Intrinsic Geometry

Andrejs Treibergs

University of Utah

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Part of a series of thee lectures on geometric analysis.

1. "Curvature and the geometry of surfaces" by Nick Korevaar,
2. "Intrinsic geometry" by Andrejs Treibergs,
3. "Calculus of variations and minimal surfaces" by Nat Smale.

The URL for these Beamer Slides: *"Geometric Analysis: Intrinsic Geometry"*

`http://www.math.utah.edu/treiberg/IntrinsicSlides.pdf`

3. Outline.

- Hyperbolic Space.
- Parabolicity.
- Complete Manifolds with Finite Total Curvature.

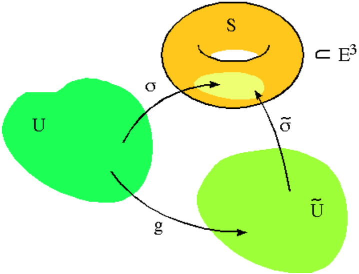

Figure: Extrinsic: Coordinate Charts for Surface in $\mathbb{E}^3$

Surfaces $S \subset \mathbb{R}^3$ inherit the metric ds^2 from $\mathbb{E}^3$. By the Korn - Lichtenstein theorem, $\exists$ local isothermal charts, $\sigma : U \to S$, $\tilde{\sigma} : \tilde{U} \to S$

$$\phi(z)^2 \, |dz|^2 = \sigma^* \left(ds^2 \right), \qquad \tilde{\phi}(\tilde{z})^2 \, |d\tilde{z}|^2 = \tilde{\sigma}^* \left(ds^2 \right)$$

where $z = x + iy$ so $|dz|^2 = dx^2 + dy^2$.

The induced metrics are consistently defined. The transition function $g : U \to \tilde{U}$ given by $g = \tilde{\sigma}^{-1} \circ \sigma$ is holomorphic (if orientation preserved) which identifies local metrics by a change of variables

$$\phi(z)^2 \, |dz|^2 = \tilde{\phi}(g(z))^2 \left| \frac{dg}{dz} \right|^2 |dz|^2 = g^* \left(\tilde{\phi}(\tilde{z})^2 \, |d\tilde{z}|^2 \right).$$

Thus, oriented surfaces with a Riemannian metric have the structure of a Riemann Surface.

We don't need to embed the surface in Euclidean Space as long as we have a cover S by charts and define the INTRINSIC METRIC of S chartwise in a consistent way.

The Riemannian metric gives length and angles of vectors and lengths of curves. If $\gamma : [\alpha, \beta] \to S$ then

$$\mathsf{L}(\gamma) = \int_\alpha^\beta \phi\big(\gamma(t)\big)\, |\dot\gamma(t)|\, dt$$

The Riemannian metric induces a distance function on S. If $P, Q \in S$,

$$\mathsf{d}(P,Q) = \inf\left\{ \mathsf{L}(\gamma) : \begin{array}{c} \gamma : [\alpha,\beta] \to S \text{ is piecewise } C^1, \\ \gamma(\alpha) = P,\ \gamma(\beta) = Q \end{array} \right\}$$

Theorem

(S, d) *is a metric space.*

7. Hyperbolic Space invented to show independence of 5th Postulate.

Euclid's Postulates are the following:

1. any two points may be joined by a line segment;
2. any line segment may be extended to form a line;
3. a circle may be drawn with any given center and distance;
4. any two right angles are equal;
5. (Playfair's Version) Given any line m and a point p, there is a unique line through p and parallel to m.

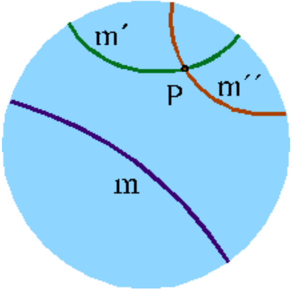

Figure: m' and m'' are parallels to m through P. This is Poincaré's model of the Hyperbolic Plane $\mathbb{H}^2$. The space is the unit disk. Lines are diameters or arcs of circles that are perpendicular to the boundary circle.

In letters found after his death, Gauß had already realized in 1816 that there are geometries in which the Fifth Postulate fails. J. Bolyai and N. Lobachevski independently proved it in 1823 and 1826 by essentially constructing Poincaré's model. They assumed an axiom of Saccheri, who tried to reach a contradiction from it to prove the Fifth postulate.

5. Given any line m and a point p not in m, there are at least two lines through p and parallel to m.

This axiom is also known as the hyperbolic axiom. In 1854, Riemann showed a consistent geometry may also be constructed assuming instead that no lines are parallel.

9. The metric of the Poincaré's Model. $\mathbb{H}^2 = (\mathbb{D}, ds^2)$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk. The Poincaré metric is

$$ds^2 = \phi(z)^2 \, |dz|^2 \qquad \text{where} \qquad \phi(z) = \frac{2}{1 - |z|^2}.$$

Theorem (Hilbert, 1901)

There is no C^2 isometric immersion $\sigma : \mathbb{H}^2 \to \mathbb{E}^3$.

The metric is invariant under rotation about the origin $z \mapsto e^{i\alpha} z$ ($\alpha \in \mathbb{R}$) and reflection $z \mapsto \bar{z}$. It is also invariant under the holomorphic self-maps of $\mathbb{D}$. Such maps $f : \mathbb{D} \to \mathbb{D}$ that fix the circle and map $p \in \mathbb{D}$ to 0 have the form

$$w = f(z) = \frac{e^{i\alpha}(z - p)}{1 - \bar{p}z}$$

They are isometries of the Poincaré plane because the pulled-back metric

$$
\begin{aligned}
f^*(ds^2) &= \phi(w)^2|dw|^2 \\
&= \frac{4}{\left(1 - \frac{|z-p|^2}{|1-\bar{p}z|^2}\right)^2} \frac{(1-|p|^2)^2|dz|^2}{|1-\bar{p}z|^4} \\
&= \frac{4(1-|p|^2)^2|dz|^2}{\left(|1-\bar{p}z|^2 - |z-p|^2\right)^2} \\
&= \frac{4(1-|p|^2)^2|dz|^2}{\left((1-\bar{p}z)(1-p\bar{z}) - (z-p)(\bar{z}-\bar{p})\right)^2} \\
&= \frac{4(1-|p|^2)^2|dz|^2}{\left(1 - p\bar{z} - \bar{p}z + |p|^2|z|^2 - |z|^2 + \bar{p}z + p\bar{z} - |p|^2\right)^2} \\
&= \frac{4(1-|p|^2)^2|dz|^2}{(1-|p|^2)^2(1-|z|^2)^2} = \frac{4|dz|^2}{(1-|z|^2)^2} = \phi(z)^2|dz|^2.
\end{aligned}
$$

A geodesic is a curve that locally minimizes the length. A calculus of variations argument shows geodesics satisfy a 2nd order ODE.

If $\zeta : [a, b] \to U$ is minimizing in an isothermic patch, and $\eta : [a, b] \to \mathbb{C}$ is a varation such that $\eta(a) = \eta(b) = 0$, then the length $\mathsf{L}(\zeta + \epsilon\eta)$ is least when $\epsilon = 0$ so

$$
\begin{aligned}
0 &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathsf{L}(\zeta + \epsilon\eta) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_a^b \phi(\zeta + \epsilon\eta) \left|\dot{\zeta}(t) + \epsilon\dot{\eta}(t)\right| \, dt \\
&= \left.\left(\int_a^b \nabla\phi(\zeta + \epsilon\eta) \bullet \eta |\dot{\zeta} + \epsilon\dot{\eta}| + \phi(\zeta + \epsilon\eta) \frac{(\dot{\zeta} + \epsilon\dot{\eta}) \bullet \dot{\eta}}{|\dot{\zeta} + \epsilon\dot{\eta}|} \, dt \right)\right|_{\epsilon=0} \\
&= \int_a^b \left(\nabla\phi(\zeta)|\dot{\zeta}| - \frac{d}{dt}\left[\phi(\zeta)\frac{\dot{\zeta}}{|\dot{\zeta}|}\right] \right) \bullet \eta
\end{aligned}
$$

Since η is arbitrary, the geodesic satisfies the 2nd order ODE system

$$\frac{d}{dt}\left[\phi(\zeta)\frac{\dot{\zeta}}{|\dot{\zeta}|}\right] - \nabla\phi(\zeta)|\dot{\zeta}| = 0. \tag{1}$$

Theorem

For every $P \in S$ there is a neighborhood U such that if $Q_1, Q_2 \in U$ there there is a unique smooth distance realizing curve $\zeta : [\alpha, \beta] \to S$ from Q_1 to Q_2 such that $d(Q_1, Q_2) = L(\zeta)$, $\zeta([\alpha, \beta]) \subset U$ and ζ satisfies (1). Moreover, solutions of (1) are locally distance realizing.

For example, in $\mathbb{H}^2$, $\zeta(t) = (t, 0)$ is geodesic. $|\dot{\zeta}| = 1$,

$$\phi(\zeta(t)) = \frac{2}{1 - t^2}, \qquad \nabla\phi = \frac{4(u, v)}{(1 - u^2 - v^2)^2}.$$

Substituting

$$\frac{d}{dt}\left[\frac{2(1, 0)}{1 - t^2}\right] - \frac{4(t, 0)}{(1 - t^2)^2} = 0.$$

The length is independent of parametrization. Thus we may convert to arclength

$$s = \int_\alpha^t \phi\big(\zeta(t)\big)\,|\dot{\zeta}(t)|\,dt$$

so

$$\phi|\dot{\zeta}|\frac{d}{ds} = \frac{d}{dt}, \qquad \phi\zeta' = \frac{\dot{\zeta}}{|\dot{\zeta}|},$$

writing "$'$" for arclength derivatives.

$$\frac{d}{ds}\left[\phi(\zeta)\frac{\dot{\zeta}}{|\dot{\zeta}|}\right] - \nabla \ln \phi(\zeta)$$

or

$$\zeta'' + 2(\nabla \ln \phi \bullet \zeta')\zeta' - |\zeta'|^2 \nabla \ln \phi = 0. \tag{2}$$

One checks that $\phi|\zeta'|$ is constant along integral curves of (2) so solutions have constant speed.

We shall assume our surfaces are complete. The geodesic equation is

$$\zeta'' + 2(\nabla \ln \phi \bullet \zeta')\zeta' - |\zeta'|^2 \nabla \ln \phi = 0. \tag{3}$$

S is complete if solutions of (3) can be infinitely extended.

Theorem (Hopf - Rinow)

S is complete if and only if (S, d) with the induced distance is a complete metric space. Completeness implies that for all $Q_1, Q_2 \in S$ there there is a distance realizing geodesic $\zeta : [\alpha, \beta] \to S$ from Q_1 to Q_2 such that $d(Q_1, Q_2) = L(\zeta)$.

16. Polar coordinates.

Fix $P \in S$. The exponential map $\exp_P : T_p S \to S$ takes a vector $V \in T_P S$ with length r and maps it to the endpoint of a geodesic of length r which starts at P and heads in the direction V. (So if $r = \phi(p)|V|$, let $\zeta(t)$ be the solution of (2) with $\zeta(0) = P$ and $\zeta'(0) = \frac{V}{r}$. Define $exp_P(V) = \zeta(r)$.)

For example, if $P = 0$ in $\mathbb{H}^2$ then the length of the segment from $(0, 0)$ to $(t, 0)$ in $\mathbb{H}^2$ is

$$\rho = \int_0^t \frac{2\,dt}{1 - t^2} = \ln\left(\frac{1+t}{1-t}\right) \qquad \Longleftrightarrow \qquad t = \tanh\left(\frac{\rho}{2}\right).$$

The exponential map takes (ρ, θ) in polar coordinates of $\mathbb{E}^2 = T_P\mathbb{H}^2$ to $(t, \theta) \in \mathbb{H}^2$. Pulling back the Poincaré metric

$$d\rho^2 + \sinh^2\rho\, d\theta^2 = \frac{\operatorname{sech}^4\left(\frac{\rho}{2}\right) d\rho^2 + 4\tanh^2\left(\frac{\rho}{2}\right) d\theta^2}{\left(1 - \tanh^2\left(\frac{\rho}{2}\right)\right)^2} = \exp_P^*\left(\frac{4(dt^2 + t^2 d\theta^2)}{(1 - t^2)^2}\right)$$

17. Area of a disk. Length of a circle.

In polar coordinates, $\mathbb{H}^2 = (\,\mathbb{R}^2,\; d\rho^2 + \sinh^2\!\rho\, d\theta^2\,)$. Let $B(0,r)$ be the disk about the origin of radius r (measured in $\mathbb{H}^2$.). Then

$$\mathrm{L}\big(\partial B(0,r)\big) = \int_0^{2\pi} \sinh r\, d\theta = 2\pi \sinh r,$$

$$\mathrm{A}\big(B(0,r)\big) = \int_0^{2\pi}\int_0^{r} \sinh r\, dr\, d\theta = 2\pi(\cosh r - 1).$$

The Taylor expansion near $r=0$ gives

$$\mathrm{A}\big(B(0,r)\big) = \pi r^2 \left(1 + \frac{r^2}{12} + \cdots\right) = \pi r^2 \left(1 - \frac{K(0)\, r^2}{12} + \cdots\right)$$

thus $K(0) = -1$.

If S is complete, then $\exp_p : T_P S \to S$ is onto. Let $\mathbf{e}_1, \mathbf{e}_2 \in T_p S$ be orthonormal vectors. Let $V(\theta) = \cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2$. Consider the unit speed geodesic $\gamma(t, \theta) = \exp_P(tV(\theta))$ from P in the $V(\theta)$ dierection. For each $r > 0$ let $\Theta(r) \in \mathbb{S}^1$ be the set of directions $V(\theta)$ such that $\gamma(\bullet, \theta)$ is minimizing over $[0, r]$. Thus if $r_1 < r_2$ we have $\Theta(r_2) \subset \Theta(r_1)$. Let $\mathcal{U} = \cup_{r>0} r\Theta(r)$. It turns out that $\exp_P(\mathcal{U})$ covers all of S except for a set of measure zero.
The metric of S in polar coordinates becomes

$$ds^2 = dr^2 + J(r, \theta)^2 \, d\theta \tag{4}$$

where $J \geq 0$ in $\mathcal{U}$.
(The fact that circles of radius r about P cross the geodesic rays emanating from P orthogonally, hence no cross term, is a lemma of Gauß.)

19. Jacobi Equation.

The variation vector field measures the spread of geodesics as they are rotated about $P \in S$.

$$V = \frac{d}{d\theta}\gamma(t,\theta) \tag{5}$$

is perpendicular to $\dot{\gamma}(t,\theta)$ and has length $J(t,\theta)$ as in the metric in polar coordinates (4). By differentiating the geodesic equation (2) with respect to θ one finds the Jacobi Equation

$$J_{ss}(s,\theta) + K\big(\gamma(s,\theta)\big)\, J(s,\theta) = 0 \tag{6}$$

with initial conditions, $J(0,\theta) = 0$ and $J_s(0,\theta) = 1$.

For example, in $\mathbb{H}^2$, $J(s,\theta) = \sinh s$ and $K \equiv -1$.
For $\mathbb{E}^2$, $J(s,\theta) = s$ and $K \equiv 0$.

Let $ds^2 = \phi(z)^2\,|dz|^2$ be the metric in an isothermal coordinate patch. The intrinsic area form, gradient and Laplacian are given by the formulas

$$dA = \phi^2\,dx\,dy; \qquad |\nabla u|^2 = \frac{u_x^2 + u_y^2}{\phi^2}; \qquad \Delta u = \frac{u_{xx} + u_{yy}}{\phi^2} = \frac{4u_{z\bar{z}}}{\phi^2}.$$

Let $u \in C^1(\Omega)$ be a function on a domain $\Omega \subset S$. Then the energy or Dirichlet integral is invariant under conformal change of metric

$$\int_\Omega |\nabla u|^2\,dA = \int_\Omega u_x^2 + u_y^2\,dx\,dy.$$

A function is harmonic if $\Delta u = 0$ and subharmonic if $\Delta u \geq 0$ (at least weakly.) These notions agree irregardless of conformal metric $\phi^2\,|dz|^2$.

21. Parabolicity.

We seek generalizations of the Riemann mapping theorem to surfaces.

Theorem (Riemann Mapping Theorem)

Let $\Omega \subset \mathbb{R}^2$ be a simply connected open set that is not the whole plane. Then there is an analytic, one-to-one mapping onto the disk $f : \Omega \to \mathbb{D}$.

A noncompact, simply connected surface S is said to be parabolic if it is conformally equivalent to the plane. That is, there is a global isothermal coordinate chart $\sigma : \mathbb{C} \to S$. Otherwise the surface is called hyperbolic. The sphere $\mathbb{S}^2$ is compact, so it is neither parabolic nor hyperbolic.

Theorem (Koebe's Uniformization Theorem)

Let S be a simply connected Riemann Surface. Then S is conformally equivalent to the disk, the plane or to the sphere.

Conceivably, the topological disk could have many conformal structures, but the uniformization theorem tells us there are only two. The topological sphere has only one conformal structure.

For each $p \in S$, the positive Green's function $z \mapsto g(z,p)$ is harmonic for $z \in S - \{p\}$, $g(z,p) > 0$, $\inf_z g(z,p) = 0$ and in a isothermal patch around p, $g(z,p) + \ln|z-p|$ has a harmonic extension to a neighborhood of p (so $g(z,p) \to \infty$ as $z \to p$.)

Theorem

Let S be a simply connected noncompact Riemann surface. Then the following are equivalent.

- *S is hyperbolic.*
- *S has a positive Green's function.*
- *S has a negative nonconstant subharmonic function.*
- *S has a bounded nonconstant harmonic function.*

e. g., $u = ax + by$ is a bounded harmonic function on $\mathbb{D}$ hence on $\mathbb{H}^2$, but there are no bounded harmonic functions on $\mathbb{E}^2$.

23. A theorem relating curvature and function theory.

A surface s is said to have finite total curvature if

$$\int_S |K|\, dA < \infty.$$

Theorem (Blanc & Fiala, Huber)

Let S be a noncompact, complete surface with finite total curvature. Then S is conformally equivalent to a closed Riemann surface of genus g with finitely many punctures $\Sigma_g - \{p_1, \ldots, p_k\}$.

For example, $\mathbb{E}^2$ has zero total curvature and is conformal to $\mathbb{S}^2 - \{S\}$ via stereographic projection.

To illustrate something of the ways of geometric analysis, we sketch the proof for the simply connected case.

Lemma

Assume that S is a complete, noncompact, simply connected surface with finite total curvature

$$\int_S |K|\, dA = C < \infty.$$

Then

$$\mathrm{L}\big(\partial B(p,r)\big) \le (2\pi + C)r.$$

Proof Idea. By the Jacobi equation (5),

$$\begin{aligned} J_r(r,\theta) - 1 &= J_r(r,\theta) - J_r(0,\theta) \\ &= \int_0^r J_{rr}(r,\theta)\, dr \\ &= -\int_0^r K\big(\gamma(r,\theta)\big)\, J(r,\theta)\, dr. \end{aligned}$$

25. Growth of a geodesic circle proof.

The length $L(r) = \mathrm{L}\big(\partial B(p,r)\big)$ grows at a rate

$$\begin{aligned}
\frac{dL}{dr} &= \lim_{h\to 0+} \frac{L(r+h) - L(r)}{h} \\
&= \lim_{h\to 0+} \frac{1}{h}\left(\int_{\Theta(r+h)} J(r+h,\theta)\,d\theta - \int_{\Theta(r)} J(r,\theta)\,d\theta\right) \\
&= \lim_{h\to 0+} \left(\int_{\Theta(r+h)} \frac{J(r+h,\theta) - J(r,\theta)}{h}\,d\theta - \frac{1}{h}\int_{\Theta(r)-\Theta(r+h)} J(r,\theta)\,d\theta\right) \\
&\le \int_{\Theta(r)} J_r(r,\theta)\,d\theta \\
&= \int_{\Theta(r)} \left(1 - \int_0^r K\big(\gamma(r,\theta)\big)\,J(r,\theta)\,dr\right) d\theta \\
&\le 2\pi + \int_0^r \int_{\Theta(r)} \big|K\big(\gamma(r,\theta)\big)\big|\; J(r,\theta)\,d\theta\,dr \\
&= 2\pi + \int_{B(P,r)} |K|\,dA \quad = \quad \le 2\pi + C.
\end{aligned}$$

Theorem (Blanc & Fiala)

Let S be a complete, noncompact, simply connected Riemann surface of finite total curvature. Then S is parabolic.

Proof idea. Suppose not. Then S has a global isothermal chart $\sigma : \mathbb{D} \to S$. Then $u = -\ln|z|$ is a positive Green's function on S. Let $\epsilon > 0$ and $K \subset \mathbb{D} - \overline{B(0,\epsilon)}$ be any compact domain. On the one hand, the energy is uniformly bounded

$$\mathcal{E}(K) = \int_K u_x^2 + u_y^2 \, dx \, dy \leq -2\pi \ln \epsilon.$$

This estimate is done in the background $\mathbb{D}$ metric. But energy is conformally invariant, so it will hold in the surface metric also.

Indeed, let $R = \sup\{|z| : z \in K\} < 1$ and $\mathcal{A} = \{z \in \mathbb{D} : \epsilon \leq |z| \leq R\}$ be an annulus containing K. Using the fact that $u > 0$ and $u_r < 0$ on K, by integrating by parts,

$$\begin{aligned}
\mathcal{E}(K) \leq \mathcal{E}(\mathcal{A}) &= -\int_{\mathcal{A}} u \Delta_0 u \, dx \, dy + \oint_{|z|=\epsilon} u \frac{\partial u}{\partial n} + \oint_{|z|=R} u \frac{\partial u}{\partial n} \\
&\leq 0 - \oint_{|z|=\epsilon} \frac{\ln \epsilon}{\epsilon} + 0 \\
&\leq -2\pi \ln \epsilon
\end{aligned}$$

since on $|z| = \epsilon$, $u = -\ln \epsilon$ and $u_r = -\frac{1}{\epsilon}$ and $\mathrm{L}(\{z : |z| = \epsilon\}) = 2\pi\epsilon$.

On the other hand, bounded total curvature will imply that the energy will grow to infinity. For the remainder of the argument, work in intrinsic polar coordinates. Let $B(0,r)$ denote the intrinsic ball for $r \geq 1$ and

$$\mathcal{E}(r) = \int_{B(0,r)-\overline{B(0,1)}} |Du|^2 \, dA = \int_1^r \int_{\partial B(0,r)} |Du|^2 \, ds \, dr$$

where s is length along $\partial B(0,r)$. By the Schwartz inequality,

$$L(r)\frac{d\mathcal{E}}{dr} = \int_{\partial B(0,r)} ds \int_{\partial B(0,r)} |Du|^2 \, ds \geq \left(\int_{\partial B(0,r)} |Du| \, ds \right)^2$$

$$\geq \left(\oint_{\partial B(0,r)} \frac{\partial u}{\partial n} \right)^2 = \left(\int_{B(0,r)-\overline{B(0,\delta)}} \Delta u \, dA - \oint_{\partial B(0,\delta)} \frac{\partial u}{\partial n} \right)^2 = c_0$$

for any fixed $\delta \in (0,1]$ where $c_0 > 0$ is independent of r.
In fact, $c_0 \to 4\pi^2$ as $\delta \to 0$.

Now, using the length lemma,

$$\frac{d\mathcal{E}}{dr} \geq \frac{c_0}{(2\pi + C)r}.$$

Integrating, this says

$$\mathcal{E}(r) \geq \frac{c_0 \ln r}{(2\pi + C)} \to \infty \text{ as } r \to \infty.$$

This is a contradiction because the energy is invariant under conformal change and is uniformly bounded. $\square$

- To understand the argument, try using it to prove $\mathbb{E}^2$ is parabolic!

A harmonic map locally minimizes energy of a map between surfaces, generalizing harmonic functions and geodesics.
If $h : (S_1, \phi(z)^2 \, |dz|^2) \to (S_2, \psi(w)^2 \, |dw|^2)$ is harmonic, then it satisfies the PDE system

$$h_{z\bar{z}} + 2 \frac{\psi_w(h)}{\psi(h)} \, h_z \, h_{\bar{z}} = 0.$$

Theorem

If $f : \tilde{S} \to S_1$ is a conformal diffeomorphism and $h : S_1 \to S_2$ is harmonic, then $h \circ f : \tilde{S} \to S_2$ is harmonic.

Theorem (Treibergs 1986)

Let $\mathcal{I} \subset \partial\mathbb{D}$ be any closed set with at least three distinct points and $Conv(\mathcal{I})$ its convex hull in $\mathbb{H}^2$, then there is a complete, spacelike entire constant mean curvature surface $\mathcal{S}$ in Minkowski Space such that the Gauß map $\mathcal{G} : S \to Conv(\mathcal{I})$ is a harmonic diffeomorphism.

Minkowski Space is $\mathbb{E}^{2,1} = (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$.
An *entire, spacelike surface* $\mathcal{S} \subset \mathbb{E}^{2,1}$ is the graph of a function $\mathcal{S} = \left\{ \big(x, y, u(x, y)\big) : (x, y) \in \mathbb{R}^2 \right\}$ such that $u_x^2 + u_y^2 < 1$.
The Gauß map is the map given by the future-pointing unit normal

$$\mathcal{G} = \frac{(u_x, u_y, 1)}{\sqrt{1 - u_x^2 - u_y^2}} : \mathcal{S} \to \mathcal{H}.$$

The hyperboloid $\mathcal{H} = \{(x, y, x) : x^2 + y^2 - z^2 = -1,\ z > 0\}$ consists of all future-pointing vectors of length -1.
The *hyperboloid model* $(\mathcal{H}, dx^2 + dy^2 - dz^2)$ is isometric to $\mathbb{H}^2$.

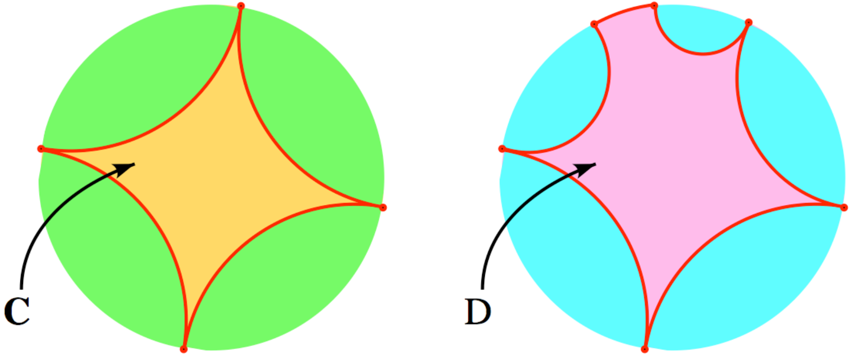

Figure: Harmonic diffeomorphisms

Corollary

Let $\mathcal{I} \subset \partial\mathbb{B}$ be a closed set with at least three points.

- *If $\mathcal{I}$ is finite, there is a harmonic diffeomorphism*

$$h : \mathbb{C} \to Conv(\mathcal{I}).$$

- *If $\mathcal{I}$ has nonempty interior, then there is a harmonic diffeomorphism*

$$h : \mathbb{B} \to Conv(\mathcal{I}).$$

Since $\mathcal{S}$ is convex, its total curvature is $-\mathrm{A}\big(\mathcal{G}(\mathcal{S})\big) = \pi(2 - \sharp\mathcal{I})$ by the Gauß-Bonnet Theorem in $\mathbb{H}^2$. By Blanc-Fiala's Theorem, $\mathcal{S}$ is conformal to the plane if $\mathcal{I}$ is finite.

If $\mathcal{I}$ has nonempty interior then one can construct a nonconstant bounded harmonic function on $\mathcal{S}$ so it is conformal to the disk.

Thanks!