On the Compression of Elastic Tubes

Feng Liu & Andrejs Treibergs

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Abstract

We find explicit formulas for the modulus of compression for all postbuckled elastic tube geometries using Levy's solution for closed thin elastic rings under pressure. The equivalent geometric problem is to to determine the rate of deformation of a plane curve of given length, enclosing a given area and minimizing bending energy due to a change in area. The variational problem is solved. The solution is compared to the minimizer from a simple restricted class of curves consisting of four arcs of circles.

We are interested in the geometric deformation of a carbon nanotube under hydrostatic pressure. This problem arises in the design of a nanotube electromechanical pressure sensor [28]. Single walled carbon nanotubes were first created in the laboratory over a decade ago [14, 15]. Modeled as elastic tubes, hydrostatic pressure forces the volume reduction of a nanotube. Its walls keep a fixed cross section length, have area depending on pressure, but resist by minimizing bending energy. The electrical response to a large deformation is a metal to semiconductor transition and the resulting decrease in conductance. Since the amount of deformation for different pressures depends on size, by devising an array of nanotubes of various sizes, any conductance response can be engineered into the sensor. It is therefore of interest to determine the modulus of deformation due to pressure.

The problem of minimizing the bending energy for plane curves with fixed endpoints and given length was proposed by J. & D. Bernoulli and studied by Euler, thus energy minimizing curves are called Euler elastica. This problem spurred the development of the calculus of variations and the theory of elliptic functions [26]. The solution for thin rings deforming under hydrostatic pressure was found by M. Levy [21]. The buckling of a circular ring under hydrostatic pressure has been studied by Carrier [5], Chaskalovic and Naili [6] who determine bifurcation points, as well as many others, *e.g.,* [2, 3, 16, 17, 23, 24, 25]. It is now a standard example in mechanics texts, *e.g.* [7, Pages 274–281] and geometry texts, *e.g.* [22] which gives an elementary discussion of elastica with given turning angle. Similar models describe the shape of red blood cells [4, 8, 9]. Elastica in three space and other spaceforms [18, 19], as well as dynamical deformations [20] have been studied. Another problem equivalent to minimizing $\sup|K|$ for fixed A and L is discussed in [13].

We formulate the variational problem. Let s denote arclength along a curve Γ. The position vector is then $X(s) = (x(s), y(s))$. Since we are parameterizing by arclength, the unit tangent vector is given by

$$T(s) = (x'(s), y'(s)) = (\cos\theta(s), \sin\theta(s)), \tag{1}$$

where $\theta(s)$ is the angle T makes with the positive x-axis and prime denotes differentiation with respect to arclength. The position may be recovered by integrating

$$X(s) = X_0 + \int_0^s (\cos\theta(\sigma), \sin\theta(\sigma))\, d\sigma.$$

We'll take $X_0 = (0,0)$. The curvature of the curve is given by

$$K = \theta'(s).$$

The cross section of the tube is to be regarded as an inextensible elastic rod in the plane which is subject to a constant normal hydrostatic pressure $\mathcal{P}$ along its outer boundary. The section is assumed to have a uniform wall thickness h_0 and elastic properties. The centerline of the wall is given by a smooth embedded closed curve in the plane $\Gamma \subset \mathbf{R}^2$ which bounds a compact region Ω whose boundary has given length L_0 and which encloses a given area Area(Ω). Among such curves we seek one, Γ_0, that minimizes the energy

$$\mathrm{E}(\Gamma) = \frac{\mathcal{B}}{2} \int_\Gamma (K - K_0)^2 \, ds + \mathcal{P} \left(\mathrm{Area}(\Omega) - A_0 \right),$$

where $\mathcal{B} = Eh_0^3 / \{12(1-\nu^2)\}$ is the flexural rigidity modulus of the section, E is Young's modulus, ν is Poisson's ratio, K denotes the curvature of the curve and K_0 is the undeformed curvature ($= 2\pi/L_0$ for the circle.)

This is equivalent to the problem of minimizing

$$\mathrm{E}(\Gamma) = \int_\Gamma K^2 \, ds, \tag{2}$$

among curves of fixed length L_0 that enclose a fixed area $A_0 = \mathrm{Area}(\Omega)$. We are interested in the relation between the geometry of the minimizer and the values of A_0 and L_0. The problem is invariant under a homothetic scaling of Γ_0. Thus if the curve is scaled to $\tilde{\Gamma}_0 = c\Gamma_0$, its area, length and energy change by $\tilde{A}_0 = c^2 A_0$, $\tilde{L}_0 = cL_0$ and $\tilde{E} = c^{-1}E$ for $c > 0$. Since the shape of the minimizer is independent of the scaled data, it suffices to find the relation between the *Isoperimetric Ratio,* $\mathcal{I}$, and other dimensionless measures of the shape of Γ_0. The isoperimetric ratio

$$\mathcal{I} = \frac{4\pi A}{L^2}$$

satisfies $0 < \mathcal{I} \le 1$ by the isoperimetric inequality, which says that the area of any figure with fixed boundary length does not exceed the area of a circle with that boundary length. Moreover, the only figure with $\mathcal{I} = 1$ is the circle.

Assuming that the minimizing curve has reflection symmetry in both the x and y-directions, which is the principal mode ($n = 2$) of buckling, we only need to find θ for $0 \le s \le L$ where $4L = L_0$, over a quarter of the curve, and then reflect to get the closed curve. We are assuming that Γ is a closed C^1 curve. By rotation and translation, we assume $x(0) = y(0) = 0 = x(L_0) = y(L_0)$. In order for the curve not to have a corner at the endpoints, it is necessary that $\theta(0) = 0$ and $\theta(L_0) = 2\pi$. It is also assumed, that for $\theta(s)$, the minimizer, the resulting curve $\gamma = X([0, L])$ remains an embedded curve. Let $\hat{\gamma}$ denote the closed curve γ followed by the line segment from $X(L)$ to $(0, y(L))$ followed by the line segment back to $(0,0)$. Then by Green's theorem, the area is bounded by Γ is given by

$$\frac{1}{4}\mathrm{Area}(\Gamma) = \frac{1}{4}\int_\Gamma x \, dy = \oint_{\hat{\gamma}} x \, dy = \int_\gamma x \, dy, \tag{3}$$

because $dy = 0$ on the horizontal segment and $x = 0$ on the vertical segment. The variational problem is to find a function $\theta : [0, L] \to \mathbf{R}$ such that $\theta(0) = 0$, $\theta(L) = \pi/2$ satisfying Area$(\theta) = A_0$ which minimizes (2). In fact, since it takes energy to squeeze the curve, it suffices to find an energy minimizer among such curves that satisfy Area$(\theta) \le A_0$.

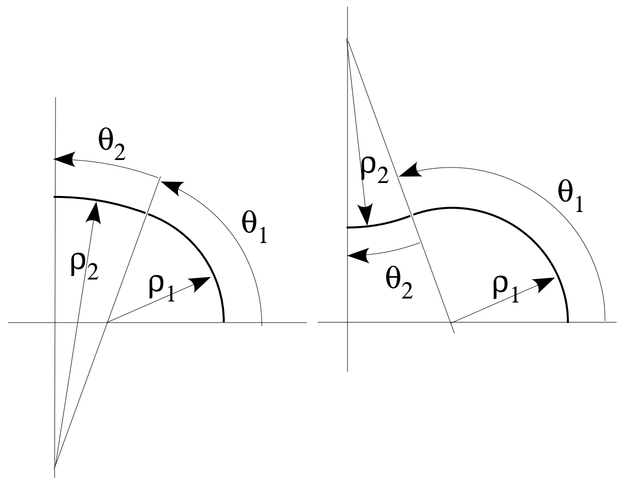

Figure 1: Quarter Peanut Domains.

1. Warmup: Peanut Example.

Let us illustrate the computation of the modulus of deformation in a family of curves, the peanuts. These curves arise when trying to minimize the sup-norm of curvature instead of the bending energy (the L^2 norm of the curvature). [13].

The domains have reflection symmetry on their x and y-axes. Each quarter of the domain remains in a coordinate quadrant and consists of two arcs. The arc in the first quadrant starts perpendicular to the x-axis and has curvature $k_1 > 0$ for a length $\ell_1 > 0$ followed by a second arc tangent to the first whose curvature is k_2, which is allowed to be negative, of length $\ell_2 > 0$. The end or the second arc is on the y-axis and perpendicular to it. Thus the length and total curvature of the arc in the first quadrant is

$$\begin{aligned} \ell_1 + \ell_2 = &= L, \\ k_1\ell_1 + k_2\ell_2 &= \frac{\pi}{4}. \end{aligned}$$

If the second curvature is negative then the total figure is peanut shaped. The total angle along each arc is given by $\theta_i = k_i\,\ell_i$. We shall suppose that $0 < \theta_1 < \pi$ and that the entire arc remains in the first quadrant. The radii of curvature are thus $r_i = 1/k_i$. It is convenient to introduce the coordinates $\xi = 2\theta_2$ and $r = r_1 > 0$. The area of the figure in the first quadrant may be computed as follows. There are four cases, which have to be analyzed slightly differently, namely when $0 < k_1 < k_2$, when $0 < k_2 < k_1$, when $k_2 = 0$ and when $k_2 < 0 < k_1$. For example, in the second case (Fig. 1a), the area is the sum of the areas of sectors of angles $\theta_1 > 0$ and radius r_1 and angle θ_2 with radius r_2 minus the triangle of the second sector in the fourth quadrant. The hypotenuse has length $r_2 - r_1$. Solving in terms of (r, ξ) we get

$$\begin{aligned} \theta_1 &= \frac{\pi - \xi}{2} \\ \theta_2 &= \frac{\xi}{2} \\ \ell_1 &= \theta_1 r_1 = \frac{(\pi - \xi)r}{2} \\ \ell_2 &= \theta_2 r_2 \;=\; L - \ell_1 \;=\; \frac{2L - (\pi - \xi)r}{2} \end{aligned}$$

$$r_2 = \frac{L-\theta_1 r_1}{\theta_2} = \frac{2L-(\pi-\xi)r}{\xi} = \frac{2\left(L-\frac{\pi}{2}r\right)}{\xi} + r$$

Hence the area in the peanut is

$$\begin{aligned}\frac{1}{4}\text{Area}(\Gamma) &= \frac{1}{2}\theta_1 r_1^2 + \frac{1}{2}\theta_2 r_2^2 - \frac{1}{2}(r_1-r_2)^2 \cos\theta_2 \, \sin\theta_2 \\ &= Lr - \frac{\pi}{4}r^2 + \left(L-\frac{\pi}{2}r\right)^2 f(\xi)\end{aligned}$$

where

$$f(\xi) = \frac{\xi - \sin\xi}{\xi^2}.$$

This is the expression in the first and fourth cases as well. Note that the function $f(\xi)$ is an odd bounded function which is increasing on $-\pi \le \xi \le \pi$. In the third case, the area is the sum of the quarter circle plus the rectangle so

$$\frac{1}{4}\text{Area}(\Gamma) = \frac{\pi}{4}r_1^2 + \ell_2 r_1 = Lr - \frac{\pi}{4}r^2$$

In fact, one can check that when $k_2 \to 0$ then in all cases the former expression converges to the latter. Also, if $\xi \ge 0$ then as $r \to 0$ the area converges to the area of the football shaped domain with its pointed ends on the x-axis, $4L^2 f(\xi)$.

We may equally easily write the bending energy in these parameters

$$\begin{aligned}\frac{1}{4}\text{E}(\gamma) &= \ell_1 k_1^2 + \ell_2 k_2^2 = \frac{\theta_1}{r_1} + \frac{\theta_2}{r_2} \\ &= \frac{\pi-\xi}{2r} + \frac{\xi^2}{4L-2(\pi-\xi)r}\end{aligned}$$

The Lagrange functional is

$$\begin{aligned}\mathcal{L} &= \frac{\mathcal{B}}{2}\text{E}(\Gamma) - \mathcal{P}\left(A_0 - \text{Area}(\Gamma)\right) \\ \frac{2\pi}{\mathcal{B}}\mathcal{L} &= \frac{2\pi(\pi-\xi)}{r} + \frac{2\pi\xi^2}{2L-(\pi-\xi)r} + \lambda\left\{\mathcal{D}^2 - \left(L-\frac{\pi}{2}r\right)^2 (1-\pi f(\xi))\right\}\end{aligned}$$

Where $A_0 = 4A$, $\lambda = 8\mathcal{P}/\mathcal{B}$ is the normalized pressure and $\mathcal{D}^2 = L^2 - \pi A$ is the isoperimetric difference for the quarter figure. $\mathcal{D} \ge 0$ and $\mathcal{D} = 0$ if and only if the peanut is a circle by the isoperimetric inequality.

For given A and L, we wish to find the E-minimizing configuration r, ξ. We formulate the problem using the Lagrange multiplier λ whose value is normalized pressure. Equivalently, we fix the pressure λ and determine the configuration that minimizes the energy of deformation $\mathcal{L}$. We are interested in the modulus of deformation area due to pressure, or the quantity

$$M = \frac{\partial\mathcal{P}}{\partial \log \text{Area}(\Gamma)} = \frac{\mathcal{B}}{8}\frac{\partial\lambda}{\partial\left(\log\frac{\text{Area}}{4}\right)}.$$

The constrained maximization problem satisfies the Euler-Lagrange equations.

$$\begin{aligned}0 &= \frac{1}{\mathcal{B}}\frac{\partial\mathcal{L}}{\partial r} = -\frac{\pi-\xi}{r^2} + \frac{\xi^2(\pi-\xi)}{(2L-(\pi-\xi)r)^2} + \frac{\lambda}{2}\left(L-\frac{\pi}{2}r\right)[1-\pi f(\xi)] \\ 0 &= \frac{1}{\mathcal{B}}\frac{\partial\mathcal{L}}{\partial\xi} = -\frac{1}{r} + \frac{\xi(4L-2\pi r+\xi r)}{(2L-(\pi-\xi)r)^2} + \frac{\lambda}{2}\left(L-\frac{\pi}{2}r\right)^2 f'(\xi) \\ 0 &= \frac{2\pi}{\mathcal{B}}\frac{\partial\mathcal{L}}{\partial\lambda} = \mathcal{D}^2 - \left(L-\frac{\pi}{2}r\right)^2 (1-\pi f(\xi))\end{aligned}$$

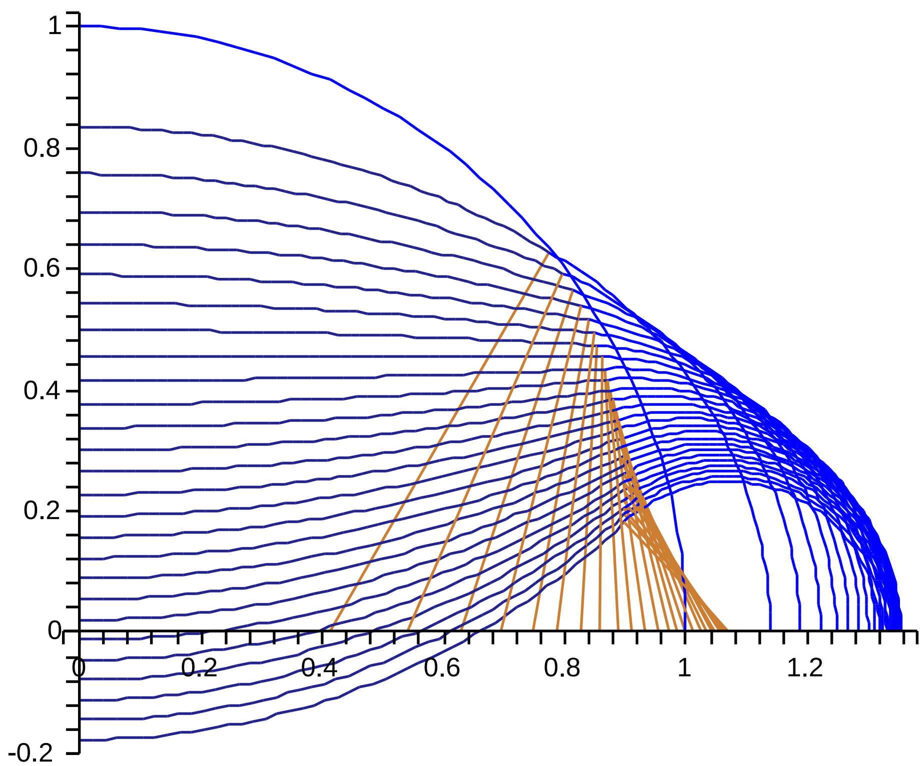

Figure 2: Deformation of optimal quarter peanuts of length $\frac{\pi}{2}$.

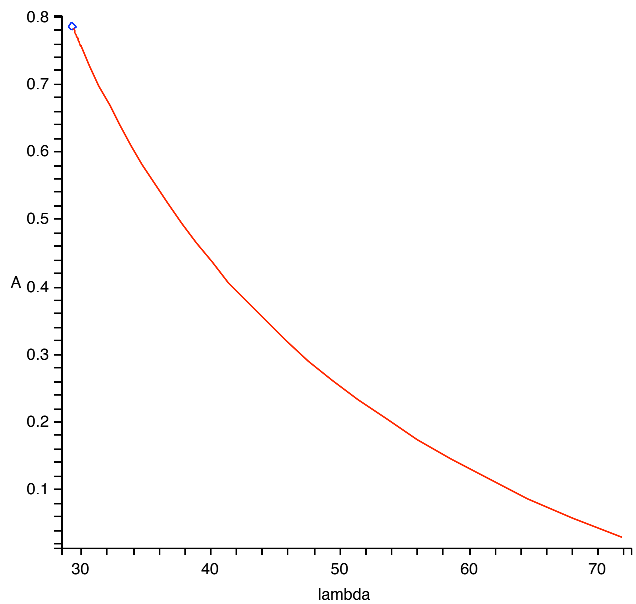

Figure 3: Pressure λ vs area A for peanuts.

The system simplifies to

$$
\begin{aligned}
0 &= \frac{\pi r - 2L - 2\xi r}{(2L-(\pi-\xi)r)^2} + \frac{1}{4}\lambda r^2 \frac{1-\pi f(\xi)}{\pi - \xi} && (4) \\
0 &= -\frac{2}{(2L-(\pi-\xi)r)^2} + \frac{1}{4}\lambda r\, f'(\xi) && (5) \\
0 &= \mathcal{D}^2 - \left(L - \frac{\pi}{2} r\right)^2 (1-\pi f(\xi)) && (6)
\end{aligned}
$$

The system may be solved as follows. Geometrically, the variables must satisfy $0 < \theta_1 < \pi$ so that $|\xi| < \pi$. But this implies that $\pi |f(\xi)| < 1$. Moreover, the isoperimetric inequality says $L^2 - \pi A > 0$ unless the curve is a circle and equality holds, which we rule out. (Remember that A and L are the area and boundary length of the quarter peanut.) The last equation yields

$$\left(L - \frac{\pi}{2} r\right)^2 = \frac{\mathcal{D}^2}{1-\pi f(\xi)} \tag{7}$$

which is always positive. Thus, when $\theta_2 = \xi/2 < 0$ the peanut is not convex. When $\theta_1 = \pi/2$, the second arc is a straight line and the boundary includes the quarter circle of radius r so that $\pi r < 2L$. In fact, in this case the area is the sum of the quarter circle plus the box $A = \frac{\pi}{4} r^2 + r(L - \frac{\pi}{2} r)$ as it should be. As $r(\xi)$ depends continuously on ξ and that the quantity (7.) is strictly positive for all ξ, then r is a continuation of $r(0)$ and so the negative root is required for all ξ. It follows that

$$r = \frac{2}{\pi}\left\{ L - \frac{\mathcal{D}}{\sqrt{1-\pi f(\xi)}} \right\}.$$

Furthermore, this gives a condition for ξ given area and length. Indeed, since $r > 0$ we must have

$$f(\xi) < \frac{A}{L^2} = \frac{\mathcal{I}}{\pi},$$

an inequality which implies $\xi < \xi_0(\mathcal{I}) < \pi$, where ξ_0 is a constant depending on the isoperimetric ratio. λ is eliminated from the first two Euler-Lagrange equations. By substiting $r(\xi)$ into the resulting equation gives a single equation for ξ. This equation is solved numerically and the corresponding solutions are drawn with sector lines, Fig. 2, using MAPLE. In our plots, we assume $L = \pi/2$. We also plot the area vs. λ (pressure) curve in Fig. 3.

Now let's compute the modulus. The Euler Lagrange equations can be thought of as giving a mapping $F : (\lambda, \rho, \xi; a) \mapsto \mathbf{R^3}$. Writing $x = (\rho, \xi, \lambda)$, for each A, the parameters are determined by solving the equation $F(x; A) = 0$. The derivative $\frac{\partial x_i}{\partial A}$ may be computed using the chain rule. Differentiating by A, for each $j = 1, 2, 3$,

$$\sum_{i=1}^{3} \frac{\partial F_j}{\partial x_i}\frac{dx_i}{dA} + \frac{\partial F_j}{\partial A} = 0$$

Hence for each $k = 1, 2, 3$, the derivative may be found using the inverse of the Jacobean matrix

$$\frac{dx_k}{dA} = -\sum_{j=1}^{3} \left[\left(\frac{\partial F_j}{\partial x_i}\right)^{-1}\right]_{kj} \frac{\partial F_j}{\partial A} = 0$$

Computing, we find

$$\left(\frac{\partial F_j}{\partial x_i}\right) = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} & \frac{r^2(1-\pi f(\xi))}{4(\pi-\xi)} \\ \mathfrak{C} & \mathfrak{D} & \frac{r\, f'(\xi)}{4} \\ \pi(L - \frac{\pi}{2}r)[1 - \pi f(\xi)] & \pi(L - \frac{\pi}{2}r)^2 f'(\xi) & 0 \end{pmatrix}$$

where

$$\begin{aligned}
\mathfrak{A} &= \frac{-2\pi L + \pi^2 r - 3\pi r\xi + 2r\xi^2}{(2L - (\pi - \xi)r)^3} + \frac{1}{2}\lambda r \frac{1 - \pi f(\xi)}{\pi - \xi}, \\
\mathfrak{B} &= \frac{2r^2\xi}{(2L - (\pi - \xi)r)^3} + \frac{1}{4}\lambda r^2 \left[\frac{1 - \pi f(\xi)}{(\pi - \xi)^2} - \frac{\pi f'(\xi)}{\pi - \xi}\right] \\
\mathfrak{C} &= \frac{-4(\pi - \xi)}{(2L - (\pi - \xi)r)^3} + \frac{1}{4}\lambda\, f'(\xi) \\
\mathfrak{D} &= \frac{4r}{(2L - (\pi - \xi)r)^3} + \frac{1}{4}\lambda r\, f''(\xi)
\end{aligned}$$

Similarly, $\frac{\partial F}{\partial A} = (0, 0, -\pi)^T$. Thus, using Cramer's rule,

$$\begin{aligned}
\frac{d\lambda}{dA} &= \frac{1}{\det\left(\frac{\partial F_j}{\partial x_i}\right)} \begin{vmatrix} \mathfrak{A} & \mathfrak{B} & 0 \\ \mathfrak{C} & \mathfrak{D} & 0 \\ \pi(L - \frac{\pi}{2}r)[1 - \pi f(\xi)] & \pi(L - \frac{\pi}{2}r)^2 f'(\xi) & \pi \end{vmatrix} \\
&= \frac{4(\mathfrak{A}\mathfrak{D} - \mathfrak{B}\mathfrak{C})(\pi - \xi)}{\left(L - \frac{\pi}{2}r\right)\left\{ \begin{array}{lcl} -\mathfrak{A}r\left(L - \frac{\pi}{2}r\right)(\pi - \xi)(f')^2 & + & \mathfrak{B}r(1 - \pi f)(\pi - \xi)f' \\ +\mathfrak{C}r^2(1 - \pi f)\left(L - \frac{\pi}{2}r\right)f' & - & \mathfrak{D}r^2(1 - \pi f)^2 \end{array} \right\}}
\end{aligned}$$

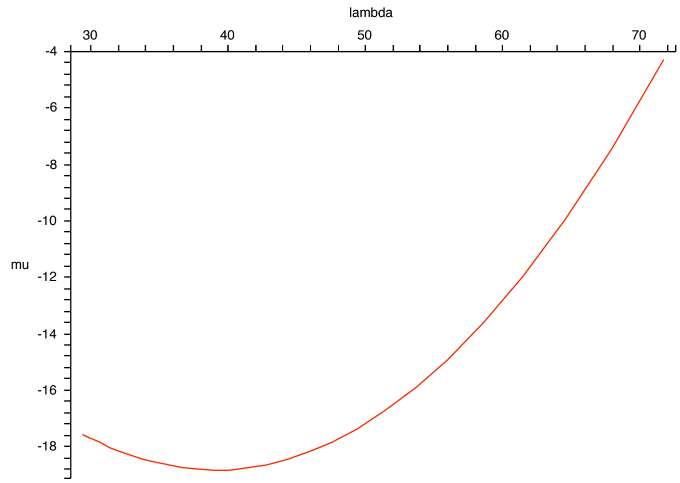

Figure 4: Modulus $d\lambda/d\ln A$ vs. λ for extremal peanuts.

The result of the MAPLE computation of $\mathrm{d}\lambda/\mathrm{d}\ln\mathrm{A}$ is plotted in Fig. 4.

We can compute the limiting pressure and modulus at the circle from these formulas. First, as the isoperimetric difference $\mathcal{D} \to 0$, we see from (6) that either $r \to 2L/\pi$ or $\xi \to \pi$. Assuming that ξ does not limit to π, after eliminating λ in (5) and (6) we find

$$\left[L - \frac{\pi}{2}r + \xi r\right] f'(\xi) = \frac{[1 - \pi f(\xi)]r}{\pi - \xi} \tag{8}$$

so that in the limit,

$$\xi\, f'(\xi) = \frac{1 - \pi f(\xi)}{\pi - \xi}$$

which has a unique solution $\xi = \pi/2$ in $(-\pi, \pi)$. Taking the limit and solving (4) we get that the pressure to deform the peanut at the circle is

$$\mathcal{P} = \frac{\mathcal{B}}{8}\lambda = \frac{\pi\mathcal{B}}{(4-\pi)r^3} \approx 3.659792369\frac{\mathcal{B}}{r^3}$$

so $\lambda \to 8\pi(4-\pi)^{-1}r^{-3} \approx 29.27833894\, r^{-3}$ which says that the peanuts are harder to buckle than the elastica.

To compute the modulus at the circle, let us compute $d\lambda/d\ln A$ for the circle or radius one. Then the modulus at the circle of radius r_0 is given by $M = \frac{1}{8}\mathcal{B}r_0^{-3} d\lambda/d\ln A$. Let us expand the quantities near the circle in terms of

$$\epsilon = L - \frac{\pi}{2}r.$$

First, $r = 1 - 2\epsilon/\pi$ holds exactly. Thus we may find the second order expansion of ξ near zero using equation (8)

$$\xi \approx \frac{\pi}{2} - \frac{2(4-\pi)\epsilon}{\pi^2 - 8} - \frac{8(4-\pi)^2\epsilon^2}{\pi(\pi^2-8)^2} + \dots$$

Substituting this into equation (5), we obtain the expansion of λ near $\epsilon = 0$,

$$\lambda \approx \lambda_0 + \lambda_1\epsilon + \lambda_2\epsilon^2 + \dots = \frac{8\pi}{4-\pi} + \frac{64(\pi^2 + 2\pi - 16)\epsilon^2}{(4-\pi)\pi(\pi^2-8)} + \dots$$

Also, using equation (6) we find the expansion of the area near $\epsilon = 0$,

$$A \approx a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots = \frac{\pi}{4} - \frac{(4-\pi)\epsilon^2}{\pi^2} + \ldots$$

Thus, using L'Hospitals's rule, the modulus at the circle is

$$A\frac{d\lambda}{dA} = \frac{\pi\lambda_2}{4a_2} = -\frac{16\pi(\pi^2+2\pi-16)}{(4-\pi)^2(\pi^2-8)} \approx -17.51367398.$$

Finally, we remark that the family of peanuts stays embedded for $\xi > \xi_0$ where the critical $-\pi < \xi_0 < 0$. The pinching is at the origin when $(r+|r_2|)\cos\xi_0/2 = |r_2|$ or in other words, $r_2 = -r\cos(\xi_0/2)/(1-\cos\xi/2)$. Using $\xi r_2 = 2L - (\pi - \xi)r$ in equation (8), we solve to find $\xi_0 \approx -1.387393003$ where $r \approx 0.09103308488$ and $\mathcal{I} = 0.09989963316$. On the other hand, the convexity changes from negative to positive at $\xi_c = 0$. Then (8) implies $r_c = \pi^2/(12+\pi^2) \approx 0.4512932289$, when $\mathcal{I} \approx .2627521721$.

2. Euler Lagrange Equation for the energy minimizing curve.

Since we are looking to minimize E subject to Area$(\theta) \le A_0/4 = A$, the Lagrange Multiplier $\lambda = 8\mathcal{P}/\mathcal{B} \ge 0$ is nothing more than scaled pressure such that at the minimum, the variations satisfy $4\,\delta E = -\lambda\,\delta$Area The corresponding Lagrange Functional is thus

$$\begin{aligned}\mathcal{L}[\gamma] &= 4\int_\gamma K(s)^2\,ds - \lambda\left\{A - \int_\gamma x\,dy\right\} \\ &= 4\int_0^L \dot\theta(s)^2\,ds - \lambda\left\{A - \int_0^L\int_0^s \cos\theta(\sigma)\,d\sigma\,\sin\theta(s)\,ds\right\}.\end{aligned}$$

Assuming that the minimizer is the function $\theta(s)$ with $\theta(0) = 0$ and $\theta(L) = 2/\pi$, we make a variation $\theta + \epsilon v$ where $v \in C^1([0, L])$ with $v(0) = v(L) = 0$. Then

$$\begin{aligned}&0 = \delta\mathcal{L} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\mathcal{L} = \\ &= 8\int_0^L \dot\theta\dot v\,ds - \lambda\int_0^L\left\{\int_0^s v(\sigma)\sin\theta(\sigma)\,d\sigma\,\sin\theta(s) - \int_0^s \cos\theta(\sigma)\,d\sigma\,\cos\theta(s)v(s)\right\}ds\end{aligned}$$

Integrating by parts, and reversing the order of integration in the second integral, $\delta\mathcal{L} =$

$$-8\int_0^L \ddot\theta v\,ds - \lambda\left\{\int_0^L\int_\sigma^L \sin\theta(s)\,ds\,v(\sigma)\sin\theta(\sigma)\,d\sigma - \int_0^L\int_0^s \cos\theta(\sigma)\,d\sigma\,\cos\theta(s)v(s)\,ds\right\}.$$

Switching names of the integration variables in the second term yields

$$\delta\mathcal{L} = \int_0^L\left[-8\ddot\theta(s) - \lambda\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\,\sin\theta(s) - \int_0^s \cos\theta(\sigma)\,d\sigma\,\cos\theta(s)\right\}\right]v(s)\,ds.$$

Since $v \in C_0^1([0, L])$ was arbitrary, the minimizer satisfies the integro-differential equation

$$\ddot\theta(s) = -\frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\,\sin\theta(s) - \int_0^s \cos\theta(\sigma)\,d\sigma\,\cos\theta(s)\right\} \tag{9}$$

Thus if $\lambda = 0$ we must have $\theta(s) = \frac{\pi s}{2L}$ and γ is a circle of radius L/π. Thus if $\mathcal{I} < 1$ then $\lambda > 0$. To see the differential equation implied by (9), we assume that $\dot{\theta} \neq 0$ and differentiate

$$\begin{aligned}
\theta''' &= \frac{\lambda}{8}\left\{\sin\theta(s)\,\sin\theta(s) + \cos\theta(s)\,\cos\theta(s)\right\} \\
&\quad -\frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\,\cos\theta(s) + \int_0^s \cos\theta(\sigma)\,d\sigma\,\sin\theta(s)\right\}\theta'(s) \\
&= \frac{\lambda}{8} - \frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\,\cos\theta(s) + \int_0^s \cos\theta(\sigma)\,d\sigma\,\sin\theta(s)\right\}\theta'(s) \qquad (10) \\
\theta'''' &= \frac{\lambda}{8}\left\{\sin\theta(s)\,\cos\theta(s) - \cos\theta(s)\,\sin\theta(s)\right\}\theta'(s) + \\
&\quad +\frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\,\sin\theta(s) - \int_0^s \cos\theta(\sigma)\,d\sigma\,\cos\theta(s)\right\}(\theta'(s))^2 \\
&\quad -\frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\,\cos\theta(s) + \int_0^s \cos\theta(\sigma)\,d\sigma\,\sin\theta(s)\right\}\theta''(s) \qquad (11)
\end{aligned}$$

from which we get

$$\theta''''\theta' = -\theta''\,(\theta')^3 + \left[\theta''' - \frac{\lambda}{8}\right]\theta''(s). \qquad (12)$$

This differential equation may be integrated as follows:

$$\frac{\theta''''\theta' - \theta'''\theta''}{(\theta')^2} = \left[\frac{\theta'''}{\theta'}\right]' = -\theta'\theta'' - \frac{\lambda\theta''}{8(\theta')^2} = \left[-\frac{1}{2}(\theta')^2 + \frac{\lambda}{8\theta'}\right]'$$

so there is a constant c_1 so that

$$\theta''' = c_1\theta' - \frac{1}{2}(\theta')^3 + \frac{\lambda}{8}. \qquad (13)$$

In other words, the curvature $K = \theta'$ satisfies

$$K'' = c_1 K + \frac{\lambda}{8} - \frac{1}{2}K^3. \qquad (14)$$

Multiplying by K' and integrating, we find a first integral. For some constant H,

$$(K')^2 = c_1 K^2 + H + \frac{\lambda K - K^4}{4} = F(K). \qquad (15)$$

The Euler-Lagrange equations have the following immediate consequence. The area is given using (10) and (13),

$$\begin{aligned}
A &= \frac{1}{2}\int_\gamma x\,dy - (y - y(L))\,dx \\
&= \frac{1}{2}\int_0^L \left\{\sin\theta(s)\int_0^s \cos(\theta\sigma)\,d\sigma - \cos\theta(s)\int_s^L \sin\theta(\sigma)\,d\sigma\right\} ds \\
&= \frac{4}{\lambda}\int_0^L \left\{\frac{\frac{\lambda}{8} - \theta'''}{\theta'}\right\} ds \\
&= \frac{2}{\lambda}\int_0^L K^2\,ds - \frac{4c_1 L}{\lambda}. \qquad (16)
\end{aligned}$$

3. Solution of Euler Lagrange Equation.

Since the curve closes, the curvature is a L_0-periodic function which satisfies (14), the nonlinear spring equation [11]. As we expect that the curvature to continue analytically beyond the endpoints of the quarter curve, and as we assume that the curve have reflection symmetries at the endpoints, the curvature would continue as an even function at the endpints. In particular, the boundary conditions on θ imply $K'(0) = K'(L) = 0$ from (9). As we have differentiated (9) twice, the solutions of (14) have two extra constants of integration which have to be satisfied by virtue of being solutions of (9). Furthermore, expecting buckling to occur in the $n = 2$ mode, the optimal curves will be elliptical or peanut shaped, the endpoints of the quarter curves will be the minima and maxima of the curvature around the curve, and these to be the only critical points of curvature. Since the minimum K may be negative, as in peanut shaped regions, the embeddedness of the reflection is more likely to be satisfied if $K(0) = K_1$ is the maximum of the curvature and $K(L) = K_2$ is the minimum of curvature around the curve.

One degree of freedom in the problem is homothety, which will be irrelevant to deducing nondimensional measures, as we've already remarked. Indeed, if the curve is scaled $\tilde{X} = cX$ then $\tilde{K} = c^{-1}K$, $d\tilde{K}/d\tilde{s} = c^{-2}K'$, $\tilde{c}_1 = c^{-2}c_1$, $\tilde{H} = c^{-4}H$ and $\tilde{\lambda} = c^{-3}\lambda$. For convenience, as $\lambda > 0$ for noncircular regions, we set $\lambda = 1$ to fix the scaling.

As K and K' vary, they satisfy (15), thus the parameters c_1, H, λ must allow solvability of (15). Moreover, $0 = F(K_1) = F(K_2)$ and the points $(K_1, 0)$ and $(K_2, 0)$ must be in the same component of the solution curve of (15) in phase (K, K') space. Thus, given K_1, K_2 we can solve for c_1 and H,

$$c_1 = \frac{1}{4}\left(K_1^2 + K_2^2 - \frac{\lambda}{K_1 + K_2}\right), \tag{17}$$

$$H = -\frac{K_1K_2}{4}\left(K_1K_2 + \frac{\lambda}{K_1 + K_2}\right), \tag{18}$$

provided $K_2 \neq -K_1$. A solution would have a minimum and maximum curvature with appropriate c_1 and H so we assume the solvability condition. Then $4F(K) = Q_1(K)Q_2(K)$ can be factored into quadratic polynomials, where

$$\begin{aligned} Q_1 &= (K_1 - K)(K - K_2); \\ Q_2 &= K^2 + (K_1 + K_2)K + K_1K_2 + \frac{\lambda}{K_1 + K_2} \end{aligned}$$

Since we've assumed that $F(K)$ is positive in the interval $K_2 < K < K_1$, this forces other inequalities among the c_1, H and λ. For example, if $K_2 = 0$, then $H = 0$ and $Q_2 > 0$ near $K = 0$ only if $\lambda = 1$, which we assume to be true. For $K_2 < 0$, then $Q_2 > 0$ near $K = 0$ for some K_1 only if $K_1 + K_2 > 0$, which we also assume.

Since the possible homotheties and translations of the same solution (shifts like $K(s + c)$) have been eliminated, the remaining indeterminacy coming from the constants of integration is to ensure that the direction angle Θ changes by exactly $\pi/2$ over γ. Thus given K_2, we solve for K_1 so that

$$\Theta(L) = \frac{\pi}{2} \tag{19}$$

where

$$\Theta(L) = \int_0^{L_0} K(s)\, ds = \int_{K_2}^{K_1} \frac{K\, dK}{\sqrt{F(K)}}.$$

We have used equation (15) to change variables from s to $K(s)$. In fact, this integral can be reduced to a complete elliptic integral. Similarly

$$L = \int_0^{L_0} ds = \int_{K_2}^{K_1} \frac{dK}{\sqrt{F(K)}} \tag{20}$$

is a complete elliptic integral. In order to tabulate and graph closed solutions of (14), we choose K_2, then find c_1 and H using (17,18). Then find K_1 so that (19) holds. Then compute L using (20) and integrate (2,1,3,14) numerically on $0 \le s \le L$. K_1 is found using a simple root finder to solve (19).

We may also express the quarter area in this way. (16) becomes

$$A = \frac{2}{\lambda} \int_0^L K^2 \, ds - \frac{4c_1 L}{\lambda} = \frac{2}{\lambda} \int_{K_2}^{K_1} \frac{K^2 \, dK}{\sqrt{F(K)}} - \frac{4c_1 L}{\lambda} \tag{21}$$

4. Reduction to Elliptic Integrals.

We now describe the reduction of (19,20) to complete elliptic integrals, following the procedure [1], [12]. Choose a constant μ so that $Q_2 - \mu Q_1$ is a perfect square. This happens upon the vanishing of the discriminant

$$\Delta = D^2(\mu+1)^2 - 4S^2\mu - 4(\mu+1)\frac{\lambda}{S} \tag{22}$$

where $S = K_1 + K_2$, $D = K_1 - K_2$ and $P = K_1 K_2$. It is zero when μ equals one of

$$\mu_1, \mu_2 = \frac{S^3 + 4PS + 2\lambda \pm 2\sqrt{(\lambda + 2K_1 S^2)(\lambda + 2K_2 S^2)}}{SD^2}. \tag{23}$$

where, say, $\mu_1 > \mu_2$. The factors are

$$(1+\mu_1)K^2 + (1-\mu_1)SK + (1+\mu_1)P + \frac{\lambda}{S} = Q_2 - \mu_1 Q_1 \quad = \quad F_1^2 = (\alpha K - \beta)^2 \tag{24}$$

$$(1+\mu_2)K^2 + (1-\mu_2)SK + (1+\mu_2)P + \frac{\lambda}{S} = Q_2 - \mu_2 Q_1 \quad = \quad F_2^2 = (\eta K + \delta)^2. \tag{25}$$

The signs were chosen based on numerical values. It follows that

$$\alpha = \sqrt{1+\mu_1} \tag{26}$$

$$\beta = \sqrt{(1+\mu_1)P + \frac{\lambda}{S}} \tag{27}$$

$$\eta = \sqrt{1+\mu_2} \tag{28}$$

$$\delta = \sqrt{(1+\mu_2)P + \frac{\lambda}{S}}, \tag{29}$$

which turn out to be positive. These variables satisfy a relation verified in section **5.**

$$0 = \alpha\beta\left(2 - \eta^2\right) + \delta\eta\left(2 - \alpha^2\right) \tag{30}$$

We can now solve for the factors as sums of squares.

$$Q_1 = \frac{F_1^2 - F_2^2}{\mu_2 - \mu_1}$$

$$Q_2 = \frac{\mu_2 F_1^2 - \mu_1 F_2^2}{\mu_2 - \mu_1}$$

The idea is to change variables in the integral according to

$$T = \frac{F_1}{F_2} = \frac{\alpha K - \beta}{\eta K + \delta}, \qquad K = \frac{\beta + \delta T}{\alpha - \eta T}, \qquad \frac{dT}{dK} = \frac{\alpha\delta + \beta\eta}{(\eta K + \delta)^2}.$$

The function T is increasing. Since $Q_1(K_1) = Q_1(K_2) = 0$ it follows that $T = 1$ when $K = K_1$ and $T = -1$ when $K = K_2$. Moreover,

$$Q_1 Q_2 = \frac{(F_1^2 - F_2^2)(\mu_2 F_1^2 - \mu_1 F_2^2)}{(\mu_2 - \mu_1)^2} = \frac{(T^2 - 1)(\mu_2 T^2 - \mu_1) F_2^4}{(\mu_2 - \mu_1)^2}$$

Therefore, the integral (20) becomes

$$L = \frac{2(\mu_1 - \mu_2)}{(\alpha\delta + \beta\eta)\sqrt{\mu_1}} \int_{-1}^{1} \frac{dT}{\sqrt{(1 - T^2)(1 - \frac{\mu_2}{\mu_1} T^2)}} = \frac{4(\mu_1 - \mu_2)}{(\alpha\delta + \beta\eta)\sqrt{\mu_1}} \mathcal{K}(m) \tag{31}$$

where $m = \sqrt{\mu_2/\mu_1}$ is imaginary and

$$\mathcal{K}(m) = \int_0^1 \frac{dT}{\sqrt{(1 - T^2)(1 - m^2 T^2)}}$$

is the complete elliptic integral of the first kind.

To find $\Theta(L)$ we express K by partial fractions

$$K = \frac{\beta + \delta T}{\alpha - \eta T} = \frac{(\alpha\delta + \beta\eta) T}{\alpha^2 - \eta^2 T^2} + \frac{\frac{\delta}{\eta} + \frac{\beta}{\alpha}}{1 - \frac{\eta^2}{\alpha^2} T^2} - \frac{\delta}{\eta}$$

Because the first term is odd, we get

$$\begin{aligned} \Theta &= \frac{2(\mu_1 - \mu_2)}{(\alpha\delta + \beta\eta)\sqrt{\mu_1}} \int_{-1}^{1} \frac{K\, dT}{\sqrt{(1 - T^2)(1 - m^2 T^2)}} \\ &= \frac{4(\mu_1 - \mu_2)}{\alpha\eta\sqrt{\mu_1}} \Pi\left(\frac{\eta^2}{\alpha^2}, m\right) - \frac{\delta}{\eta} L \end{aligned} \tag{32}$$

where

$$\Pi(n, m) = \int_0^1 \frac{dT}{(1 - nT^2)\sqrt{(1 - T^2)(1 - m^2 T^2)}}$$

is the complete elliptic integral of the third kind.

To find (21), we note using partial fractions that

$$\begin{aligned} K^2 &= \frac{(\beta + \delta T)^2}{(\alpha - \eta T)^2} = \frac{(\beta + \delta T)^2 (\alpha + \eta T)^2}{(\alpha^2 - \eta^2 T^2)^2} \\ &= \frac{\delta^2\eta^2 T^4 + (\alpha^2\delta^2 + 4\alpha\beta\delta\eta + \beta^2\eta^2) T^2 + \alpha^2\beta^2}{(\alpha^2 - \eta^2 T^2)^2} + \frac{2(\alpha\delta + \beta\eta)(\delta\eta T^3 + \alpha\beta T)}{(\alpha^2 - \eta^2 T^2)^2} \\ &= \frac{\delta^2}{\eta^2} - \frac{(\alpha\delta + \beta\eta)(3\alpha\delta + \beta\eta)}{\eta^2(\alpha^2 - \eta^2 T^2)} + \frac{2\alpha^2(\alpha\delta + \beta\eta)^2}{\eta^2(\alpha^2 - \eta^2 T^2)^2} + \frac{2(\alpha\delta + \beta\eta)(\delta\eta T^3 + \alpha\beta T)}{(\alpha^2 - \eta^2 T^2)^2}. \end{aligned} \tag{33}$$

The third term is handled by the standard reduction (*e.g.*, [27], p. 515).

$$\frac{d}{dT}\left(\frac{\eta^4 T\sqrt{(1-T^2)(1-m^2T^2)}}{\alpha^2-\eta^2T^2}\right) = \frac{\alpha^2\eta^4 + (\eta^6 - 2\alpha^2\eta^4[m^2+1])T^2 + 3m^2\alpha^2\eta^4T^4 - m^2\eta^6T^6}{(\alpha^2-\eta^2T^2)^2\sqrt{(1-T^2)(1-m^2T^2)}}$$
$$= \frac{2\alpha^2(\alpha^2-\eta^2)(m^2\alpha^2-\eta^2)}{(\alpha^2-\eta^2T^2)^2\sqrt{(1-T^2)(1-m^2T^2)}} - \frac{3\alpha^4m^2 - 2(m^2+1)\alpha^2\eta^2 + \eta^4}{(\alpha^2-\eta^2T^2)\sqrt{(1-T^2)(1-m^2T^2)}}$$
$$+\frac{m^2\alpha^2-\eta^2}{\sqrt{(1-T^2)(1-m^2T^2)}} + \eta^2\frac{\sqrt{1-m^2T^2}}{\sqrt{1-T^2}}$$

Thus we find after dropping the odd integral and boundary term that (16) becomes,

$$A = \frac{4(\mu_1-\mu_2)}{\lambda(\alpha\delta+\beta\eta)\sqrt{\mu_1}}\int_{-1}^{1}\frac{(K^2-2c_1)\,dT}{\sqrt{(1-T^2)(1-m^2T^2)}}$$
$$= \frac{4(\mu_1-\mu_2)}{\lambda\eta^2(\alpha\delta+\beta\eta)\sqrt{\mu_1}}\int_{-1}^{1}\left\{\delta^2 - 2c_1\eta^2 - \frac{(\alpha\delta+\beta\eta)(3\alpha\delta+\beta\eta)}{(\alpha^2-\eta^2T^2)}\right.$$
$$\left.+\frac{2\alpha^2(\alpha\delta+\beta\eta)^2}{(\alpha^2-\eta^2T^2)^2}\right\}\frac{dT}{\sqrt{(1-T^2)(1-m^2T^2)}}$$
$$= \frac{8(\mu_1-\mu_2)(\delta^2-2c_1\eta^2)}{\lambda\eta^2(\alpha\delta+\beta\eta)\sqrt{\mu_1}}\mathcal{K}(m) - \frac{8(\mu_1-\mu_2)}{\alpha^2\lambda\eta^2\sqrt{\mu_1}}(3\alpha\delta+\beta\eta)\Pi\left(\frac{\eta^2}{\alpha^2},m\right)$$
$$+\frac{4(\mu_1-\mu_2)(\alpha\delta+\beta\eta)}{\lambda\eta^2(\alpha^2-\eta^2)(m^2\alpha^2-\eta^2)\sqrt{\mu_1}}\int_{-1}^{1}\left\{\frac{3\alpha^4m^2-2(m^2+1)\alpha^2\eta^2+\eta^4}{(\alpha^2-\eta^2T^2)\sqrt{(1-T^2)(1-m^2T^2)}}\right.$$
$$\left.-\frac{m^2\alpha^2-\eta^2}{\sqrt{(1-T^2)(1-m^2T^2)}} - \eta^2\frac{\sqrt{1-m^2T^2}}{\sqrt{1-T^2}}\right\}dT$$
$$= \frac{8(\mu_1-\mu_2)}{\eta^2\lambda\sqrt{\mu_1}}\left[\frac{\delta^2-2c_1\eta^2}{\alpha\delta+\beta\eta} - \frac{\alpha\delta+\beta\eta}{\alpha^2-\eta^2}\right]\mathcal{K}(m)$$
$$+\frac{8(\mu_1-\mu_2)}{\alpha^2\lambda\eta^2\sqrt{\mu_1}}\left[\frac{(\alpha\delta+\beta\eta)(3\alpha^4m^2-2(m^2+1)\alpha^2\eta^2+\eta^4)}{(\alpha^2-\eta^2)(m^2\alpha^2-\eta^2)} - 3\alpha\delta-\beta\eta\right]\Pi\left(\frac{\eta^2}{\alpha^2},m\right)$$
$$-\frac{8(\mu_1-\mu_2)(\alpha\delta+\beta\eta)}{\lambda(\alpha^2-\eta^2)(m^2\alpha^2-\eta^2)\sqrt{\mu_1}}\mathcal{E}(m) \tag{34}$$

where

$$\mathcal{E}(m) = \int_0^1 \frac{\sqrt{1-m^2T^2}}{\sqrt{1-T^2}}\,dT$$

is the complete elliptic integral of the second kind.

We can also write the solution $K(s)$ in terms of elliptic integrals. Expressing the incomplete integral corresponding to (31), we find by substituting $T = -\mathrm{cn}(\nu)$ (see [1], p. 596) that

$$s = \frac{2(\mu_1-\mu_2)}{(\alpha\delta+\beta\eta)\sqrt{\mu_1}}\int_{-1}^{T}\frac{dT}{\sqrt{(1-T^2)(1-\frac{\mu_2}{\mu_1}T^2)}}$$
$$= \frac{2(\mu_1-\mu_2)}{(\alpha\delta+\beta\eta)\sqrt{\mu_1+\mu_2}}\,\mathrm{cn}^{-1}\left(T\,\middle|\,\frac{-\mu_2}{\mu_1+\mu_2}\right).$$

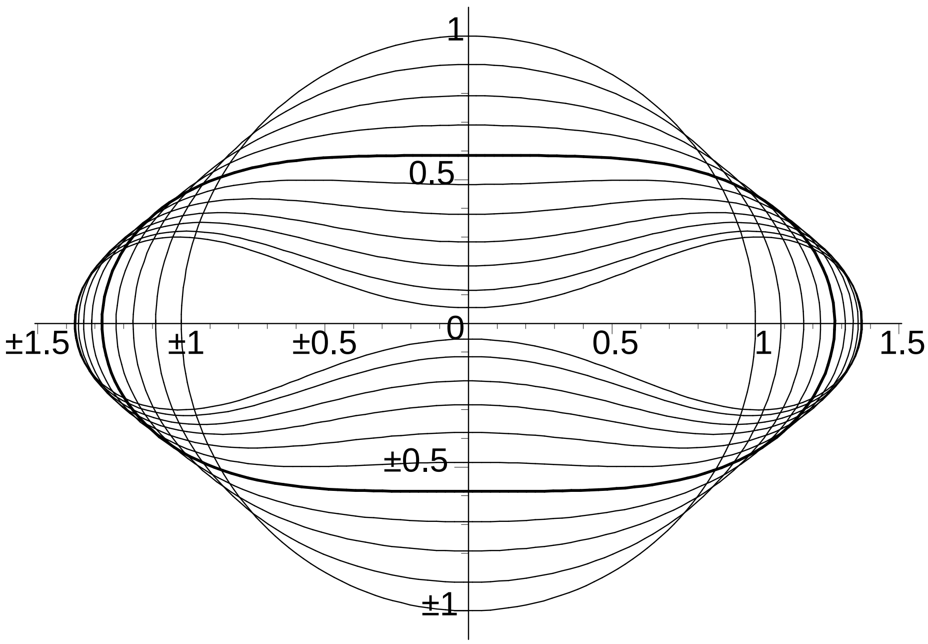

Figure 5: Mode $n = 2$ elastica for various pressures and length $L = \pi/2$.

It follows that

$$T = -\mathrm{cn}\left(\zeta s \,\middle|\, \frac{-\mu_2}{\mu_1 + \mu_2}\right)$$

so that

$$K = \frac{\beta - \delta\,\mathrm{cn}(\zeta s)}{\alpha + \eta\,\mathrm{cn}(\zeta s)}$$

where

$$\zeta = \frac{(\alpha\delta + \beta\eta)\sqrt{\mu_1 + \mu_2}}{4(\mu_1 - \mu_2)}.$$

This is the result of Levy [21] and Carrier [5]. As a check, at zero this is $K(0) = K_2 = (\beta - \delta)/(\alpha + \eta)$ as it is also a root of Q_1. Similarly at L, where $K(L) = K_1 = (\beta + \delta)/(\alpha - \eta)$.

5. Reduction formulae.

In this section we collect some formulæ that will be used to simplify (34) to one set of descriptive variables. θ satisfies the fourth order BVP (12) with multiplier λ and boundary conditions $\theta(0) = 0$ and $\theta(L) = \frac{\pi}{2}$ and $\theta''(0) = \theta''(L) = 0$ which come from the integral equation (9). (13) is its first integral with constant of integration c_1. Since the ODE is independent of θ we consider instead the second order ODE for $K = \theta'$. One constant of integration is c_0 so that $\theta = c_0 + \int_0^s K(\sigma)\,d\sigma$, which is determined by $c_0 = \theta(0) = 0$. The condition that remains on K for determining the integration constants is the *total curvature condition,* that $\theta(L) - \theta(0) = \frac{\pi}{2} = \int_0^L K(\sigma)\,d\sigma$. Thus (15) is the first integral, a first order ODE for K involving the multiplier λ and two constants of integration c_1 and H. As (15) is autonomous, the remaining constant of integration may be interpreted as translation $K(\sigma) \mapsto K(\sigma + c_2)$. The solutions $K(\sigma; \lambda, c_1, H, c_2)$ of (15) depend on four parameters which are determined by the two boundary conditions, the curvature condition and the area constraint $\int_{\hat{\gamma}} x\,dy = A$.

The second set of parameters that define solutions of (15) are (λ, K_1, K_2, c_2), where K_1 and K_2 are sup and inf of $K(\sigma)$. c_2 is determined henceforth by the condition $K(c_2) = K_1$ or $K = K_1$ when $s = 0$. Of course we may use the parameters (S, P, Λ, c_2) just as well, where $\Lambda = \lambda/S$. In order to reduce the integrals to canned elliptic functions, we expresses solutions of (15) in a new set of parameters $(\lambda, \mu_1, \mu_2, c_2)$. These in turn can be expressed in terms of another set $(\alpha, \beta, \delta, \eta, c_2)$ which satisfy the relation (30), and thus account for the same degrees of freedom. The expression for Area currently involves all of these variables. In order to analyze the expression better we convert (34) to $(\lambda, \mu_1, \mu_2, c_2)$ variables only.

Using the fact that μ_1 and μ_2 are roots of the quadratic equation (22),

$$0 = \Delta = D^2\mu^2 - \left(2S^2 + 8P + 4\Lambda\right)\mu + \left(D^2 - 4\Lambda\right)$$

we obtain

$$\begin{aligned}
\mu_1 + \mu_2 &= \frac{2S^2 + 8P + 4\Lambda}{D^2} && (35)\\
\mu_1 - \mu_2 &= \frac{2\sqrt{\Lambda^2 + 2S^2\Lambda + 4PS^2}}{D^2} = \frac{2\sqrt{(\Lambda + 2K_1S)(\Lambda + 2K_2S)}}{D^2} && (36)\\
\mu_1\mu_2 = &= \frac{D^2 - 4\Lambda}{D^2} && (37)\\
(1+\mu_1)(1+\mu_2) &= \frac{4S^2}{D^2} = \frac{4S^2}{S^2 - 4P} && (38)\\
1 - \mu_1\mu_2 &= \frac{4\Lambda}{D^2} && (39)\\
\frac{(1+\mu_1)(1+\mu_2)}{1 - \mu_1\mu_2} &= \frac{S^2}{\Lambda} = \frac{S^3}{\lambda} && (40)
\end{aligned}$$

Thus using $D^2 + 4P = S^2$, and (26)-(29) we have

$$\begin{aligned}
\alpha\eta &= \frac{2S}{D} && (41)\\
\alpha^2 - \eta^2 &= \mu_1 - \mu_2 && (42)\\
m^2\alpha^2 - \eta^2 &= m^2 - 1 && (43)\\
3\alpha^4 m^2 - 2(m^2+1)\alpha^2\eta^2 + \eta^4 &= (\mu_2\mu_1 - 2(\mu_1 - \mu_2) - 1)(1 - m^2) && (44)\\
\alpha\beta + \delta\eta &= \frac{1}{2}S(\mu_1 - \mu_2) && (45)\\
\mu_2\alpha\beta + \mu_1\delta\eta &= \frac{1}{2}S(\mu_1 - \mu_2) && (46)
\end{aligned}$$

(42), (43), (45) and (46) occur by equating coefficients of (24) and (25). Equating (45) and (46) and using (26) and (28) gives (30). Now we invert the transformation to $(\lambda, \mu_1, \mu_2, c_2)$ variables using (17)

$$\frac{\lambda^{\frac{1}{3}}(1+\mu_1)^{\frac{1}{3}}(1+\mu_2)^{\frac{1}{3}}}{(1 - \mu_1\mu_2)^{\frac{1}{3}}} = S \qquad (47)$$

$$\frac{\lambda^{\frac{2}{3}}(1-\mu_1\mu_2)^{\frac{1}{3}}}{(1+\mu_1)^{\frac{1}{3}}(1+\mu_2)^{\frac{1}{3}}} = \Lambda \tag{48}$$

$$\frac{\lambda^{\frac{2}{3}}\left(\mu_1\mu_2+\mu_1+\mu_2-3\right)}{4\left(1-\mu_1\mu_2\right)^{\frac{2}{3}}\left(1+\mu_1\right)^{\frac{1}{3}}(1+\mu_2)^{\frac{1}{3}}} = P \tag{49}$$

$$\frac{2\lambda^{\frac{1}{3}}}{\left(1-\mu_1\mu_2\right)^{\frac{1}{3}}\left(1+\mu_1\right)^{\frac{1}{6}}(1+\mu_2)^{\frac{1}{6}}} = D \tag{50}$$

$$\frac{\lambda^{\frac{2}{3}}\left[6+2\mu_1+2\mu_2+6\mu_1\mu_2\right]}{16\left(1-\mu_1\mu_2\right)^{\frac{2}{3}}\left(1+\mu_1\right)^{\frac{1}{3}}\left(1+\mu_2\right)^{\frac{1}{3}}} = \frac{S^2-2P-\Lambda}{4} = c_1 \tag{51}$$

$$\frac{\lambda^{\frac{1}{3}}(1+\mu_2)^{\frac{1}{3}}(\mu_1-1)}{2(1-\mu_1\mu_2)^{\frac{1}{3}}(1+\mu_1)^{\frac{1}{6}}} = \sqrt{(1+\mu_1)P+\Lambda} = \beta \tag{52}$$

$$\frac{\lambda^{\frac{1}{3}}(1+\mu_1)^{\frac{1}{3}}(1-\mu_2)}{2(1-\mu_1\mu_2)^{\frac{1}{3}}(1+\mu_2)^{\frac{1}{6}}} = \sqrt{(1+\mu_2)P+\Lambda} = \delta \tag{53}$$

$$\frac{\lambda^{\frac{1}{3}}(\mu_1-\mu_2)}{(1-\mu_1\mu_2)^{\frac{1}{3}}(1+\mu_1)^{\frac{1}{6}}(1+\mu_2)^{\frac{1}{6}}} = \alpha\delta+\beta\eta \tag{54}$$

$$\frac{\lambda^{\frac{1}{3}}(1-\mu_1\mu_2)^{\frac{2}{3}}}{(1+\mu_1)^{\frac{1}{6}}(1+\mu_2)^{\frac{1}{6}}} = \alpha\delta-\beta\eta \tag{55}$$

$$-\frac{\lambda^{\frac{2}{3}}\left[\mu_1\mu_2^2+3\mu_1\mu_2+3\mu_2+1\right]}{2(1-\mu_1\mu_2)^{\frac{2}{3}}(1+\mu_1)^{\frac{1}{3}}(1+\mu_2)^{\frac{1}{3}}} = \delta^2-2c_1\eta^2 \tag{56}$$

We have used the fact that $\mu_1 > 1$ and $\mu_2 < 1$.

We now record the expressions for L, Θ and A in $(\lambda, \mu_1, \mu_2, c_2)$ variables. By (31), we obtain

$$L = \frac{4(1-\mu_1\mu_2)^{\frac{1}{3}}(1+\mu_1)^{\frac{1}{6}}(1+\mu_2)^{\frac{1}{6}}}{\lambda^{\frac{1}{3}}\sqrt{\mu_1}}\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right) = \frac{8\mathcal{K}(m)}{D\sqrt{\mu_1}}. \tag{57}$$

Similarly (32) by (53) and (20) becomes

$$\Theta = \frac{4(\mu_1-\mu_2)}{(1+\mu_1)^{\frac{1}{2}}(1+\mu_2)^{\frac{1}{2}}\sqrt{\mu_1}}\Pi\left(\frac{1+\mu_2}{1+\mu_1}, \sqrt{\frac{\mu_2}{\mu_1}}\right) - \frac{2(1+\mu_1)^{\frac{1}{2}}(1-\mu_2)}{(1+\mu_2)^{\frac{1}{2}}\sqrt{\mu_1}}\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right) \tag{58}$$

It is noteworthy that this expression is independent of λ. It means that the the deformations of the elastic ring go through the same shapes, irregardless of size. Using (42) and (43) we simplify the expression (34) to yield

$$\begin{aligned} A &= \frac{8}{\eta^2\lambda\sqrt{\mu_1}}\left[\frac{(\delta^2-2c_1\eta^2)(\mu_1-\mu_2)}{\alpha\delta+\beta\eta} - \alpha\delta-\beta\eta\right]\mathcal{K}(m) \\ &\quad -\frac{8\left[(\alpha\delta+\beta\eta)(\mu_1\mu_2-1)+(\alpha\delta-\beta\eta)(\mu_1-\mu_2)\right]}{\alpha^2\lambda\eta^2\sqrt{\mu_1}}\Pi\left(\frac{\eta^2}{\alpha^2}, m\right) - \frac{8(\alpha\delta+\beta\eta)}{\lambda(m^2-1)\sqrt{\mu_1}}\mathcal{E}(m). \end{aligned}$$

Then (54), (55) and (56) imply

$$A = \frac{8\mu_1\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right) - 4\left(\mu_1\mu_2+2\mu_1+1\right)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)}{\lambda^{\frac{2}{3}}(1-\mu_1\mu_2)^{\frac{1}{3}}(1+\mu_1)^{\frac{1}{6}}(1+\mu_2)^{\frac{1}{6}}\sqrt{\mu_1}}. \tag{59}$$

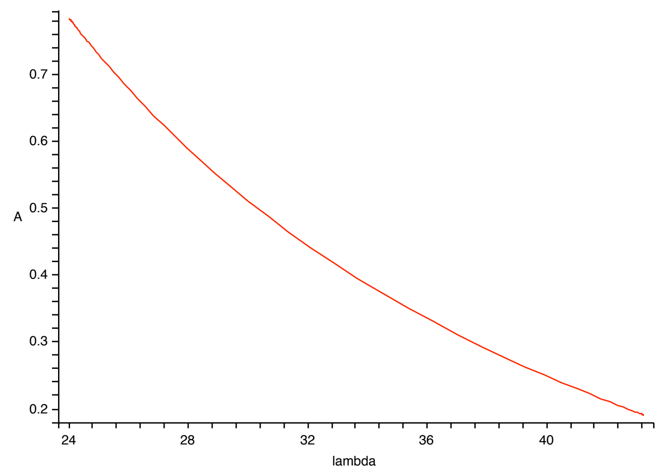

Figure 6: Elastica pressure λ vs. quarter area A.

6. Computation of deformation moduli.

The explicit formulæ allow differentiation to obtain explicit rates of change. For example let us compute the pressure modulus of area $\mathrm{d}\lambda/\mathrm{d}\ln\mathrm{A}$. Then there is a mapping $F(\mu_1,\mu_2,\lambda) = (\Theta(\mu_1,\mu_2), L(\mu_1,\mu_2,\lambda))$ implicitly defines (μ_1,μ_2) in terms of λ so the result follows from differentiating

$$\frac{d\ln A}{d\lambda} = \frac{\partial \ln A}{\partial\mu_1}\frac{\partial\mu_1}{\partial\lambda} + \frac{\partial \ln A}{\partial\mu_2}\frac{\partial\mu_2}{\partial\lambda} + \frac{\partial \ln A}{\partial\lambda}$$

Since Θ and L are constant, differentiating F, we find

$$\begin{aligned} 0 &= \frac{\partial\ln\Theta}{\partial\lambda} &= \frac{\partial\ln\Theta}{\partial\mu_1}\frac{\partial\mu_1}{\partial\lambda} + \frac{\partial\ln\Theta}{\partial\mu_2}\frac{\partial\mu_2}{\partial\lambda} \\ 0 &= \frac{\partial\ln L}{\partial\lambda} &= \frac{\partial\ln L}{\partial\mu_1}\frac{\partial\mu_1}{\partial\lambda} + \frac{\partial\ln L}{\partial\mu_2}\frac{\partial\mu_2}{\partial\lambda} + \frac{\partial\ln L}{\partial\lambda} \end{aligned}$$

so by Cramer's rule,

$$\begin{pmatrix} \frac{\partial\mu_1}{\partial\lambda} \\ \frac{\partial\mu_2}{\partial\lambda} \end{pmatrix} = -\begin{pmatrix} \frac{\partial\ln\Theta}{\partial\mu_1} & \frac{\partial\ln\Theta}{\partial\mu_2} \\ \frac{\partial\ln L}{\partial\mu_1} & \frac{\partial\ln L}{\partial\mu_2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{\partial\ln L}{\partial\lambda} \end{pmatrix} = \frac{\frac{\partial\ln L}{\partial\lambda}}{\frac{\partial\ln\Theta}{\partial\mu_1}\frac{\partial\ln L}{\partial\mu_2} - \frac{\partial\ln\Theta}{\partial\mu_2}\frac{\partial\ln L}{\partial\mu_1}} \begin{pmatrix} \frac{\partial\ln\Theta}{\partial\mu_2} \\ -\frac{\partial\ln\Theta}{\partial\mu_1} \end{pmatrix}$$

which means that

$$\frac{d\ln A}{d\lambda} = \frac{\frac{\partial\ln L}{\partial\lambda}\left(\frac{\partial\ln A}{\partial\mu_1}\frac{\partial\ln\Theta}{\partial\mu_2} - \frac{\partial\ln A}{\partial\mu_2}\frac{\partial\ln\Theta}{\partial\mu_1}\right)}{\frac{\partial\ln\Theta}{\partial\mu_1}\frac{\partial\ln L}{\partial\mu_2} - \frac{\partial\ln\Theta}{\partial\mu_2}\frac{\partial\ln L}{\partial\mu_1}} + \frac{\partial\ln A}{\partial\lambda}. \tag{60}$$

Let us compute the six partial derivatives in this formula. The basic differentiation formulæ for the elliptic functions are

$$\frac{d\mathcal{E}(m)}{d\,m^2} = \frac{\mathcal{E}(m) - \mathcal{K}(m)}{2m^2} \tag{61}$$

$$\frac{d\mathcal{K}(m)}{d\,m^2} = \frac{\mathcal{E}(m)}{2m^2(1-m^2)} - \frac{\mathcal{K}(m)}{2m^2} \tag{62}$$

$$\frac{d}{d\,q}\left[\frac{1}{\sqrt{q}}\mathcal{K}\left(\frac{p}{q}\right)\right] = -\frac{1}{2(q-p)\sqrt{q}}\mathcal{E}\left(\frac{p}{q}\right) \tag{63}$$

$$\frac{d\Pi(n,m)}{d\,m^2} = \frac{\mathcal{E}(m)}{2(m^2-n)(1-m^2)} - \frac{\Pi(n,m)}{2(m^2-n)} \tag{64}$$

$$\frac{d\Pi(n,m)}{d\,n} = \frac{(m^2-n^2)\Pi(n,m)}{2(1-n)(m^2-n)n} - \frac{\mathcal{K}(m)}{2(1-n)n} - \frac{\mathcal{E}(m)}{2(1-n)(m^2-n)}. \tag{65}$$

Differentiating (57), we find using (62) or (63),

$$\ln L = \ln 4 + \frac{\ln(1-\mu_1\mu_2)}{3} + \frac{\ln(1+\mu_1)}{6} + \frac{\ln(1+\mu_2)}{6} - \frac{\ln\lambda}{3} - \frac{\ln\mu_1}{2} + \ln\left(\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\right)$$

$$\frac{\partial \ln L}{d\,\mu_1} = \frac{1-2\mu_2-3\mu_1\mu_2}{6(1-\mu_1\mu_2)(1+\mu_1)} - \frac{\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)}{2(\mu_1-\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)} \tag{66}$$

$$\frac{\partial \ln L}{d\,\mu_2} = \frac{\mu_1\mu_2-2\mu_2-3}{6(1-\mu_1\mu_2)(1+\mu_2)\mu_2} + \frac{\mu_1\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)}{2\mu_2(\mu_1-\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)} \tag{67}$$

$$\frac{\partial \ln L}{d\,\lambda} = -\frac{1}{3\lambda}. \tag{68}$$

Differentiating (58), we find using (62), (64) and (65),

$$\begin{aligned}
\Theta &= \frac{2}{(1+\mu_1)^{\frac{1}{2}}(1+\mu_2)^{\frac{1}{2}}\sqrt{\mu_1}}\left[2(\mu_1-\mu_2)\Pi\left(\frac{1+\mu_2}{1+\mu_1},\sqrt{\frac{\mu_2}{\mu_1}}\right) - (1+\mu_1)(1-\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\right] \\
\ln\Theta &= \ln 2 - \frac{\ln(1+\mu_1)}{2} - \frac{\ln(1+\mu_2)}{2} - \frac{\ln\mu_1}{2} \\
&\quad + \ln\left[2(\mu_1-\mu_2)\Pi\left(\frac{1+\mu_2}{1+\mu_1},\sqrt{\frac{\mu_2}{\mu_1}}\right) - (1+\mu_1)(1-\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\right] \\
\frac{\partial\ln\Theta}{d\,\mu_1} &= -\frac{1}{2(1+\mu_1)} \\
&\quad + \frac{2(\mu_1-\mu_2)^2\Pi\left(\frac{1+\mu_2}{1+\mu_1},\sqrt{\frac{\mu_2}{\mu_1}}\right) + (1+\mu_1)(\mu_1-\mu_2)\mu_2\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right) + (1-\mu_1^2)(1-\mu_2)\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)}{2(\mu_1-\mu_2)(1+\mu_1)\left[2(\mu_1-\mu_2)\Pi\left(\frac{1+\mu_2}{1+\mu_1},\sqrt{\frac{\mu_2}{\mu_1}}\right) - (1+\mu_1)(1-\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\right]} \\
&= \frac{(\mu_1-\mu_2)(1+\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right) + (1-\mu_1)(1+\mu_2)\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)}{2(\mu_1-\mu_2)\left[2(\mu_1-\mu_2)\Pi\left(\frac{1+\mu_2}{1+\mu_1},\sqrt{\frac{\mu_2}{\mu_1}}\right) - (1+\mu_1)(1-\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\right]} \qquad (69) \\
\frac{\partial\ln\Theta}{d\,\mu_2} &= -\frac{1}{2(1+\mu_2)} \\
&\quad + \frac{\left\{\begin{array}{c} 2(\mu_1-\mu_2)^2\mu_2\Pi\left(\frac{1+\mu_2}{1+\mu_1},\sqrt{\frac{\mu_2}{\mu_1}}\right) + (1+\mu_1)(1+\mu_2^2)(\mu_1-\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right) \\ -(1-\mu_2^2)(1+\mu_1)\mu_1\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\end{array}\right\}}{2(\mu_1-\mu_2)(1+\mu_2)\mu_2\left[2(\mu_1-\mu_2)\Pi\left(\frac{1+\mu_2}{1+\mu_1},\sqrt{\frac{\mu_2}{\mu_1}}\right) - (1+\mu_1)(1-\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\right]}
\end{aligned}$$

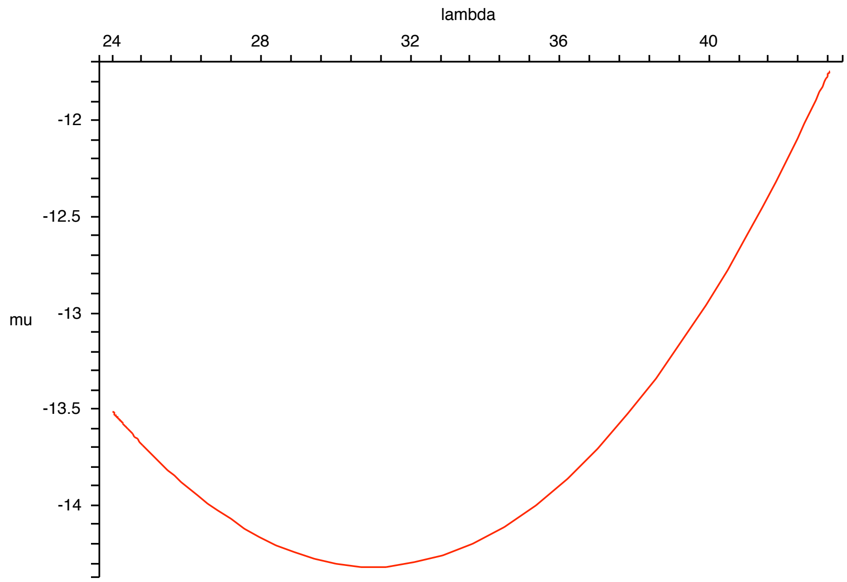

Figure 7: Modulus $\mathrm{d}\lambda/\mathrm{d}\ln A$ versus λ.

$$= \frac{(1+\mu_1)\left[(\mu_1-\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)-(1-\mu_2)\mu_1\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\right]}{2(\mu_1-\mu_2)\mu_2\left[2(\mu_1-\mu_2)\Pi\left(\frac{1+\mu_2}{1+\mu_1},\sqrt{\frac{\mu_2}{\mu_1}}\right)-(1+\mu_1)(1-\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\right]} \tag{70}$$

(71)

Differentiating (59), we find using (62) and (64),

$$\begin{aligned}
A &= \frac{8\mu_1\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)-4\left(\mu_1\mu_2+2\mu_1+1\right)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)}{\lambda^{\frac{2}{3}}(1-\mu_1\mu_2)^{\frac{1}{3}}(1+\mu_1)^{\frac{1}{6}}(1+\mu_2)^{\frac{1}{6}}\sqrt{\mu_1}}. \\
\ln A &= \ln 4-\frac{2\ln\lambda}{3}-\frac{\ln(1-\mu_1\mu_2)}{3}-\frac{\ln(1+\mu_1)}{6}-\frac{\ln(1+\mu_2)}{6} \\
&\quad +\ln\left[\frac{2\mu_1\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)-(\mu_1\mu_2+2\mu_1+1)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)}{\sqrt{\mu}_1}\right] \\
\frac{\partial\ln A}{d\,\mu_1} &= \frac{3\mu_1\mu_2+2\mu_2-1}{6(1-\mu_1\mu_2)(1+\mu_1)} \\
&\quad +\frac{(\mu_1\mu_2+2\mu_1+1)\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)-2(\mu_1-\mu_2)(1+\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)}{2(\mu_1-\mu_2)\left[2\mu_1\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)-(\mu_1\mu_2+2\mu_1+1)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\right]} \qquad (72) \\
\frac{\partial\ln A}{d\,\mu_2} &= \frac{3\mu_1\mu_2+2\mu_1-1}{6(1-\mu_1\mu_2)(1+\mu_2)}
\end{aligned}$$

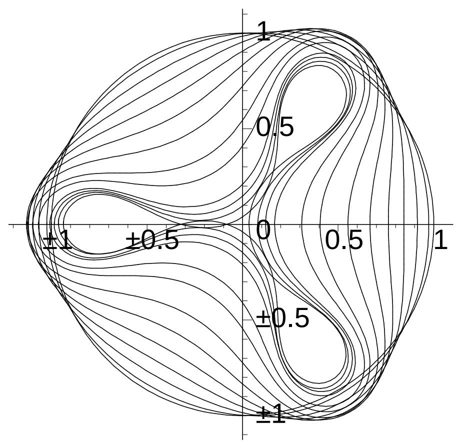

Figure 8: Mode $n = 3$ elastica for various pressure and length $L = \pi/2$.

$$-\frac{(\mu_1\mu_2 + 2\mu_2 + 1)\mu_1 \mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right) - (1 - \mu_1\mu_2)(\mu_1 - \mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)}{2(\mu_1 - \mu_2)\mu_2\left[2\mu_1\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right) - (\mu_1\mu_2 + 2\mu_1 + 1)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\right]} \tag{73}$$

$$\frac{\partial \ln A}{\partial \lambda} = -\frac{2}{3\lambda}. \tag{74}$$

These expressions are used in equation (60) to obtain the modulus.

7. Numerical results.

First we observe that the circle is the limiting figure as $D \to 0$. The formulas (23) are not effective for computation for small D, however, the expressions (20,32) may be recomputed in terms of $D^2\mu_i$ and become nonsingular as $D \to 0$. To see the limiting circle, make the change of variable

$$K = \frac{S}{2} + \frac{D}{2}T,$$

in equation (32) to find

$$\Theta = \int_{-1}^{1} \frac{2(S + DT)\sqrt{S}\,dT}{\sqrt{(1 - T^2)(4S^3 - SD^2 + 4\lambda + 4S^2TD + ST^2D^2)}} \to \frac{\pi\sqrt{S^3}}{\sqrt{S^3 + \lambda}} \tag{75}$$

as $D \to 0$. Since $\pi/n = \Theta(L)$ for n-th mode buckling and since the initial circle has radius $R_0 = 1/K_0 = 2/S_0$, it follows that the pressure needed to deform the ring is given by

$$\lambda = \frac{8(n^2 - 1)}{R_0^3} \qquad \Longleftrightarrow \qquad \mathcal{P} = \frac{\mathcal{B}(n^2 - 1)}{R_0^3}$$

Of course this is the familiar pressure needed to buckle the ring!

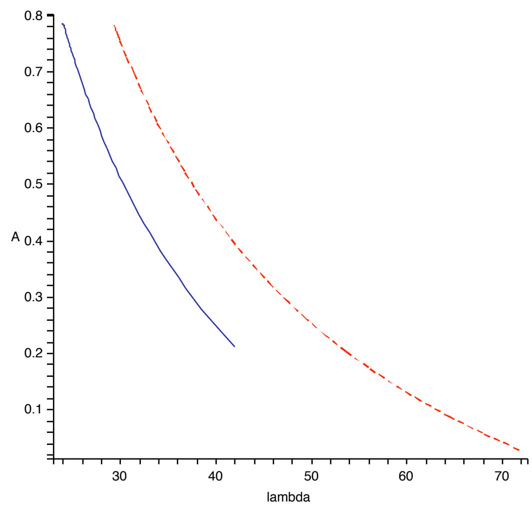

Figure 9: Pressure λ vs. quarter area A for elastica (solid line) and peanuts (dashed line).

Therefore, we may identify the limiting values of μ_1 and μ_2 from (23) also. As S is bounded away from zero, we see that $\mu_1 \to \infty$ as $D \to 0$ since the numerator stays away from zero. Rationalizing the denominator in (23), we see that

$$\mu_2 = \frac{D^2 S - 4\lambda}{2S^3 - D^2 S + 2\lambda + 2\sqrt{(\lambda + S^3)^2 - S^4 D^2}} \to -\frac{\lambda}{S^3 + \lambda} = -\frac{3}{4},$$

if $\Theta(L) = \frac{\pi}{2}$ as is the case here.

We may also compute the modulus at the circle by expanding S and λ in powers of D near $D = 0$. Expanding $S^{1/2}$ and $(4S^3 - SD^2 + 4\lambda + 4S^2 TD + ST^2 D^2)^{-1/2}$ in powers of D in (75) and (20), integrating and equating $\Theta = \pi/2$ and $L = \pi/2$, we may solve for the coefficients

$$\begin{aligned} S &\approx S_0 + S_1 D + S_2 D^2 + \dots = 2 + \frac{1}{32} D^2 + \dots, \\ \lambda &\approx \lambda_0 + \lambda_1 D + \lambda_2 D^2 + \dots = 24 + \frac{9}{16} D^2 + \dots. \end{aligned}$$

Using these and (17) which implies $4c_1 = (S^2 + D^2)/2 + \lambda/S$ in λA given by (21), after expanding and integrating we find

$$A \approx a_0 + a_1 D + a_2 D^2 + \dots = \frac{\pi}{4} - \frac{\pi}{96} D^2 + \dots.$$

Thus, using L'Hospitals's rule, the modulus of compression for the elastica at the circle is

$$A \frac{d\lambda}{dA} = \frac{\pi \lambda_2}{4 a_2} = -13.5.$$

Using the described procedure, we list dimensions for several closed curves in Fig. 11. We also plot some $n = 2$ elastic rings in Fig. 5. These were obtained from the choices of K_2's indicated.[1] $\mathcal{I} = 4\pi A_0 / L_0^2$ is the

[1]This table was computed by using the MAPLE computer algebra system.

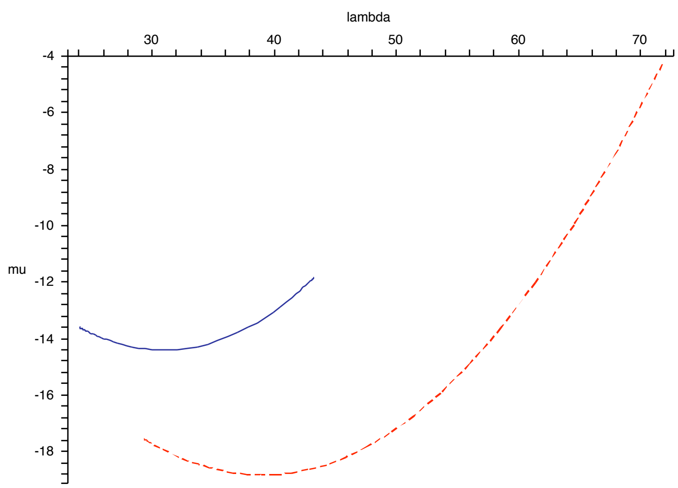

Figure 10: Pressure λ vs. modulus $d\lambda/d\ln A$ for elastica (solid line) and peanuts (dashed line).

isoperimetric ratio. The coordinates $X = x(L)$ and $Y = y(L)$ are the endpoints of the solution segment γ. Thus X/Y is the ratio of the neckwidth to the wingspan. $E_0 = 4\mathbb{E}(L)$ is the energy of the closed figure. $L_0 = 4L$ is the length of the full loop, A_0 is its enclosed area. $K_{\max} = K_1$ and $K_{\min} = K_2$ are the minimum and maximum curvatures. λ is the Lagrange multiplier parameter in (14), and is always $\lambda = 1$ for these computations. The first row corresponds to the circle.

The rings remain embedded for $K_2 > -.2878$, suggesting that the embedded minimizer of the variational problems is not given by these figures for isoperimetric ratios below the critical $\mathcal{I}_0 = .270949$. The ratio $\mathcal{I}_c = .819469$ is the transition point between convex and nonconvex minimizers. Observe that when $K_2 = -.2878$ then $K1 = 1.1282$ so that $(\mu_1, \mu_2) = (2.364, -.5811)$. The figures become nonconvex at $K_2 = 0$ when $K_1 = .7189988$ and $(\mu_1, \mu_2) = (13.485428, -.72386)$.

The area response to pressure can be computed as follows: for $\mu_1 \in [2.364, \infty)$ given, we solve (58) $\Theta = \frac{\pi}{2}$ for μ_2. Then for fixed $\lambda = 1$, say, we can compute L and A from equations (57) and (59). Then by scaling by a factor $c = L_0/L$, where $L_0 = \frac{\pi}{2}$ for the unit circle, $\tilde{L} = cL = L_0$ is held fixed and we get the values $\tilde{A} = 4c^2 A$ and $\tilde{\lambda} = 1/c^3$ for the corresponding area and dimensionless pressure. We obtain the area response to pressure, computed theoretically, plotted on Fig. 6. Note that since $R \to 2 \cdot 3^{1/3}$ as $D \to 0$, we see that $L \to \pi 3^{1/3}$ so $c \to 2^{-1}3^{-1/3}$ so $\tilde{\lambda} \to 24$. It is tabulated in Fig. 11

Continuing in this fashion, we now show the plot of the deformation modulus of the ring. As the modulus $\mathrm{d}\lambda/\mathrm{d}\ln\mathrm{A}$ is expressible in terms of the μ_i's through formula (60), whose terms are computed using (66)–(74), we display the result in Fig. 7. A short tabulation is in Fig. 13.

What evidence is there that the other closed curves that satisfy (14) are not the minimizers such as if K is periodic of period L_0/n where the mode $n \neq 2$? We must have at least $n = 2$ (four critical points of curvature) because of the Four Vertex Theorem for closed plane curves [10]. For example, there are closed curves with $\Theta(L) = \frac{\pi}{3}$. Then $L_0 = 6L$ and the other variables are suitably increased. The curve $\gamma = \gamma([0, L])$ makes up one sixth of the boundary. The area inside Γ is then six times the area between γ and the y-axis plus the area of the equilateral triangle whose base is $2x(L)$. Thus $A_0 = 6A(L) + \sqrt{3}[x(L)]^2$. This time, the ratio $\mathcal{I}_c = .935405$ is the transition point between convex and nonconvex minimizers and the figures remain embedded for $K_2 > -.516$. Fig. 12 has short table of closed $n = 3$ solutions. R_i and R_o are the distances to the center from points of max curvature and minimum curvature. Notice that the energy is higher for this family of solutions than for the $n = 2$ family. Several examples are plotted in Figure 8.

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Department of Materials Science and Engineering
University of Utah,
Salt Lake City, UT 84112
Tel: (801)-587-7719; Fax: (801)-581-4816
`fliu@eng.utah.edu`

Department of Mathematics
University of Utah,
Salt Lake City, UTAH 84112-0090
Tel: (801)-581-8350; Fax: (801)-581-4148
`treiberg@math.utah.edu`

Dimensions of some closed $n=2$ solutions of the Euler Lagrange equation.							
$\mathcal{I}$	$L_0E_0/16$	X/Y	$4X/L_0$	$L_0K_{\max}$	$L_0K_{\min}$	$L_0^3\lambda$	K_2
1.000000	2.467401	1.000000	.636620	6.283185	6.283185	93.0188	.3467
.996952	2.489981	.913645	.607197	7.140493	5.440264	93.1786	.3000
.986799	2.565479	.827725	.573754	8.086784	4.542239	93.7152	.2500
.969301	2.696626	.747252	.538229	9.067561	3.645911	94.6560	.2000
.944192	2.887138	.671051	.500500	10.089812	2.747722	96.0428	.1500
.911147	3.142166	.598089	.460398	11.162443	1.843779	97.9370	.1000
.869755	3.468829	.527406	.417693	12.297110	.929642	100.4284	.0500
.819469	3.877073	.458052	.372066	13.509586	.000000	103.6482	.0000
.759535	4.381159	.389001	.323060	14.822106	-.951833	107.7935	-.0500
.688845	5.002399	.319027	.269997	16.267738	-1.934825	113.1737	-.1000
.605652	5.774732	.246471	.211802	17.899348	-2.961980	120.3075	-.1500
.558499	6.234491	.208474	.180246	18.811504	-3.498161	124.8031	-.1750
.506900	6.757775	.168724	.146597	19.810861	-4.054290	130.1591	-.2000
.449965	7.361994	.126516	.110312	20.925359	-4.635973	136.6769	-.2250
.386270	8.075069	.080740	.070527	22.200262	-5.251849	144.8561	-.2500
.313266	8.947093	.029439	.025704	23.717686	-5.916782	155.6248	-.2750
.297285	9.146632	.018307	.015977	24.060199	-6.056768	158.2139	-.2800
.290858	9.227832	.013835	.012072	24.199177	-6.112519	159.2813	-.2819
.283012	9.327692	.008381	.007311	24.369799	-6.180138	160.6050	-.2842
.280424	9.360815	.006582	.005741	24.426323	-6.202339	161.0468	-.2850
.273533	9.449446	.001796	.001566	24.577413	-6.261195	162.2355	-.2870
.271792	9.471947	.000586	.000511	24.615733	-6.276010	162.5389	-.2875
.270919	9.483250	-.000020	-.000018	24.634978	-6.283433	162.6915	-.2878
.270043	9.494589	-.000628	-.000548	24.654280	-6.290866	162.8448	-.2880
.262969	9.586610	-.005539	-.004828	24.810797	-6.350717	164.0946	-.2900
.225459	10.086698	-.031575	-.027462	25.657864	-6.661573	171.0751	-.3000

Figure 11: Table of selected $n=2$ solutions.

Dimensions of some closed $n=3$ solutions of the Euler Lagrange equation.							
$\mathcal{I}$	$L_0E_0/16$	R_i/R_o	$6R_i/L_0$	$L_0K_{\max}$	$L_0K_{\min}$	$L_0^3\lambda$	K_2
1.000000	2.467401	1.000000	.636620	6.283185	6.283185	248.0502	.2500
.997486	2.517074	.951033	.619948	7.551076	5.029361	248.4668	.2000
.977085	2.923221	.857998	.581947	10.170077	2.526227	251.9051	.1000
.935405	3.770823	.768858	.537321	12.943128	0.000000	259.2699	.0000
.871105	5.128532	.680524	.485215	15.943671	-2.590495	271.6242	-.1000
.781953	7.122885	.589344	.424075	19.292342	-5.302010	291.1063	-.2000
.663623	10.005778	.489695	.350832	23.219891	-8.228261	322.3892	-.3000
.505282	14.399987	.369229	.257937	28.300094	-11.572790	378.4027	-.4000
.400689	17.751277	.289918	.197048	31.828129	-13.578226	429.2513	-.4500
.249354	23.514553	.166285	.106787	37.626785	-16.262423	537.6081	-.5000
.200059	25.722796	.121863	.076431	39.824509	-17.084904	587.4178	-.5100
.159972	27.640560	.082986	.050930	41.760762	-17.680307	635.9165	-.5140
.113622	30.134387	.035783	.021363	44.274668	-18.399463	704.3099	-.5170

Figure 12: Table of selected $n=3$ solutions.

The modulus $\mathrm{d}\lambda/\mathrm{d}\ln \mathrm{A}$ when $L=\pi/2$ for some selected $n=2$ shapes.			
λ	μ_1	$\mathcal{I}$	Modulus
43.214273	2.236000	0.244317	-11.722405
43.150076	2.242250	0.245657	-11.749869
42.712516	2.286000	0.254904	-11.933148
41.620321	2.404750	0.278880	-12.360812
39.823458	2.636000	0.321361	-12.971827
37.548166	3.017250	0.381373	-13.580778
35.157118	3.586000	0.453420	-14.023526
32.953310	4.379750	0.529819	-14.254401
31.090308	5.436000	0.603562	-14.318649
29.595549	6.792250	0.670028	-14.284643
28.429449	8.486000	0.727167	-14.205908
27.530759	10.554750	0.774806	-14.114369
26.839726	13.036000	0.813806	-14.025888
26.306443	15.967250	0.845432	-13.946938
25.892036	19.386000	0.870992	-13.879212
25.567207	23.329750	0.891660	-13.822248
25.310187	27.836000	0.908424	-13.774771
25.104864	32.942250	0.922085	-13.735322
24.939288	38.686000	0.933282	-13.702527
24.804548	45.104750	0.942513	-13.675195
24.693951	52.236000	0.950173	-13.652329

Figure 13: Table of Modulus for selected shapes.