}{2}x\,dx = \left[w_0 x + \frac{\delta}{4}x^2\right]_0^{20}$$
$$= 20w_0 + \frac{\delta}{4}20^2 = 20 \cdot 10 + \frac{\frac{1}{2}}{4}20^2 = \boxed{250 \text{ foot-pounds}}$$

11. *A space capsule weighing 5000 pounds is propelled to an altitude of 200 miles above the surface of the earth. How much work is done against the force of gravity? Assume that the earth is a sphere of radius 4000 miles and that the force of gravity is $f(x) = -k/x^2$, where x is the distance between the center of the earth to the capsule (inverse square law). Thus the lifting force required is k/x^2, and equals 5000 pounds when $x = 4000$ miles. (Text problem 306[21]).*

The force is close to 5000 pounds for the whole trip and the distance is 200 miles, so an estimate of the work is 1000000 mile-pounds. Let us work out the constant. On the ground, the weight in pounds is

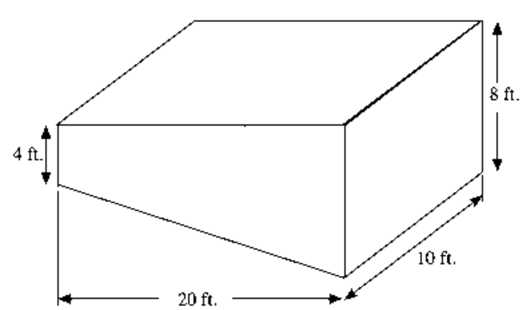
$$5000 = f(4000) = \frac{k}{4000^2}$$

which implies $k = 8 \times 10^{10}$ mile^2 lb. The work to lift another 200 miles is

$$W = \int_a^b f(x)\,dx = \int_{4000}^{4200} \frac{k}{x^2}\,dx = \left[-\frac{k}{x}\right]_{4000}^{4200}$$
$$= -\frac{k}{4200} - \left(-\frac{k}{4000}\right) = \boxed{952381 \text{ mile-pounds}}$$

The book solution has an error.

12. *Find the total force exerted by the fluid against the bottom of the swimming pool shown in the figure, assuming it is full of water. (Text problem 306[33].)*



Let x be distance along the length. Then the depth $g(x)$ is a linear function that is 4 ft. at the $x = 0$ end and 8 ft. at the $x = 20$ end. Hence

$$g(x) = 4 + \frac{1}{5}x$$

The force on the bottom depends on the pressure which depends on depth. The force on a horizontal area A sqare feet at depth $g(x)$ feet equals the weight of water over A so

$$pA = \text{force} = \delta A g(x)$$

or

$$p(x) = \delta g(x)$$

in pounds per square foot, where $\delta = 62.4$ pounds per cubic foot is the density of water. The pressure on a little strip across the bottom is

$$dF = pA = \delta g(x) w_0 ds$$

where $w_0 = 10$ ft is the width of the pool and

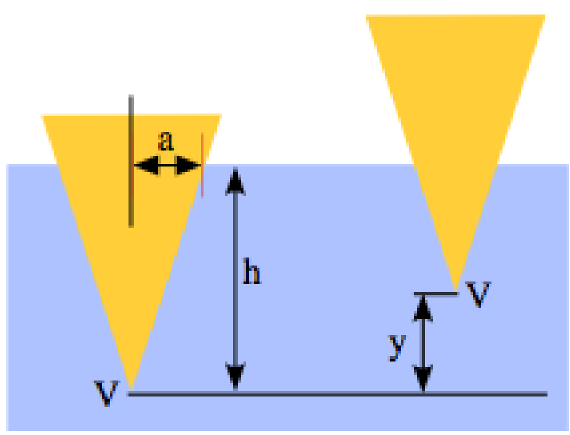
$$ds = \sqrt{1 + \dot{g}^2(x)}\, dx = \sqrt{1 + \frac{1}{25}}\, dx = \frac{\sqrt{26}}{5}\, dx$$

is the slant length of the strip. Integrating, we find the total force

$$\begin{aligned} F &= \frac{\sqrt{26}}{5}\delta w_0 \int_0^{20} 4 + \frac{1}{5}x\, dx = \frac{\sqrt{26}}{5}\delta w_0 \left[4x + \frac{1}{10}x^2\right]_0^{20} \\ &= \frac{\sqrt{26}}{5}\delta w_0 \left[4 \cdot 20 + \frac{1}{10} \cdot 20^2\right] = 122.3765\delta w_0 = \boxed{76362.92 \text{ pounds}} \end{aligned}$$

Ths differs from the text solution, which didn't take the slant of the bottom into account.

13. *A conical buoy weighs m pounds and floats with its vertex down and h feet below the surface of the water as in the figure. A boat crane lifts the buoy to the deck so that V is 15 feet above the water surface. How much work is done? Hint: use Archimedes Principle, which says the force required to hold the buoy y feet above its original position $0 \leq y \leq h$ is equal to the weight minus the weight of the water displaced by the buoy. (Text problem 306[35].)*



Let D cubic feet be the amount of water displaced by the buoy when it is lifted y feet, $0 \le y \le h$. It is the volume of a cone with radius $r = \frac{a}{h}(h-y)$ at the water's surface and height $h-y$ from the surface to V

$$D = \frac{\pi}{3} r^2 (h-y) = \frac{\pi a^2}{3h^2}(h-y)^3$$

Since the buoy is floating when $y = 0$, the weight of the water displaced equals the weight of the buoy, or

$$m = \frac{\pi a^2 \delta h}{3} \text{ lbs}$$

where δ pounds per cubic foot is the density of water. As long as some of the buoy is in the water, by Archimedes Principle, the force needed to to hold the buoy at y ft, where $0 \le y \le h$ is the weight of the buoy minus the weight of the water displaced. Solving for δ,

$$F(y) = m - \delta D = m - \left(\frac{3m}{\pi a^2 h}\right)\left\{\frac{\pi a^2}{3h^2}(h-y)^3\right\} = m\left\{1 - \frac{(h-y)^3}{h^3}\right\}$$

Once the tip is out of the water, there is no more buoyant force so for $y \ge h$,

$$F(y) = m.$$

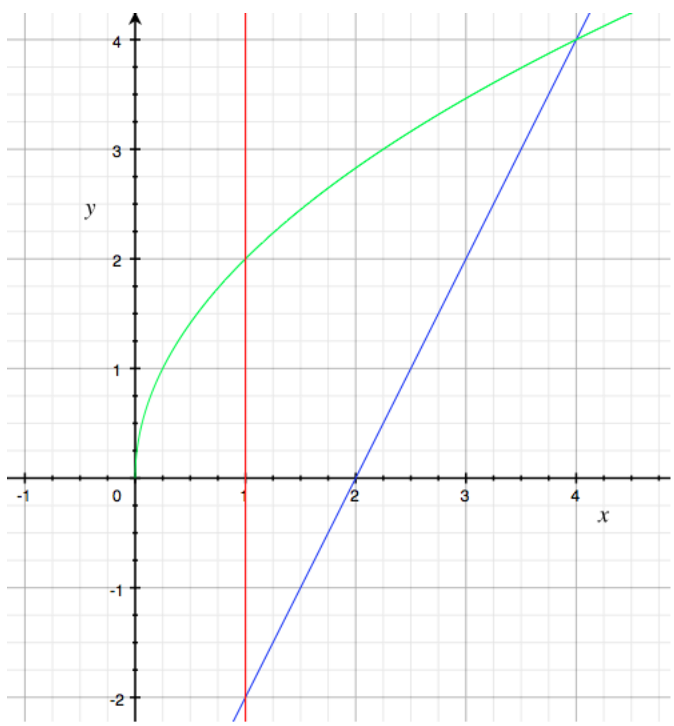
Thus the total work to lift the buoy $h + 15$ ft is

$$W = \int_0^{h+15} F(x)\,dx = \int_0^h m\left\{1 - \frac{(h-y)^3}{h^3}\right\}dx + \int_h^{h+15} m\,dx$$

$$= m\left[y + \frac{(h-y)^4}{4h^3}\right]_0^h + 15m = m\left[h - \frac{h^4}{4h^3}\right] + 15m = \boxed{\frac{3}{4}mh + 15m}$$

14. *Find the centroid of the region bounded by the given curves. (Text problem 313[13].)*

$$y = 2x - 4; \qquad y = 2\sqrt{x}; \qquad x = 1$$



The intersection is at

$$2\sqrt{x} = y = 2x - 4$$

or

$$0 = (2x-4)^2 - 4x = 4x^2 - 20x + 16 = 4(x^2 - 5x + 4) = 4(x-1)(x-4)$$

At $x = 1$ the value $2x - 4 < 0$ so can't equal $2\sqrt{x}$. Thus the only root is $x = 4$. The figure is roughly a triangle with vertices $(1, 2)$,$(1, -2)$ and $(4, 4)$ so the average of three points $(2, \frac{4}{3})$ is an estimate for the centroid.

The centroid is the center of mass assuming that the density is constant, $\delta = 1$, say. The total mass is the area between the upper and lower curves

$$m = \int_1^4 2\sqrt{x} - (2x-4)\,dx = \left[\frac{4}{3}x^{\frac{3}{2}} - x^2 + 4x\right]_1^4 = \left[\frac{4}{3}\cdot 8 - 16 + 16 - \frac{4}{3} + 1 - 4\right] = \frac{19}{3}$$

The moment about the y-axis is

$$\begin{aligned} M_y &= \int_1^4 x\left[2\sqrt{x} - (2x-4)\right]\,dx = \left[\frac{4}{5}x^{\frac{5}{2}} - \frac{2}{3}x^3 + 2x^2\right]_1^4 \\ &= \frac{4}{5}\cdot 32 - \frac{2}{3}\cdot 64 + 2\cdot 16 - \frac{4}{5} + \frac{2}{3} - 2 = \frac{64}{5} \end{aligned}$$

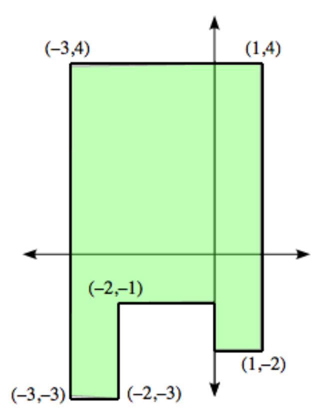
The moment about x axis is more conveniently found from the alternate formula of the weighted sums of centroids of vertical strips

$$\begin{aligned} M_x &= \frac{1}{2}\int_1^4 \left(2\sqrt{x}\right)^2 - (2x-4)^2\,dx = \frac{1}{2}\int_1^4 4x - 4x^2 + 16x - 16\,dx \\ &= \frac{1}{2}\left[10x^2 - \frac{4}{3}x^3 - 16x\right]_1^4 = \frac{1}{2}\left[10\cdot 16 - \frac{4}{3}\cdot 64 - 16\cdot 4 - 10 + \frac{4}{3} + 16\right] = 9 \end{aligned}$$

The centroid is thus

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{\frac{64}{5}}{\frac{19}{3}}, \frac{9}{\frac{19}{3}}\right) = \boxed{\left(\frac{192}{95}, \frac{27}{19}\right)}$$

15. *Find the centroid of the region in the diagram. (Text problem 313[22].)*



The mass and moments may be found for sub-rectangles whose union is the region and added. Let us subdivide the region into three pieces. Other subdivisions are possible too.

Let $\mathcal{R}_1$ be the rectangle whose vertices are $(-3,4)$, $(-3,-3)$, $(-2,-3)$ and $(-2,4)$. Let $\mathcal{R}_2$ be the rectangle whose vertices are $(-2,4)$, $(-2,-1)$, $(0,-1)$ and $(0,4)$. Let $\mathcal{R}_3$ be the rectangle whose vertices are $(0,4)$, $(0,-2)$, $(1,-2)$ and $(1,4)$. The width and height of the rectangles are 1×7, 2×5 and 1×6, respectively. Therefore their areas (masses) are

$$m_1 = 7, \qquad m_2 = 10, \qquad m_3 = 6$$

The centers of mass are the centers of the rectangles. Rearranging the equations

$$M_i^y = m_i \bar{x}_i, \qquad M_i^x = m_i \bar{y}_i.$$

The moments are the length of the lever arm at the center times the area. The centers of the rectangles are $(-2.5, .5)$, $(-1, 1.5)$ and $(.5, 1)$, respectively. Thus the moments are

$$M_1^y = 7\cdot(-2.5) = -17.5, \qquad M_2^y = 10\cdot(-1) = -10, \qquad M_3^y = 6\cdot 0.5 = 3,$$
$$M_1^x = 7\cdot 0.5 = 3.5, \qquad M_2^x = 10\cdot 1.5 = 15, \qquad M_3^x = 6\cdot 1 = 6.$$

The moments and masses add so we may find the centroid of the total region

$$\bar{x} = \frac{M_1^y + M_2^y + M_3^y}{m_1+m_2+m_3} = \frac{-17.5 - 10 + 3}{7+10+6} = \boxed{\frac{-49}{46}},$$
$$\bar{y} = \frac{M_1^x + M_2^x + M_3^x}{m_1+m_2+m_3} = \frac{3.5 + 15 + 6}{7+10+6} = \boxed{\frac{49}{46}}.$$

16. *Use Pappus's Theorem to find the centroid of a semicircular region or radius a. (Text problem 313[25].)*

The region in question is bounded by the curves

$$y = \sqrt{a^2 - x^2}, \qquad y = 0$$

which is the upper semicircle. By left-right symmetry, $\boxed{\bar{x} = 0}$. If $\bar{y}$ is the y-coordinate of the centroid, then according to Pappus's Theorem, the volume of the solid generated by revolving the region about the x-axis is given by the length that the centroid travels in revolution times the area A of the region

$$V = 2\pi \bar{y} A$$

We know that the volume of a sphere and the area of the semicircle are

$$V = \frac{4}{3}\pi a^3, \qquad A = \frac{\pi}{2}a^2.$$

Solving for $\bar{y}$ we find

$$\bar{y} = \frac{V}{2\pi A} = \frac{\frac{4}{3}\pi a^3}{\pi^2 a^2} = \boxed{\frac{4a}{3\pi}} = 0.4244132a.$$

17. *Let $f(x)$ be a nonnegative continuous function on $[0,1]$. Show that*

$$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$

Use this formula to evaluate the integral. (Text problem 313[32].)

$$\int_0^\pi x \sin x \, \cos^4 x\,dx$$

This is just the formula $M_y = \bar{x}m$. The mass (area) under the curve $y = f(\sin x)$ is

$$m = \int_0^{\pi} f(\sin x)\,dx$$

The center of mass is $\bar{x} = \frac{\pi}{2}$ because the function $y = f(\sin x) = g(x)$ is symmetric about the line $x = \frac{\pi}{2}$.

To see the whole proof based on the symmetry of $f(\sin x)$, note that for $0 \leq x \leq \pi$,

$$g(x) = f(\sin x) = f\big(\sin(\pi - x)\big) = g(\pi - x).$$

But then by alternately substituting $u = \pi - x$ so $x = \pi - u$, $du = -dx$, $u = \frac{\pi}{2}$ when $x = \frac{\pi}{2}$ and $u = 0$ when $x = \pi$, and then substituting $u = x$,

$$\begin{aligned}
\int_{x=0}^{\pi} xg(x)\,dx &= \int_{x=0}^{\frac{\pi}{2}} xg(x)\,dx + \int_{x=\frac{\pi}{2}}^{\pi} xg(x)\,dx \\
&= \int_0^{\frac{\pi}{2}} xg(x)\,dx - \int_{u=\frac{\pi}{2}}^{0} (\pi - u)g(\pi - u)\,du \qquad \text{[Here we let } u = \pi - x.\text{]} \\
&= \int_0^{\frac{\pi}{2}} xg(x)\,dx + \int_{u=0}^{\frac{\pi}{2}} (\pi - u)g(u)\,du \qquad \text{[Use } g(\pi - u) = g(u).\text{]} \\
&= \int_0^{\frac{\pi}{2}} xg(x)\,dx + \int_{x=0}^{\frac{\pi}{2}} (\pi - x)g(x)\,dx \qquad \text{[Here we let } x = u.\text{]} \\
&= \int_0^{\frac{\pi}{2}} \{x + \pi - x\}g(x)\,dx \\
&= \pi \int_0^{\frac{\pi}{2}} g(x)\,dx \\
&= \frac{\pi}{2} \int_{x=0}^{\frac{\pi}{2}} g(x)\,dx + \frac{\pi}{2} \int_0^{\frac{\pi}{2}} g(x)\,dx \\
&= \frac{\pi}{2} \int_{x=0}^{\frac{\pi}{2}} g(x)\,dx + \frac{\pi}{2} \int_{u=0}^{\frac{\pi}{2}} g(u)\,du \qquad \text{[Here we let } u = x.\text{]} \\
&= \frac{\pi}{2} \int_{x=0}^{\frac{\pi}{2}} g(x)\,dx - \frac{\pi}{2} \int_{x=\pi}^{\frac{\pi}{2}} g(\pi - x)\,dx \qquad \text{[Here we let } x = \pi - u.\text{]} \\
&= \frac{\pi}{2} \int_{x=0}^{\frac{\pi}{2}} g(x)\,dx + \frac{\pi}{2} \int_{x=\frac{\pi}{2}}^{\pi} g(x)\,dx \qquad \text{[Use } g(\pi - x) = g(x).\text{]} \\
&= \frac{\pi}{2} \int_{x=0}^{\pi} g(x)\,dx.
\end{aligned}$$

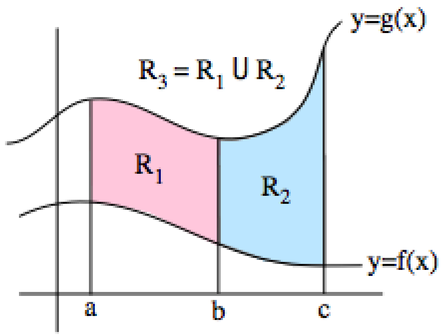
To see the application, consider $f(x) = x(1 - x^2)^2$ and apply the formula. Thus

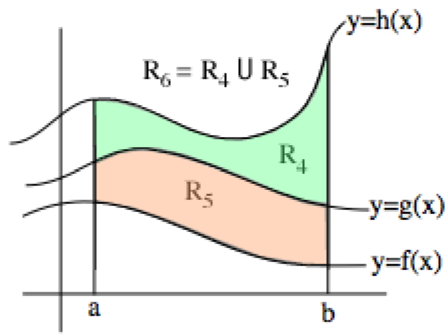
$$\begin{aligned}
\int_0^{\pi} x \sin x \, \cos^4 x \,dx &= \int_0^{\pi} x \sin x \, \left(1 - \sin^2 x\right)^2 \,dx \\
&= \int_0^{\pi} xf(\sin x)\,dx \\
&= \frac{\pi}{2} \int_0^{\pi} f(\sin x)\,dx \\
&= \frac{\pi}{2} \int_0^{\pi} \sin x \, \left(1 - \sin^2 x\right)^2 \,dx
\end{aligned}$$

$$= \frac{\pi}{2}\int_0^{\pi} \sin x\, \cos^4 x\, dx$$
$$= \frac{\pi}{2}\left[-\frac{1}{5}\cos^5 x\right]_0^{\pi} = \frac{\pi}{10}[-(-1)-(-1)] = \boxed{\frac{\pi}{5}}$$

18. *Consider the homogeneous laminas R_1, R_2, R_4, and R_5 shown in the figure and homogeneous laminas R_3 and R_6 which are unions R_1 and R_2, R_4 and R_5, respectively. Show that the mass and moments are additive. (Text problems 313[19,20].)*

$$m(R_3) = m(R_1) + m(R_2) \qquad m(R_6) = m(R_4) + m(R_5)$$
$$M_y(R_3) = M_y(R_1) + M_y(R_2) \qquad M_y(R_6) = M_y(R_4) + M_y(R_5)$$
$$M_x(R_3) = M_x(R_1) + M_x(R_2) \qquad M_x(R_6) = M_x(R_4) + M_x(R_5)$$





The additivity follows from additivities of the integral. For the masses interval additivity implies

$$m(R_3) = \int_a^c g(x) - f(x)\, dx = \int_a^b g(x) - f(x)\, dx + \int_b^c g(x) - f(x)\, dx = m(R_1) + m(R_2)$$

For moments about the y-axis, interval additivity implies

$$M_y(R_3) = \int_a^c x\big[g(x) - f(x)\big]\, dx$$
$$= \int_a^b x\big[g(x) - f(x)\big]\, dx + \int_b^c x\big[g(x) - f(x)\big]\, dx = M_y(R_1) + M_y(R_2)$$

Also from the alternate formula for moments about the x-axis, interval additivity implies

$$M_x(R_3) = \frac{1}{2}\int_a^c g^2(x) - f^2(x)\, dx$$
$$= \frac{1}{2}\int_a^b g^2(x) - f^2(x)\, dx + \frac{1}{2}\int_b^c g^2(x) - f^2(x)\, dx = M_x(R_1) + M_x(R_2)$$

For the second graph's masses, additivity of integrands implies

$$m(R_6) = \int_a^b h(x) - f(x)\, dx = \int_a^b \big[h(x) - g(x)\big] + \big[g(x) - f(x)\big]\, dx$$
$$= \int_a^b h(x) - g(x)\, dx + \int_a^b g(x) - f(x)\, dx = m(R_5) + m(R_6)$$

For the moments about the y-axis, additivity of integrands implies

$$\begin{aligned} M_y(R_6) &= \int_a^b x\big[h(x) - f(x)\big]\,dx = \int_a^b x\big[h(x) - g(x)\big] + x\big[g(x) - f(x)\big]\,dx \\ &= \int_a^b x\big[h(x) - g(x)\big]\,dx + \int_a^b x\big[g(x) - f(x)\big]\,dx = M_y(R_5) + M_y(R_6) \end{aligned}$$

For the moments about the x-axis using the alternate formula, additivity of integrands implies

$$\begin{aligned} M_x(R_6) &= \frac{1}{2}\int_a^b h^2(x) - f^2(x)\,dx = \frac{1}{2}\int_a^b \big[h^2(x) - g^2(x)\big] + \big[g^2(x) - f^2(x)\big]\,dx \\ &= \frac{1}{2}\int_a^b h^2(x) - g^2(x)\,dx + \frac{1}{2}\int_a^b g^2(x) - f^2(x)\,dx = M_x(R_5) + M_x(R_6) \end{aligned}$$

19. *Find the general solution for the differential equations. Then find a particular solution that satisfies the initial condition. (Problems 208[8,10,12].)*

   (a) $\dfrac{dy}{dx} = \sqrt{\dfrac{x}{y}}$; $\quad y = 4$ at $x = 1$.

   (b) $\dfrac{dy}{dt} = y^4$; $\quad y = 1$ at $t = 1$.

   (c) $\dfrac{du}{dt} = u^3(t^3 - t)$; $\quad y = 4$ at $t = 0$.

For (a.), assuming $x > 0$ and $y > 0$, separating variables we find

$$y^{\frac{1}{2}}\,dy = x^{\frac{1}{2}}\,dx$$

Integrating yields

$$\frac{2}{3}y^{\frac{3}{2}} = \frac{2}{3}x^{\frac{3}{2}} + C.$$

Solving for y gives the general solution

$$y = \boxed{\left(x^{\frac{3}{2}} + C'\right)^{\frac{2}{3}}}$$

where we put $C' = \frac{3}{2}C$. At the initial point $y = 4$ when $x = 1$ we solve for C'

$$4^{\frac{3}{2}} = 1^{\frac{3}{2}} + C'$$

so $C' = 7$. Thus the particular solution is

$$y = \boxed{\left(x^{\frac{3}{2}} + 7\right)^{\frac{2}{3}}}$$

For (b.), assuming $y > 0$, separating variables we find

$$y^{-4}\,dy = dt$$

Integrating yields

$$-\frac{1}{3}y^{-3} = t + C.$$

Solving for y gives the general solution

$$y = \boxed{(C' - 3t)^{-\frac{1}{3}}}$$

where we put $C' = -3C$. At the initial point $y = 1$ when $t = 1$ we solve for C'

$$1^{-3} = C' - 3 \cdot 1.$$

so $C' = 4$. Thus the particular solution is

$$y = \boxed{(4 - 3t)^{-\frac{1}{3}}}$$

For (c.), assuming $u > 0$, separating variables we find

$$u^{-3}\, du = (t^3 - t)\, dt$$

Integrating yields

$$-\frac{1}{2}u^{-2} = \frac{1}{4}t^4 - \frac{1}{2}t^2 + C.$$

Solving for y gives the general solution

$$u = \boxed{\left(C' - \frac{1}{2}t^4 + t^2\right)^{-\frac{1}{2}}}$$

where we put $C' = -2C$. At the initial point $u = 4$ when $t = 0$ we solve for C'

$$4^{-2} = C' - \frac{1}{2} \cdot 0 + 0^2.$$

so $C' = \frac{1}{16}$. Thus the particular solution is

$$u = \boxed{\left(\frac{1}{16} - \frac{1}{2}t^4 + t^2\right)^{-\frac{1}{2}}}$$

20. *A ball is thrown upward from the surface of the earth with an initial velocity of* 96 *feet per second. What is the maximum height it reaches? (Text problem 208[22].)*

    The acceleration due to gravity near the surface is $g = -32$ feet per second per second. Newton's Law gives the equation of motion of the ball starting at the surface with upward velocity. Let y be the height in feet at time t seconds.

$$ma = m\frac{d^2y}{dt^2} = -mg, \qquad y(0) = 0, \quad y'(0) = 96.$$

    Its integrals (the general solutions) are

$$v = \frac{dy}{dt} = -gt + C_1, \qquad y = -\frac{1}{2}gt^2 + C_1 t + C_2.$$

    From the initial velocity $96 = v(0) = -g \cdot 0 + C_1$ so $C_1 = 96$. From the initial position, $0 = y(0) = -\frac{1}{2}g0^2 + 96 \cdot 0 + C_2$ so $C_2 = 0$. The position of the ball is thus

$$y(t) = -16t^2 + 96t.$$

The ball reaches the maximum at time T when $v(T) = 0$ or

$$0 = \frac{dy}{dt}(T) = -32T + 96$$

or at $T = 3$ sec. At that time, the maximum height is

$$y(3) = -16 \cdot 3^2 + 96 \cdot 3 = \boxed{144 \text{ feet}}$$

21. *The rate of change of volume V of a melting snowball is proportional to the surface area S of the ball. If the radius of the ball at $t = 0$ is $r = 2$ and at $t = 10$ is $r = 0.5$, find the radius as a function of time. (Text problem 208[25].)*

The rate of change of volume is given by the equation

$$\frac{dV}{dt} = -kS$$

where k is the proportionality constant. Relating this to the radius of the snowball, we find

$$V = \frac{4}{3}\pi r^3; \qquad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}; \qquad S = 4\pi r^2.$$

Thus the differential equation in terms of radius reads

$$4\pi r^2 \frac{dr}{dt} = \frac{dV}{dt} = -kS = -4k\pi r^2$$

Separating variables we find

$$dr = -k\,dt$$

whose solution is

$$r(t) = -kt + C.$$

At time $t = 0$, we have

$$2 = r(0) = -k \cdot 0 + C$$

so $C = 2$. At $t = 10$, we have

$$0.5 = r(10) = -k \cdot 10 + 2$$

or $k = \frac{3}{20}$. Thus the formula for the radius is

$$r(t) = \boxed{-\frac{3}{20}t + 2}$$

22. *determine the escape velocity for an object launched from each of the following celestial bodies. Here $g \approx 32$ feet per second per second. (Text problem 208[27].)*

Celestial Body	Acceleration of Gravity	Radius (Miles)
Moon	$-0.165g$	1080
Venus	$-0.85g$	3800
Jupiter	$-2.6g$	43000
Sun	$-28g$	432000

Let's recall the computation of the escape velocity. The derivation in the text computes the velocity of a projectile shot from the surface. The escape velocity is the smallest velocity for which gravitational attraction fails to return the projectile back to the surface. The force of attraction for an object of mass m at a distance s from the center of the planet is given by the inverse square law

$$F = -\frac{m\tilde{g}R^2}{s^2}$$

where $\tilde{g}$ miles per second per second is the acceleration of gravity at the surface of the planet and R miles is the radius of the planet. We deduce the differential equation for the velocity $v(t)$ in miles per second of a projectile launched from the surface at initial velocity $v(0) = v_0$. The calculation neglects atmospheric resistance. Newton's Law $ma = F$ and the Chain Rule say

$$m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds} = -\frac{m\tilde{g}R^2}{s^2}.$$

Separating variables

$$v\,dv = -\frac{\tilde{g}R^2}{s^2}\,ds$$

Integrating

$$\frac{1}{2}v^2 = \frac{\tilde{g}R^2}{s} + C$$

Using the initial condition $v(R) = v_0$,

$$\frac{1}{2}v_0^2 = \frac{\tilde{g}R^2}{R} + C$$

Thus the velocity satisfies

$$v^2(t) = \frac{2\tilde{g}R^2}{s} + v_0^2 - 2\tilde{g}R$$

The projectile returns to the surface if the right side becomes negative. But this never happens if

$$v_0 \geq \sqrt{2\tilde{g}R}$$

The *escape velocity* is $\sqrt{2\tilde{g}R}$. We compute the escape velocity for the planets in the table. For example, the escape velocity for the earth is $\sqrt{2 \cdot 32 \text{ ft/sec}^2 \cdot 1/5280 \text{ miles/ft} \cdot 3960 \text{ miles}} = 6.93$ miles/sec.

Body	Accel. Gravity	Radius (Miles)	Escape Vel. (Miles per Sec.)
Earth	$-g$	3960	6.93
Moon	$-0.165g$	1080	1.49
Venus	$-0.85g$	3800	6.26
Jupiter	$-2.6g$	43000	36.8
Sun	$-28g$	432000	383

23. *According to Toricelli's Law, the time rate of change of volume V of water in a draining tank is proportional to the square root of water's depth. A cylindrical tank of radius $\frac{10}{\sqrt{\pi}}$ centimeters and height 16 centimeters, which was full initially, took 40 seconds to drain. Find the volume of water at time t. What is the volume when $t = 10$? (Text problem 208[35].)*

If y centimeters is the depth of the water in the tank, then the corresponding volume is

$$V = \pi r^2 y = \pi \left(\frac{10}{\sqrt{\pi}} \right)^2 y = 100y.$$

If initially $y = 16$ entimeters then $V(0) = 1600$ cubic centimeters. Toricelli's Law says the volume satisfies

$$\frac{dV}{dt} = k\sqrt{y} = k\sqrt{\frac{V}{100}} = k'V^{\frac{1}{2}}$$

where $k' = \frac{k}{10}$ is the proportionality constant. Separating variables

$$V^{-\frac{1}{2}}\, dV = k'\, dt$$

The general solution is

$$2V^{\frac{1}{2}} = k't + C.$$

At the initial time

$$2 \cdot 1600^{\frac{1}{2}} = k' \cdot 0 + C$$

so $C = 80$. We are also told that it took 40 seconds to reach zero volume, so

$$2 \cdot 0^{\frac{1}{2}} = k' \cdot 40 + 80.$$

Thus $k' = -2$. Then the volume of water in the tank for $0 \le t \le 40$

$$V(t) = \boxed{(40 - t)^2}$$

Finally, at $t = 10$ the volume is $V(10) = \boxed{\text{900 cubic centimeters.}}$

24. *Using the left Riemann sum, right Riemann sum, Midpoint Rule, Trapezoid Rule and Simpson's Rule with $n = 8$ to approximate the following integral. Use the Second Fundamental Theorem of Calculus to find the exact value of the integral. (text problem 268[4].)*

$$I = \int_1^3 \left(x^2 + 1\right)^{\frac{1}{2}} x\, dx$$

We first find the exact value using FTC.

$$I = \left[\frac{\left(x^2 + 1\right)^{\frac{3}{2}}}{3} \right]_1^3 = \left[\frac{\left(3^2 + 1\right)^{\frac{3}{2}}}{3} - \frac{\left(1^2 + 1\right)^{\frac{3}{2}}}{3} \right] = \boxed{\frac{10^{\frac{3}{2}} - 2^{\frac{3}{2}}}{3}} = 9.598116.$$

We make a table of function values at $x_i = a + \frac{(b-a)i}{n}$ and $m_i = a + \left(i - \frac{1}{2}\right)\frac{(b-a)}{n}$.

x	f(x)
1.000	1.414214
1.125	1.693349
1.250	2.000976
1.375	2.337753
1.500	2.704163
1.625	3.100569
1.750	3.527238
1.875	3.984375
2.000	4.472136
2.125	4.990641
2.250	5.539983
2.375	6.120235
2.500	6.731456
2.625	7.373692
2.750	8.046981
2.875	8.751353
3.000	9.486833

Note for $a = 1$, $b = 3$ and $n = 8$ we have $\frac{b-1}{n} = \frac{1}{4}$. The left Riemann sum is

$$\begin{aligned} L_n &= \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right) = \frac{1}{4}\Big(f(1) + f(1.25) + \cdots + f(2.75)\Big) \\ &= \frac{1}{4}\Big(1.414214 + 2.000976 + 2.704163 + 3.527238 + \\ &\qquad + 4.472136 + 5.539983 + 6.731456 + 8.046981\Big) = \boxed{8.609287} \end{aligned}$$

The right Riemann sum is

$$\begin{aligned} R_n &= \frac{b-a}{\sum}^{n}_{i=1} f\left(a + i\frac{b-a}{n}\right) = \frac{1}{4}\Big(f(1.25) + f(1.5) + \cdots + f(2.75) + f(3)\Big) \\ &= \frac{1}{4}\Big(2.000976 + 2.704163 + 3.527238 + 4.472136 + \\ &\qquad + 5.539983 + 6.731456 + 8.046981 + 9.486833\Big) = \boxed{10.62744} \end{aligned}$$

The midpoint Riemann sum is

$$\begin{aligned} M_n &= \frac{b-a}{n}\sum_{i=1}^{n} f\left(a + \left(1 - \frac{1}{2}\right)\frac{b-a}{n}\right) = \frac{1}{4}\Big(f(1.125) + f(1.375) + \cdots + f(2.875)\Big) \\ &= \frac{1}{4}\Big(1.693349 + 2.337753 + 3.100569 + 3.984375 + \\ &\qquad + 4.990641 + 6.120235 + 7.373692 + 8.751353\Big) = \boxed{9.587992} \end{aligned}$$

The Trapezoid Rule is the average of right and left Riemann sums

$$\begin{aligned} T_n &= \frac{b-a}{2n}\left\{f(a) + 2\sum_{i=1}^{n-1} f\left(a + i\frac{b-a}{n}\right) + f(b)\right\} \\ &= \frac{b-a}{2n}\sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right) + \frac{b-a}{2n}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right) = \frac{1}{2}(L_n + R_n) \\ &\quad \frac{1}{2}(8.609287 + 10.62744) = \boxed{9.618364} \end{aligned}$$

Simpson's Rule is a convex combination of Trapezoid and Midpoint rules

$$\begin{aligned} S_{2n} &= \frac{b-a}{6n}\left(f(a) + 4f(m_1) + 2f(x_1) + 4f(m_2) + 2f(x_2) + \cdots + 2f(x_{n-1}) + 4f(m_n) + f(b)\right) \\ &= \frac{b-a}{6n}\left(f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\right) + \\ &\qquad + \frac{b-a}{6n}\left(4f(m_1) + 4f(m_2) + \cdots + 4f(m_n)\right) \\ &= \frac{b-a}{6n}\left\{f(a) + 2\sum_{i=1}^{n-1} f\left(a + i\frac{b-a}{n}\right) + f(b)\right\} + \frac{4(b-a)}{6n}\sum_{i=1}^{n} f\left(a + \left(i - \frac{1}{2}\right)\frac{b-1}{n}\right) \\ &\qquad = \frac{1}{3}T_n + \frac{2}{3}M_n = \frac{1}{3}\cdot 9.618364 + \frac{2}{3}\cdot 9.587992 = \boxed{9.598116} \end{aligned}$$

Simpson's Rule for n points does not use the m_i's so is less accurate.

$$\begin{aligned} S_n &= \frac{b-a}{3n}\left(f(a) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2} + 4f(x_{n-1}) + f(b)\right) \\ &= \frac{b-a}{3n}\left[f(a) + 4\sum_{i=1}^{n/2} f\left(a + (2i-1)\frac{b-a}{n}\right) + 2\sum_{i=1}^{n/2-1} f\left(a + 2i\frac{b-a}{n}\right) + f(b)\right] \\ &\qquad = \frac{1}{12}(1.414214 + 4\cdot 2.000976 + 2\cdot 2.704163 + 4\cdot 3.527238 + 2\cdot 4.472136 + \\ &\qquad\quad + 4\cdot 5.539983 + 2\cdot 6.731456 + 4\cdot 8.046981 + 9.486833) = \boxed{9.598106} \end{aligned}$$

25. *Determine an n so that Simpson's Rule will approximate the integral with an error E_n satisfying $|E_n| \le 0.01$ Then using that n, approximate the integral. (Text problem 296[16].)*

$$\int_4^8 \sqrt{x+1}\,dx$$

Assuming that $f(x)$ has four continuous derivatives on $[4,8]$, the error formula for Simpson's Rule is

$$E_n = -\frac{(b-a)^5}{180n^4}f^{(4)}(c) \quad \text{for some } c \in [a,b]$$

For our integrand $f(x) = (x+1)^{\frac{1}{2}}$, the derivatives are

$$\begin{aligned} f'(x) &= \frac{1}{2}(x+1)^{-\frac{1}{2}}; \\ f''(x) &= -\frac{1}{4}(x+1)^{-\frac{3}{2}}; \\ f^{(3)}(x) &= \frac{3}{8}(x+1)^{-\frac{5}{2}}; \\ f^{(4)}(x) &= -\frac{15}{16}(x+1)^{-\frac{7}{2}}; \end{aligned}$$

Using $a = 4$ and $b = 8$, the largest value for $c \in [a, b] = [4, 8]$ of

$$|f^{(4)}(c)| = \frac{15}{16}(c+1)^{-\frac{7}{2}}$$

occurs when $c = 4$ because the function is decreasing. For all $c \in [4, 8]$,

$$|f^{(4)}(c)| \le \max_{c\in[4,8]} \frac{15}{16}(c+1)^{-\frac{7}{2}} = \frac{15}{16}(4+1)^{-\frac{7}{2}} = 0.003354102$$

Hence, the error satisfies

$$|E_n| = \frac{(b-a)^5}{180n^4}|f^{(4)}(c)| \le \frac{(8-4)^5}{180n^4} \cdot \frac{15}{16}5^{-\frac{7}{2}} = \frac{15 \cdot 4^5}{16 \cdot 180 \cdot 5^{\frac{7}{2}}\, n^4} = \frac{0.01908111}{n^4}$$

For Simpsons rule, n has to be even, so that $\boxed{n = 2}$ has an error

$$|E_n| = \frac{0.01908111}{2^4} = 0.00119257$$

Thus Simpsons Rule for the integral with $n = 2$ for $x_0 = 4$, $x_1 = 2$ and $x_2 = 8$,

$$\begin{aligned} S_2 &= \frac{b-a}{3 \cdot 2}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) \\ &= \frac{2}{3}\left(f(4) + 4f(6) + f(8)\right) = \frac{2}{3}(2.236068 + 4 \cdot 2.645751 + 3.000000) = \boxed{10.54605} \end{aligned}$$

To check, we compute

$$\int_4^8 (x+1)^{\frac{1}{2}}\, dx = \left[\frac{2}{3}(x+1)^{\frac{3}{2}}\right]_4^8 = \frac{2(9^{\frac{3}{2}} - 5^{\frac{3}{2}})}{3} = 10.54644$$

The error is $I - S_2 = 10.54644 - 10.54605 = 0.00039$.

26. *Show that Simpson's Rule is exact for any cubic polynomial in two ways. By calculation and by the error estimate. (Text problem 269[18].)*

Let $x_i = a + i(b-a)/n$ as usual. Simpson's Rule for any function is the sum over all consecutive pairs of intervals of the form

$$S_n = \frac{b-a}{3n}\sum_{i=1}^{n/2}[f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})] = \sum_{i=1}^{n/2}\frac{[f(x_{2i-2} + 4f(x_{2i-1}) + f(x_{2i})][x_{2i} - x_{2i-2}]}{6}$$

If for a general cubic $f(x) = ax^3 + bx^2 + cx + d$ we can show that

$$\frac{[f(x_{2i-2} + 4f(x_{2i-1}) + f(x_{2i})][x_{2i} - x_{2i-2}]}{6} = \int_{x_{2i-2}}^{x_{2i}} f(x)\, dx \qquad (1)$$

then

$$S_n = \sum_{i=1}^{n/2}\frac{[f(x_{2i-2} + 4f(x_{2i-1}) + f(x_{2i})][x_{2i} - x_{2i-2}]}{6} = \sum_{i=1}^{n/2}\int_{x_{2i-2}}^{x_{2i}} f(x)\, dx = \int_a^b f(x)\, dx$$

as desired. However,

$$\int_{x_{2i-2}}^{x_{2i}} f(x)\,dx = \int_{x_{2i-2}}^{x_{2i}} (ax^3 + bx^2 + cx + d)\,dx = \left[\frac{a}{4}x^4 + \frac{b}{3}x^3 + \frac{c}{2}x^2 + dx\right]_{x_{2i-2}}^{x_{2i}}$$

$$= \left[\frac{a}{4}(x_{2i}^4 - x_{2i-2}^4) + \frac{b}{3}(x_{2i}^3 - x_{2i-2}^3) + \frac{c}{2}(x_{2i}^2 - x_{2i-2}^2) + d(x_{2i} - x_{2i-2})\right]$$

$$= \left[\frac{a}{4}(x_{2i}^3 + x_{2i}^2 x_{2i-2} + x_{2i} x_{2i-2}^2 + x_{2i-2}^3) + \frac{b}{3}(x_{2i}^2 + x_{2i}x_{2i-2} + x_{2i-2}^2)\right.$$

$$\left. + \frac{c}{2}(x_{2i} + x_{2i-2}) + d\right](x_{2i} - x_{2i-2})$$

Now use the following equivalents

$$\frac{a}{4}(x^3 + x^2y + xy^2 + y^3) = \frac{ax^3 + 4a\left(\frac{x+y}{2}\right)^3 + ay^3}{6}$$

$$\frac{b}{3}(x^2 + xy + y^2) = \frac{bx^2 + 4b\left(\frac{x+y}{2}\right)^2 + by^2}{6}$$

$$\frac{c}{2}(x + y) = \frac{cx + 4c\left(\frac{x+y}{2}\right) + cy}{6}$$

$$d = \frac{d + 4d + d}{6}$$

Using $x = x_{2i-2}$, $y = x_{2i}$ and so $\frac{x+y}{2} = x_{2i-1}$ in the equivalents, we may substitute

$$\int_{x_{2i-2}}^{x_{2i}} f(x)\,dx = \left[(ax_{2i}^3 + 4ax_{2i-1}^3 + ax_{2i-2}^3) + (bx_{2i}^2 + 4bx_{2i-1}^2 + bx_{2i-2}^2)\right.$$

$$\left. + (cx_{2i} + 4cx_{2i-1} + cx_{2i-2}) + d + 4d + d\right]\frac{x_{2i} - x_{2i-2}}{6}$$

$$= \left[f(x_{2i}) + 4f(x_{2i-1}) + f(x_{2i-2})\right]\frac{x_{2i} - x_{2i-2}}{6}$$

proving (1) by direct calculation.

The error for Simpson's rule is

$$E_n = -\frac{(b-a)^5}{180n^4} f^{(4)}(c) \quad \text{for some } c \in [a, b]$$

For our integrand $f(x) = ax^3 + bx^2 + cx + d$ we have $f^{(4)}(c) = 0$ for any c so that the error $E_n = 0$ and Simpson's Rule computes the integral exactly.