

Heat Equation in Geometry

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The URL for these Beamer Slides: *“Heat Equation in Geometry”*

<http://www.math.utah.edu/~treiberg/ERDHeatEquation.pdf>

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4. Outline.

- Introduction to Riemannian Manifolds and Curvatures.
- Success of Ricci Flow Motivates Looking at Elementary Flows.
- Heat Equation on the Circle
 - Separation of Variables.
 - Maximim Principle.
 - Integral Estimates.
- Curvature Flow of a Plane Curve.
 - Arclength, Tangent Vector, Normal Vector, Curvature.
 - First Variation of Length and Area.
 - Examples of Curvature Flow.
 - Curvature Flow Rounds Out Curves.
 - Maximum Principle.
 - Integral Estimates.
 - Geometric Inequalities.
 - Curvature Flow Reduces Isoperimetric Ratio.
- Some Higher Dimensional Results.

5. Local Coordinates.

A **differentiable manifold** M^n is a space that can locally be given by **curvilinear coordinate charts**, also called a **parameterizations**. About each point P_0 there neighborhood $N \in M$ which is homeomorphic to an open set in $U \subset \mathbf{R}^n$. Let

$$X : U \rightarrow \mathbf{M}$$

be a homeomorphism to $X : U \rightarrow V$. At each point $P \in X(U)$ we can identify **tangent vectors** to the surface. If $P = X(a)$ some $a \in U$, then the tangent vectors are velocity vectors of smooth curves $u(t)$ in U (or curves $X(u(t))$ in M) that pass through a . If $u(0) = a$ then the tangent vector may be written

$$V(a) = \left(\frac{\partial u_1}{\partial t}(0), \dots, \frac{\partial u_n}{\partial t}(0) \right).$$

6. GEOMETRY: Lengths of Curves.

The **Riemannian Metric** is given by the matrix function $g_{ij}(u)$ which is a smoothly varying, symmetric and positive definite.

It gives an inner product $\langle \cdot, \cdot \rangle_g$ on tangent vectors. The length of the vector V is

$$|V(a)|_g^2 = \sum_{i=1}^n \sum_{j=1}^n g_{ij}(a) \dot{u}_i(0) \dot{u}_j(0).$$

The length of the curve $u : [a, b] \rightarrow U$ on M is determined by integrating its velocity in the coordinate patch in the metric $g_{ij}(u)$.

$$L_g = \int_a^b \sqrt{\sum_{i=1}^n \sum_{j=1}^n g_{ij}(u(t)) \dot{u}_i(t) \dot{u}_j(t)} dt$$

A **vector field** is given by vector valued functions in U .

7. Angle and Area via the Riemannian Metric.

If V and W are nonvanishing vector fields on M then their angle $\alpha = \angle(V, W)$ at a point is given using the inner product

$$\cos \alpha = \frac{\langle V, W \rangle_g}{|V|_g |W|_g}.$$

If $D \subset U$ is a piecewise smooth subdomain in the patch, the volume of $X(D) \subset M$ is also determined by the metric

$$V(X(D)) = \int_D \sqrt{\det(g_{ij}(u))} \, du_1 \cdots du_n.$$

If we endow an abstract differentiable manifold M^n with a Riemannian Metric, a smoothly varying inner product on each tangent space that is consistently defined on overlapping coordinate patches, the resulting object is a **Riemannian Manifold**.

8. Curvatures.

How can one tell Riemannian manifold with strange coordinates and metric is really just Euclidean Space pulled back under some weird diffeomorphism? This is determined by the invariant, **Riemann Curvature**, which is computed from the metric as follows. Let $g^{ij}(u) = (g_{ij}(u))^{-1}$ be the inverse matrix function. Then the **Christoffel Symbols** and the Riemann Curvature are given by the formulae

$$\Gamma_{ik}^j = \frac{1}{2} \sum_{p=1}^n g^{jp} \left\{ \frac{\partial g_{ip}}{\partial u_k} + \frac{\partial g_{kp}}{\partial u_i} - \frac{\partial g_{ik}}{\partial u_p} \right\}$$

$$R_{ik\ell}^j = \frac{\partial}{\partial u_k} \Gamma_{i\ell}^j - \frac{\partial}{\partial u_\ell} \Gamma_{ik}^j + \sum_{p=1}^n \left(\Gamma_{i\ell}^p \Gamma_{pk}^j - \Gamma_{ik}^p \Gamma_{p\ell}^j \right).$$

9. Ricci and Csalar Curvature.

The **Ricci Curvature** is a contraction of the Riemann Curvature and the **Scalar Curvature** is a contraction of the Ricci Curvature.

$$R_{ik} = \sum_{p=1}^n R_i^p{}_{kp};$$
$$R = \sum_{p=1}^n \sum_{q=1}^n g^{pq} R_{pq}.$$

Thus Ricci Curvature is a nonlinear second order operator of the metric whose symbol turns out to be weakly elliptic. Its second order terms are

$$R_{ik} = \frac{1}{2} \sum_{p=1}^n \sum_{q=1}^n g^{pq} \left\{ \frac{\partial^2 g_{kp}}{\partial u_i \partial u_q} + \frac{\partial^2 g_{iq}}{\partial u_p \partial u_k} - \frac{\partial^2 g_{ik}}{\partial u_p \partial u_q} - \frac{\partial^2 g_{pq}}{\partial u_i \partial u_k} \right\} + \dots$$

10. Recent Geometric Analysis Solution of Poincaré Conjecture.



Figure: Richard Hamilton.

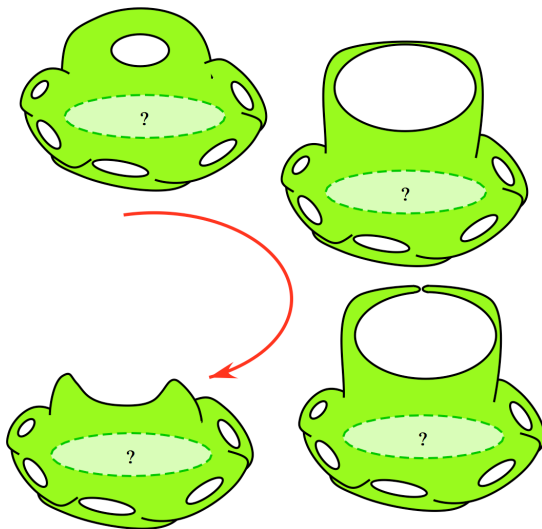
In 1904 Poincaré conjectured that a closed simply connected M^3 is homeo to the sphere. This follows from Thurston's Geometrization Conjecture.

In a series of papers beginning 1982, Hamilton perfected PDE machinery to solve the conjecture. Starting from any Riemannian metric g_0 for M , he evolved it according to Ricci Flow

$$\frac{\partial}{\partial t}g = -2\text{Ric}(g), \quad g(0) = g_0.$$

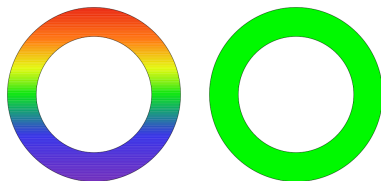
Since the solution generally encounters singularities, he proposed to intervene with surgery whenever singularities form. In 2003, Perelman found a way to control the topology at singularities enough to say the flow with surgery results in standard topological maneuvers ending at a geometrizable manifold. Topological methods imply that the original manifold was geometrizable.

11. Schematic of Ricci Flow with Surgery.



12. Simplest Example: Heat Equation on the Circle.

The space $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the circle of length 2π . We say that a function $f(\theta) \in \mathcal{C}^k(\mathbb{S}^1)$ if $f(\theta)$ is defined for all $\theta \in \mathbb{R}$, is k -times continuously differentiable and is 2π -periodic.



If $g(\theta) \in \mathcal{C}(\mathbb{S}^1)$ is the initial temperature on a thin unit circular rod. Let $u(t, \theta) \in \mathcal{C}^2([0, \infty) \times \mathbb{S}^1)$. The temperature for future times satisfies the **heat equation**

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial \theta^2}, & \text{for all } t > 0 \text{ and } \theta \in \mathbb{S}^1. \\ u(0, \theta) &= g(\theta), & \text{for all } \theta \in \mathbb{S}^1. \end{aligned}$$

13. Separation of Variables (Fourier's Method)



Figure: J. Fourier 1768–1830.

His 1822 *Théorie analytique de la chaleur* was called “a great mathematical poem” by Kelvin but Lagrange, Laplace and Legendre criticized it for a looseness of reasoning.

We make the ansatz that

$$u(t, \theta) = T(t)\Theta(\theta)$$

for a 2π -periodic Θ . Heat equation becomes

$$T'(t)\Theta(\theta) = T(t)\Theta''(\theta).$$

Separating variables implies that there is a constant λ so that

$$\frac{T'(t)}{T(t)} = -\lambda = \frac{\Theta''(\theta)}{\Theta(\theta)}.$$

This results in two equations

$$\begin{aligned}\Theta'' + \lambda\Theta &= 0, & \Theta \text{ is } 2\pi\text{-periodic on } \mathbb{R}; \\ T' + \lambda T &= 0, & \text{for all } t > 0.\end{aligned}$$

14. Separate Variables in the Heat Equation.

2π -periodicity of solutions of the first equation imply $\lambda = k^2$ for $k \in \mathbb{Z}$

$$\Theta'' + \lambda\Theta = 0$$

which yields the solution $\Theta = A_0$ if $\lambda = 0$ and

$$\Theta(\theta) = A_k \cos k\theta + B_k \sin k\theta$$

if $\lambda = k^2$ for some $k \in \mathbb{N}$ and constants A_k, B_k .

These form a complete basis for $\mathcal{L}^2(\mathbb{S}^1)$.

The corresponding solution of $T' + \lambda T = 0$ is

$$T(t) = e^{-k^2 t}.$$

Thus, the PDE is solved for $k \in \mathbb{N}$ by

$$u(t, \theta) = e^{-k^2 t} (A_k \cos k\theta + B_k \sin k\theta).$$

15. Separate Variables in the Heat Equation. -

By linearity, the solution is obtained by superposition

$$u(t, \theta) = A_0 + \sum_{k=1}^{\infty} e^{-k^2 t} (A_k \cos k\theta + B_k \sin k\theta). \quad (1)$$

The initial condition is satisfied

$$g(\theta) = u(0, \theta) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty} (A_k \cos k\theta + B_k \sin k\theta)$$

if one takes the Fourier coefficients of $g(\theta)$. For $k \in \mathbb{Z}_+$,

$$A_k = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos k\theta \, d\theta, \quad B_k = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin k\theta \, d\theta$$

Theorem (Heat Equation Properties)

Suppose that $g(\theta) \in \mathcal{C}(\mathbb{S}^1)$. Then there is a solution $u \in C([0, \infty) \times \mathbb{S}^1) \cap C^2((0, \infty) \times \mathbb{S}^1)$, given by (1), that satisfies the heat equation

$$\begin{aligned} u_t &= u_{\theta\theta}, & \text{for } (t, \theta) \in (0, \infty) \times \mathbb{S}^1; \\ u(0, \theta) &= g(\theta), & \text{for all } \theta \in \mathbb{S}^1. \end{aligned}$$

The solution has the following properties:

- 1 If u and v both satisfy the heat equation, and if $u(0, \theta) < v(0, \theta)$ for all θ then $u(t, \theta) \leq v(t, \theta)$ for all $t \geq 0$ and all θ .
- 2 $\min_{\mathbb{S}^1} g \leq u \leq \max_{\mathbb{S}^1} g$.
- 3 $\int_{\mathbb{S}^1} (u - \frac{1}{2}A_0)^2 d\theta \leq e^{-2t} \|g - A_0\|_2^2$ for all $t \geq 0$.
- 4 There are $c_1, c_2 > 0$ so that $\int_{\mathbb{S}^1} u_\theta^2 d\theta \leq c_1 e^{-2t}$ for all $t \geq c_2$.
- 5 $u(t, \theta) \rightarrow \frac{1}{2}A_0$ uniformly as $t \rightarrow \infty$.

17. Maximum Principle. (First Tool to Study Heat Equations.)

Theorem (Maximum Principle for Heat Equation on the Circle)

Suppose that $u, v \in \mathcal{C}([0, T) \times \mathbb{S}^1) \cap \mathcal{C}^2((0, T) \times \mathbb{S}^1)$ both satisfy the heat equation and initial condition

$$\begin{aligned} u_t &= u_{\theta\theta}, & v_t &= v_{\theta\theta}, & \text{on } (0, T) \times \mathbb{S}^1; \\ u &\geq v, & & & \text{on } \{0\} \times \mathbb{S}^1. \end{aligned}$$

Then

$$u \geq v, \quad \text{on } [0, T) \times \mathbb{S}^1.$$

The idea is that if there is a point (t_0, θ_0) where solutions first touch $u(t_0, \theta_0) = v(t_0, \theta_0)$ then $u(t_0, \theta) \geq v(t_0, \theta)$ for all θ for all θ . Hence

$$u_{\theta\theta}(t_0, \theta_0) \geq v_{\theta\theta}(t_0, \theta_0)$$

so $u_t(t_0, \theta_0) \geq v_t(t_0, \theta_0)$ and the solutions move apart.

However, we can't draw this conclusion unless the inequalities are strict.

18. Proof of Maximum Principle.

Proof. Choose $\epsilon > 0$. Let $w(t, \theta) = u(t, \theta) - v(t, \theta) + \epsilon t + \epsilon$. Note that w satisfies

$$w_t = u_t - v_t + \epsilon = u_{\theta\theta} - v_{\theta\theta} + \epsilon = w_{\theta\theta} + \epsilon.$$

At $t = 0$ we have $w(0, \theta) > 0$. I claim $w > 0$ for all (t, θ) . If not, there is a first time $t_1 > 0$ where $w = 0$, say at some point $w(t_1, \theta_1) = 0$.

Because $w(t, \theta_1) > 0$ for all $0 \leq t < t_1$, we have $w_t(t_1, \theta_1) \leq 0$. Because $w(t_1, \theta) \geq 0$ for all θ , we have $w_{\theta\theta}(t_1, \theta_1) \geq 0$. Plugging into the equation

$$0 \geq w_t(t_1, \theta_1) = w_{\theta\theta}(t_1, \theta_1) + \epsilon \geq 0 + \epsilon > 0$$

which is a contradiction. Thus, $w > 0$ for all (t, θ) , which implies

$$u(t, \theta) - v(t, \theta) > -\epsilon t - \epsilon.$$

But for fixed (t, θ) , by taking $\epsilon > 0$ arbitrarily small, it follows that

$$u(t, \theta) - v(t, \theta) \geq 0. \quad \square$$

To see (2), let

$$w(t, \theta) = \min_{\mathbb{S}^1} g, \quad v(t, \theta) = \max_{\mathbb{S}^1} g.$$

Since w and v are constant, they satisfy the heat equation. Since

$$w(0, \theta) \leq u(0, \theta) \leq v(0, \theta),$$

it follows from (1) that that for all (t, θ) ,

$$w(t, \theta) \leq u(t, \theta) \leq v(t, \theta).$$

Deduce co-evolution of interesting quantities such as **average temperature**.

To see (3), first notice that the average temperature remains constant in time

$$\frac{d}{dt} \left(\frac{1}{2\pi} \int_{\mathbb{S}^1} u \, d\theta \right) = \frac{1}{2\pi} \int_{\mathbb{S}^1} u_t \, d\theta = \frac{1}{2\pi} \int_{\mathbb{S}^1} u_{\theta\theta} \, d\theta = 0.$$

Thus, for all $t \geq 0$,

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} u(t, \theta) \, d\theta = \frac{1}{2} A_0 = \frac{1}{2\pi} \int_{\mathbb{S}^1} g(\theta) \, d\theta.$$

21. Co-evolution of Another Quantity: Squared Deviation.

Also the \mathcal{L}^2 deviation decreases. Using Wirtinger's inequality

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{S}^1} \left[u - \frac{1}{2} A_0 \right]^2 d\theta &= 2 \int_{\mathbb{S}^1} \left[u - \frac{1}{2} A_0 \right] u_t d\theta \\ &= 2 \int_{\mathbb{S}^1} \left[u - \frac{1}{2} A_0 \right] u_{\theta\theta} d\theta \\ &= -2 \int_{\mathbb{S}^1} u_\theta^2 d\theta \\ &\leq -2 \int_{\mathbb{S}^1} \left[u - \frac{1}{2} A_0 \right]^2 d\theta\end{aligned}$$

This says $y' \leq -2y$ so $y \leq y_0 e^{-2t}$ or

$$\int_{\mathbb{S}^1} \left[u - \frac{1}{2} A_0 \right]^2 d\theta \leq \left(\int_{\mathbb{S}^1} \left[g(\theta) - \frac{1}{2} A_0 \right]^2 d\theta \right) e^{-2t}.$$

22. Wirtinger's Inequality.

Wirtinger's Inequality bounds the L^2 norm of a function by the L^2 norm of its derivative. It is also known as the **Poincaré Inequality** in higher dimensions. We state stronger hypotheses than necessary.



Figure: Wilhelm Wirtinger
1865–1945.

Theorem (Wirtinger's inequality)

Let $f(\theta)$ be a piecewise $C^1(\mathbb{R})$ function with period 2π (for all θ , $f(\theta + 2\pi) = f(\theta)$). Let \bar{f} denote the mean value of f

$$\bar{f} = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta.$$

Then

$$\int_0^{2\pi} (f(\theta) - \bar{f})^2 d\theta \leq \int_0^{2\pi} (f'(\theta))^2 d\theta.$$

Equality holds iff for some constants a, b ,

$$f(\theta) = \bar{f} + a \cos \theta + b \sin \theta.$$

23. Proof of Wirtinger's Inequality.

Idea: express f and f' in Fourier series. Since f' is bounded and f is continuous, the Fourier series converges at all θ

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \{a_k \cos k\theta + b_k \sin k\theta\}.$$

Fourier coefficients are determined by formally multiplying by $\sin m\theta$ or $\cos m\theta$ and integrating to get

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos m\theta \, d\theta, \quad b_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin m\theta \, d\theta,$$

hence $2\bar{f} = a_0$. Sines and cosines are complete so Parseval equation holds

$$\int_0^{2\pi} (f - \bar{f})^2 = \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2). \quad (2)$$

Formally, this is the integral of the square of the series, where after multiplying out and integrating, terms like $\int \cos m\theta \sin k\theta = 0$ or $\int \cos m\theta \cos k\theta = 0$ if $m \neq k$ drop out and terms like $\int \sin^2 k\theta = \pi$ contribute π to the sum.

24. Proof of Wirtinger's Inequality..

The Fourier Series for the derivative is given by

$$f'(\theta) \sim \sum_{k=1}^{\infty} \{-ka_k \sin k\theta + kb_k \cos k\theta\}$$

Since f' is square integrable, Bessel's inequality gives

$$\pi \sum_{k=1}^{\infty} k^2 (a_k^2 + b_k^2) \leq \int_0^{2\pi} (f')^2. \quad (3)$$

Wirtinger's inequality is deduced from (2) and (3) since

$$\int_0^{2\pi} (f')^2 - \int_0^{2\pi} (f - \bar{f})^2 \geq \pi \sum_{k=2}^{\infty} (k^2 - 1) (a_k^2 + b_k^2) \geq 0.$$

Equality implies that for $k \geq 2$, $(k^2 - 1) (a_k^2 + b_k^2) = 0$ so $a_k = b_k = 0$, thus f takes the form $f(\theta) = \bar{f} + a \cos \theta + b \sin \theta$. □

25. Co-evolution of interesting quantities: Mean square heat flux.

By differentiating the equation, we get the evolution equation of heat flux u_θ

$$(u_\theta)_t = (u_t)_\theta = (u_{\theta\theta})_\theta = (u_\theta)_{\theta\theta}$$

\mathcal{L}^2 norm of heat flux decreases. Using Wirtinger's inequality

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}^1} u_\theta^2 d\theta &= 2 \int_{\mathbb{S}^1} u_\theta u_{\theta t} d\theta \\ &= 2 \int_{\mathbb{S}^1} u_\theta u_{\theta\theta\theta} d\theta \\ &= -2 \int_{\mathbb{S}^1} u_{\theta\theta}^2 d\theta \\ &\leq -2 \int_{\mathbb{S}^1} u_\theta^2 d\theta \end{aligned}$$

Thus, for any $0 < t_0 < t$,

$$\int_{\mathbb{S}^1} u_\theta^2(t, \theta) d\theta \leq \left(\int_{\mathbb{S}^1} u_\theta^2(t_0, \theta) d\theta \right) e^{-2(t-t_0)}.$$

26. Temperature converges uniformly to its average.

Since $\theta \mapsto u(t, \theta)$ is continuous, there is a point $\theta_0 \in \mathbb{S}^1$ such that $u(t, \theta_0) = \frac{1}{2}A_0$ equals its average. Let $\theta_0 \leq \theta_1 < \theta_0 + 2\pi$ be any point on the circle. By Schwarz inequality, for $t \geq c_2$,

$$\begin{aligned} |u(t, \theta_1) - u(t, \theta_0)|^2 &\leq \left| \int_{\theta_0}^{\theta_1} u_\theta(t, \theta) d\theta \right|^2 \\ &\leq (\theta_1 - \theta_0) \int_{\theta_0}^{\theta_1} u_\theta^2 d\theta \\ &\leq 2\pi \int_{\mathbb{S}^1} u_\theta^2 d\theta \\ &\leq 2\pi c_1 e^{-2t}. \end{aligned}$$

Thus temperature converges uniformly because for any θ_1 and $t \geq c_2$,

$$\left| u(t, \theta_1) - \frac{1}{2}A_0 \right| \leq (2\pi c_1)^{\frac{1}{2}} e^{-t}.$$

27. Simplest Geometric Example: How to Round Out a Curve?

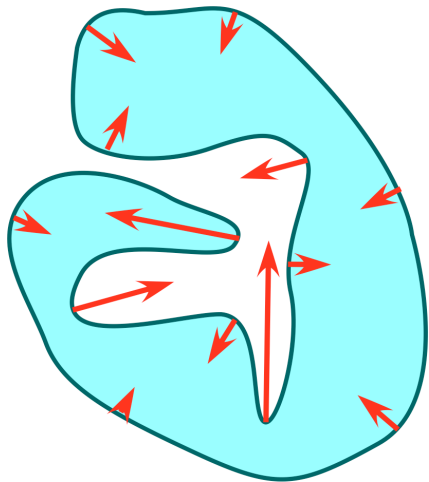


Figure: Deform Curve

Is it possible to continuously deform a curve in such a way that

- The parts that are bent the most are unbent the fastest;
- The curve doesn't cross itself;
- The deformation limits to a circle?

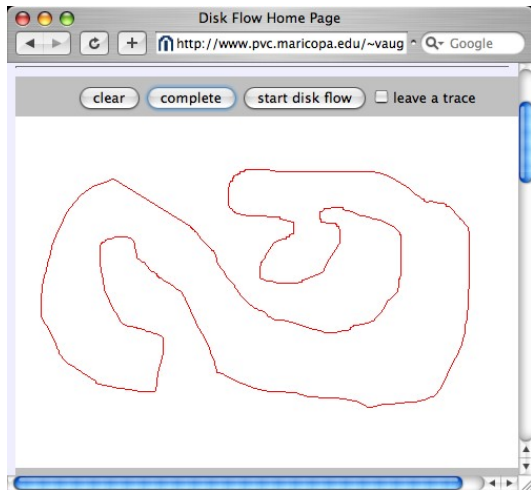
The answer is **YES!** Method: **CURVATURE FLOW** Heat Equation! Move curve with normal velocity proportional to curvature.

$$X_t = \kappa N.$$

It turns out

$$X_t = X_{ss}.$$

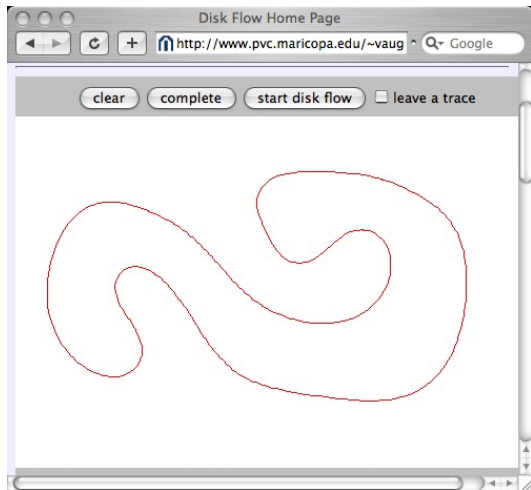
28. Vaughn's Applet.



Curvature flow applet written by Richard Vaugh, Paradise Valley CC,

<http://www.pvc.maricopa.edu/~vaughn/java/JDF/df.html>

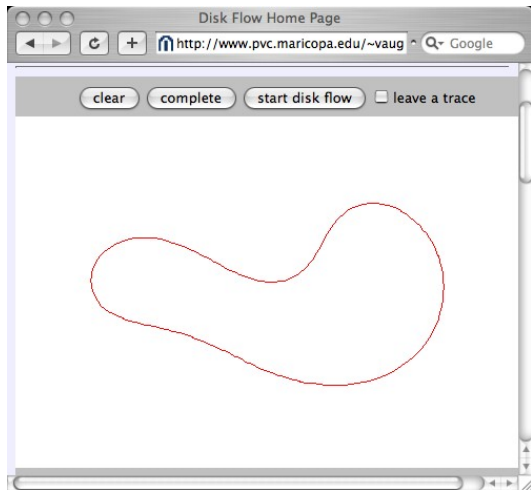
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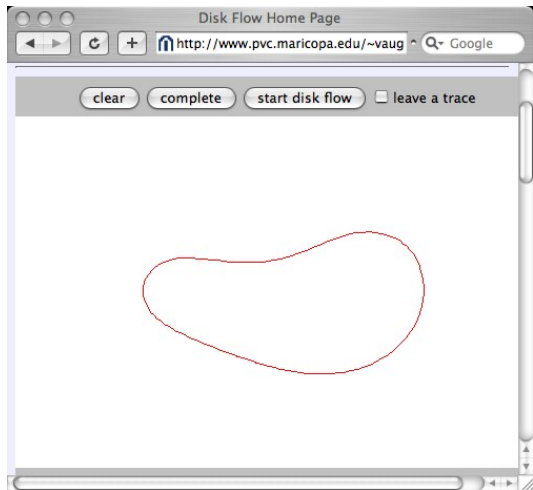
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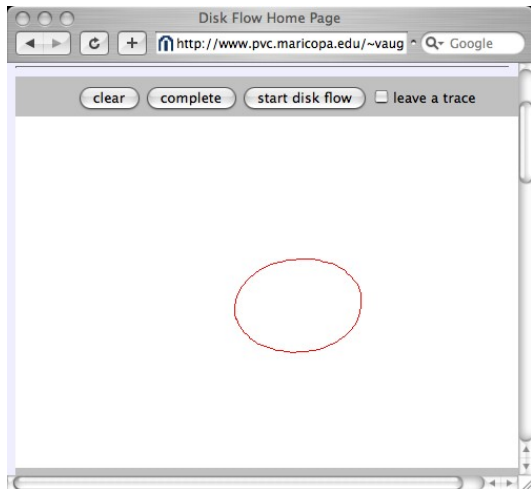
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Vaughn's link seems to have disappeared. Here is another.

Curvature flow applets written by James Sethian, University of California, Berkeley.

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http://www.math.berkeley.edu/  
~sethian/2006/Applets/EvolvingCurves/java_curve_flow.html
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30. Regular Curves

Let $X(u)$ be a regular smooth closed curve in the plane.
(Regular means $X_u \neq 0$.)

$$\begin{aligned}X &: [0, a] \rightarrow \mathbb{R}^2, \\X(0) &= X(a); \\X_u(0) &= X_u(a).\end{aligned}$$

The velocity vector is X_u . s is the arclength along the curve
 $s(u) = L(X([0, u]))$. The speed of the curve is

$$\frac{ds}{du} = |X_u|$$

The unit tangent and normal vectors are thus

$$T = \frac{X_u}{|X_u|}; \quad N = \mathcal{R}T = \mathcal{R} \left(\frac{X_u}{|X_u|} \right)$$

where $\mathcal{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the 90° rotation matrix.

31. Curvature of a Curve.

The **curvature** κ measures how fast the unit tangent vector turns relative to length along the curve.

Let θ be the angle of the tangent vector from horizontal

$$T = (\cos \theta, \sin \theta) \quad \text{so} \quad N = \mathcal{R}T = (-\sin \theta, \cos \theta)$$

Then the curvature is

$$\kappa = \frac{d\theta}{ds} \quad \text{so} \quad \frac{d}{ds}T = (-\sin \theta, \cos \theta) \frac{d\theta}{ds} = \kappa N$$

and also

$$\frac{d}{ds}N = -\kappa T.$$

32. Curvature of a Circle.

For example, for the circle of radius R ,

$$X(u) = (R \cos u, R \sin u)$$

$$X_u = (-R \sin u, R \cos u)$$

$$\frac{ds}{du} = |X_u| = |(-R \sin u, R \cos u)| = R$$

$$T = \frac{(-R \sin u, R \cos u)}{|(-R \sin u, R \cos u)|} = (-\sin u, \cos u) = (\cos \theta(u), \sin \theta(u))$$

where $\theta(u) = u + \frac{\pi}{2}$ so

$$\kappa = \frac{d\theta}{ds} = \frac{du}{ds} \frac{d\theta}{du} = \frac{1}{|X_u|} \cdot 1 = \frac{1}{R}.$$

Equivalently, since $N = \mathcal{R}T = (-\cos u, -\sin u)$,

$$\frac{dT}{ds} = \frac{du}{ds} \frac{dT}{du} = \frac{1}{|X_u|} (-\cos u, -\sin u) = \frac{1}{R} N = \kappa N. \quad \text{so} \quad \kappa = \frac{1}{R}.$$

33. Curvature of a Curve.

Note that the derivative of the speed gives

$$\frac{d}{du}|X_u| = \frac{d}{du}\sqrt{X_u \cdot X_u} = \frac{1}{2} \frac{1}{|X_u|} 2X_u \cdot X_{uu} = \frac{X_u \cdot X_{uu}}{|X_u|}.$$

κ determined from the formula

$$\begin{aligned} \frac{dT}{ds} = \kappa N &= \frac{1}{|X_u|} \frac{d}{du} \left(\frac{X_u}{|X_u|} \right) = \frac{1}{|X_u|} \left(\frac{X_{uu}}{|X_u|} - \frac{X_u \frac{d}{du}|X_u|}{|X_u|^2} \right) \\ &= \frac{1}{|X_u|} \left(\frac{X_{uu}}{|X_u|} - \frac{(X_u \cdot X_{uu})X_u}{|X_u|^3} \right). \end{aligned}$$

By the way, this formula decomposes acceleration into tangential and centripetal pieces

$$X_{uu} = \frac{d}{du}|X_u|T + \kappa|X_u|^2N$$

34. First Variation of Arclength.

Let's work out how fast length changes when we perturb the curve. The length $L(X)$ is obtained from the integral

$$L_0 = \int_{\Gamma} ds = \int_0^a |X_u| du.$$

How does the length of Γ_t given by $u \mapsto X(t, u)$ change if we deform the curve sideways at a velocity v ? In a deformation of this kind

$$\frac{\partial}{\partial t} X = vN$$

where v is velocity of deformation and N is the normal to Γ_t at $X(t, u)$. We seek the **first variation of length** which is the derivative of $L(X)$:

Theorem (First Variation of Arclength)

Suppose that $X(t, u)$ is a smooth family of regular closed curves that are deformed with normal velocity by $X_t(t, u) = v(t, u)N(t, u)$. Then

$$\frac{d}{dt} L(X) = - \int_{\Gamma_t} \kappa v \, ds. \quad (4)$$

Proof. For arbitrary function g , derivatives with respect to arclength

$$g_s = \frac{1}{|X_u|} g_u \quad \text{and} \quad ds = |X_u| du$$

Differentiating,

$$\begin{aligned} \frac{\partial}{\partial t} |X_u| &= \frac{X_u \cdot X_{ut}}{|X_u|} = \frac{X_u \cdot X_{tu}}{|X_u|} = T \cdot (X_t)_s |X_u| \\ &= T \cdot (vN)_s |X_u| = T \cdot (v_s N - v \kappa T) |X_u| = -\kappa v |X_u|. \end{aligned}$$

Hence

$$\frac{d}{dt} L(X) = \int \frac{\partial}{\partial t} |X_u| du = - \int \kappa v |X_u| du = - \int \kappa v ds \quad \square$$

36. First Variation of Arclength Example.

For example, $Y(t, u) = ((R - t) \cos u, (R - t) \sin u)$ is a circle of radius $(R - t)$, $N = (\cos u, \sin u)$ is the normal for all circles and

$$\frac{dY}{dt} = -(\sin u, \cos u) = -N$$

so $v = 1$ is constant. Since $|Y_u| = R - t$, $ds = (R - t) du$ and the curvature of X is $\kappa = \frac{1}{R - t}$, the first variation is just

$$\begin{aligned}\frac{dL}{dt} &= - \int_{\Gamma} \kappa v ds = - \int_0^{2\pi} \frac{1}{R - t} \cdot 1 (R - t) du \\ &= -2\pi = \frac{d}{dt} L(Y) = \frac{d}{dt} 2\pi(R - t).\end{aligned}$$

37. First Variation of Area.

A similar computation gives the first variation of area. Writing $X = (x, y)$, the area $A(X)$ is obtained from the line integral

$$A_0 = \frac{1}{2} \oint_{\Gamma} x \, dy - y \, dx = \frac{1}{2} \int_0^a xy_u - yx_u \, du = \frac{1}{2} \int_0^a \mathcal{R}X \cdot X_u \, du$$

since $\mathcal{R}X = (-y, x)$.

How does the area change if we deform the curve Γ_t given by $X(t, u)$ with velocity v in the N direction?

The **first variation of area** is the time derivative of $A(X)$:

Theorem (First Variation of Area.)

Suppose that $X(t, u)$ is a smooth family of regular closed curves moving such that the normal velocity is $X_t(t, u) = v(t, u)N(t, u)$. Then

$$\frac{d}{dt} A(X) = - \int_{\Gamma_t} v \, ds. \quad (5)$$

Proof.

$$\begin{aligned}
 \frac{d}{dt} A(\Gamma) &= \frac{1}{2} \frac{d}{dt} \int_0^a \mathcal{R}X \cdot X_u \, du \\
 &= \frac{1}{2} \int_0^a \mathcal{R}X_t \cdot X_u + \mathcal{R}X \cdot X_{ut} \, du \\
 &= \frac{1}{2} \int_0^a \mathcal{R}X_t \cdot X_u + \mathcal{R}X \cdot X_{tu} \, du \\
 &= \frac{1}{2} \int_0^a \mathcal{R}X_t \cdot X_u - \mathcal{R}X_u \cdot X_t \, du \\
 &= \frac{1}{2} \int_0^a [\mathcal{R}(vN) \cdot T - \mathcal{R}T \cdot (vN)] |X_u| \, du \\
 &= \frac{1}{2} \int_0^a [-(vT) \cdot T - N \cdot (vN)] \, ds \\
 &= - \int_0^a v \, ds. \quad \square
 \end{aligned}$$

For example, if Y is the circle $Y(t, u) = ((R - t) \cos u, (R - t) \sin u)$,

$$\frac{dY}{dt} = -(\cos u, \sin u) = -N$$

so $v = 1$ is constant. Then the first variation is just

$$\frac{dA}{dt} = - \int_{\Gamma} 1 \, ds = -2\pi(R - t) = \frac{d}{dt} \pi(R - t)^2.$$

Let us assume that we have a family of curves Γ_t given by $X(t, u)$ for $t \geq 0$ and $0 \leq u \leq a$. If the curve is moving normally at a velocity $v(t, u) = \kappa(t, u)$, at all points (t, u) ,

$$\frac{d}{dt}X(t, u) = \kappa(t, u)N(t, u),$$

we say that the family of curves moves by **CURVATURE FLOW**.

Since $X_s = T$ and $T_s = \kappa N$, the curvature flow satisfies

$$X_t = X_{ss}$$

suggesting that it is a heat equation. Indeed, it is a nonlinear parabolic PDE.

BUT, since arclength changes along the flow, this is a **nonlinear equation**.

41. The Circle Moving by Curvature Flow.

For example, for the family of circles $X(t, u) = \rho(t)(\cos u, \sin u)$, The points move in the normal direction because

$$\frac{d}{dt}X = \rho'(t)(\cos u, \sin u) = -\rho'N$$

Hence the circles flow by curvature if

$$-\rho' = \kappa = \frac{1}{\rho}$$

It follows that

$$\rho(t) = \sqrt{R^2 - 2t}$$

if the initial radius is $\rho(0) = R$.

The circles shrink and completely vanish when t approaches $R^2/2$.

42. Curvature of Nonparametric Curves.

In the jargon, a curve given by $y = f(x)$ is called **nonparametric**. It is still a parameterized curve $X(u) = (u, f(u))$. Thus

$$X_u = (1, \dot{f}), \quad |X_u| = \sqrt{1 + \dot{f}^2}, \quad T = \frac{(1, \dot{f})}{\sqrt{1 + \dot{f}^2}}, \quad N = \frac{(-\dot{f}, 1)}{\sqrt{1 + \dot{f}^2}}.$$

Computing curvature

$$\frac{dT}{ds} = \frac{1}{|X_u|} \frac{dT}{du} = \frac{(-\dot{f}\ddot{f}, \ddot{f})}{(1 + \dot{f}^2)^2} = \frac{\ddot{f}}{(1 + \dot{f}^2)^{\frac{3}{2}}} \frac{(-\dot{f}, 1)}{\sqrt{1 + \dot{f}^2}} = \kappa N \quad (6)$$

so the curvature of a nonparametric curve is

$$\kappa = \frac{\ddot{f}}{(1 + \dot{f}^2)^{\frac{3}{2}}}$$

43. Translating Curve Flowing by Curvature.

There is no reason to track individual points of the curve as it flows. We could reparameterize as we go and then the trajectory $t \mapsto X(t, u)$ need not be perpendicular to the curve Γ_t . We need only that the normal projection of velocity be curvature

$$N \cdot \frac{\partial X}{\partial t} = \kappa \quad (7)$$

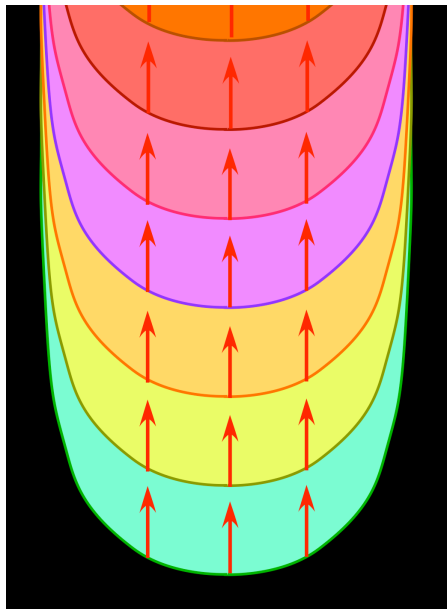
For example, suppose that a fixed curve moves by steady vertical translation. If this is also curvature flow, it is called a **soliton**. In this case

$$X = (u, f(u) + ct), \quad X_u = (1, \dot{f}), \quad |X_u| = \sqrt{1 + \dot{f}^2}, \quad \kappa = \frac{\ddot{f}}{(1 + \dot{f}^2)^{3/2}}$$

Then $\frac{\partial X}{\partial t} = (0, c)$. By (7) the equation for a translation soliton satisfies

$$\frac{c}{\sqrt{1 + \dot{f}^2}} = \frac{\ddot{f}}{(1 + \dot{f}^2)^{3/2}} \quad (8)$$

44. The Grim Reaper.



The solution of (8) gives a soliton called the **Grim Reaper**. (8) becomes

$$c = \frac{\ddot{f}}{1 + \dot{f}^2} = \frac{d}{du} \operatorname{Atn} \dot{f}$$

so

$$\operatorname{Atn} \dot{f} = cu + c_1$$

$$\dot{f} = \tan(cu + c_1)$$

$$f = c_2 - \frac{1}{c} \ln \cos(cu + c_1).$$

If $c = 1$ and $c_1 = c_2 = 0$,

$$X(t, u) = (u, t - \ln \cos(u))$$

45. κN is Independent of Parameterization.

If we change variables according to $v = h(u)$, where h' is nonvanishing, then

$$\frac{\partial}{\partial v} = \frac{\partial u}{\partial v} \frac{\partial}{\partial u} = h'(v) \frac{\partial}{\partial u}.$$

It follows that

$$\frac{\partial}{\partial s} = \frac{1}{|X_v|} \frac{\partial}{\partial v} = \frac{h'}{|h'X_u|} \frac{\partial}{\partial u} = \frac{\epsilon}{|X_u|} \frac{\partial}{\partial u}$$

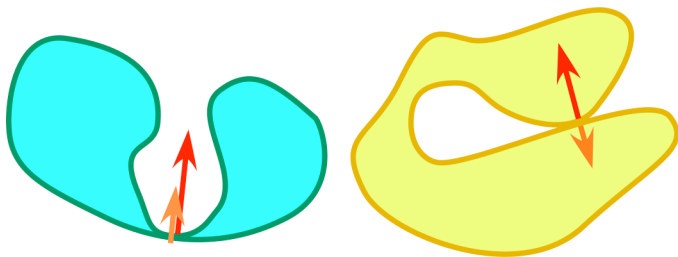
where $\epsilon = \pm 1$ according to whether $h' > 0$ or $h' < 0$. Hence X_{ss} is the same regardless of parameterization.

$$\kappa N = X_{ss} = \frac{1}{|X_v|} \frac{\partial}{\partial v} \left(\frac{1}{|X_v|} \frac{\partial}{\partial v} X \right) = \frac{1}{|X_u|} \frac{\partial}{\partial u} \left(\frac{1}{|X_u|} \frac{\partial}{\partial u} X \right).$$

Note: even the **orientation** of Γ doesn't matter.

46. Maximum Principle Prevents Flowing Curve from Touching Itself.

Initially, Γ_0 is an embedded curve. The smoothly evolving curve cannot eventually cross itself at distinct points because if they would ever touch, their motion would tend to separate the points.



Suppose at some $t_0 > 0$ the evolving curve first touches at two points $u_1 \neq u_2$ but $X(t_0, u_1) = X(t_0, u_2)$. The touch from inside Γ as if the enclosed region is pinched or from the outside as if the regions bend together.

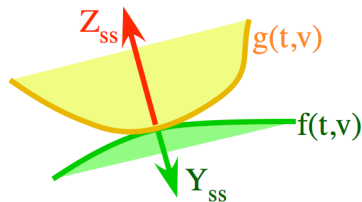


Figure: At Instant of Touching

Let us call $Y(t, u)$ the flow X near (t_0, u_1) and $Z(t, u)$ the flow X near (t_0, u_2) . Represent both curves nonparametrically as graphs over the same variable

$$Y(t, v) = (v, f(t, v)),$$

$$Z(t, v) = (v, g(t, v)).$$

Suppose that $f(t_0, v_0) = g(t_0, v_0)$ and $f(t_1, v) \leq g(t_1, v)$ for v near v_0 as would be the case at the instant of the first interior touch. It follows that

$$f_v(t_0, v_0) = g_v(t_0, v_0)$$

$$f_{vv}(t_0, v_0) \leq g_{vv}(t_0, v_0)$$

It follows from (6) that the flow velocity

$$Y_{ss} = f_{vv} \frac{(-f_v, 1)}{(1 + f_v^2)^2}$$

$$Z_{ss} = g_{vv} \frac{(-g_v, 1)}{(1 + g_v^2)^2}$$

Since the vectors are equal at (t_0, v_0) since $f_v(t_0, v_0) = g_v(t_0, v_0)$.

Because $f_{vv}(t_0, v_0) \leq g_{vv}(t_0, v_0)$, the upper curve is moving faster upward than the lower curve: the curves tend to **move apart!**

This idea can be turned into a proof.

49. Closed Curves Flowing by Curvature Vanish in Finite Time.

The same argument says that if one curve starts inside another, then they **never touch** as they flow. Thus the inside blob must extinguish before the shrinking disk does!

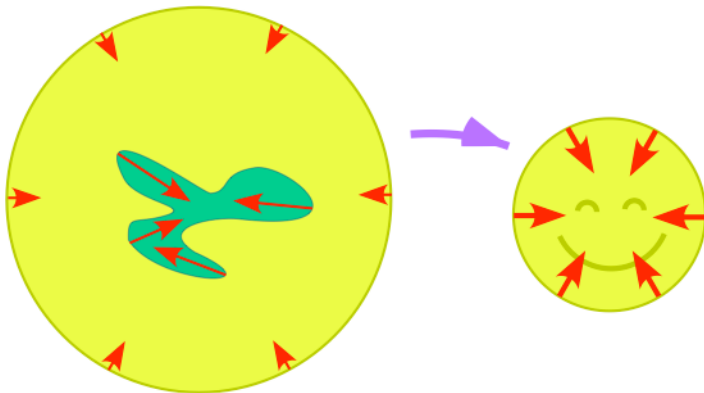


Figure: Flowing inside curve vanishes before the outside curve vanishes.

Also, closed curves inside the Grim Reaper die before it sweeps by!

50. Flow Shrinks Curves via Integral Estimate.

Let us assume that Γ_t are embedded closed curves on the interval $t \in I = [0, T)$. Then the first variation formula (4), (5) tells us that the area and length shrink.

Under curvature flow, the normal velocity is $v = \kappa$. Thus at each instant $t \in I$ we have

$$\begin{aligned}\frac{dL}{dt} &= - \int_{\Gamma_t} \kappa^2 ds, \\ \frac{dA}{dt} &= - \int_{\Gamma_t} \kappa ds = -2\pi.\end{aligned}$$

The latter integral is the total turning angle ($= 2\pi$) of an embedded closed curve. It follows that

$$A(\Gamma_t) = A_0 - 2\pi t$$

where $A_0 = A(\Gamma_0)$ is the area enclosed by the starting curve.

The flow can only exist up to vanishing time $T = \frac{A_0}{2\pi}$.

51. Inradius / Circumradius

Let K be the region bounded by γ . The radius of the smallest circular disk containing K is called the **circumradius**, denoted R_{out} . The radius of the largest circular disk contained in K is the **inradius**.

$$R_{\text{in}} = \sup\{r : \text{there is } p \in \mathbf{E}^2 \text{ such that } B_r(p) \subseteq K\}$$

$$R_{\text{out}} = \inf\{r : \text{there exists } p \in \mathbf{E}^2 \text{ such that } K \subseteq B_r(p)\}$$

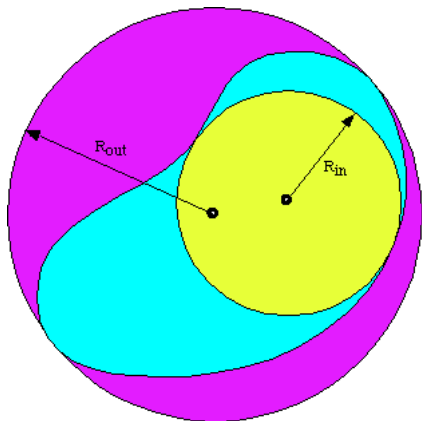




Figure: T. Bonnesen 1873–1935

Theorem (Bonnesen's Inequality [1921])

Let Ω be a convex plane domain whose boundary has length L and whose area is A . Let R_{in} and R_{out} denote the inradius and circumradius of the region Ω . Then

$$rL \geq A + \pi r^2 \quad (9)$$

for all $R_{in} \leq r \leq R_{out}$.

53. Proof of Bonnesen's Inequality.

It suffices to show (9) for polygons \mathcal{P}_n and for $R_{\text{in}} < r < R_{\text{out}}$ and then pass (9) to the limit as $\mathcal{P}_n \rightarrow \Omega$. Let $B_r(x, y)$ be the closed disk of radius r and center (x, y) . Let E_r denote the set of centers whose balls touch \mathcal{P}_n .

$$E_r = \{(x, y) \in \mathbb{R}^2 : B_r(x, y) \cap \mathcal{P}_n \neq \emptyset\}$$

Let $n(x, y)$ denote the number of points in $\partial B_r(x, y) \cap \partial \mathcal{P}_n$. The theorem follows by computing

$$I = \int_{E_r} n(x, y) \, dx \, dy$$

in two ways.

Since $B_r(x, y)$ can neither contain, nor be contained by \mathcal{P}_n , except on a set of measure zero, n is finite and

$$n(x, y) \geq 2 \quad \text{for almost all } (x, y) \in E_r.$$

54. Area of E_r for Polygons.

Hence E_r consists of all points in the interior of \mathcal{P}_n as well as all points within a distance r .

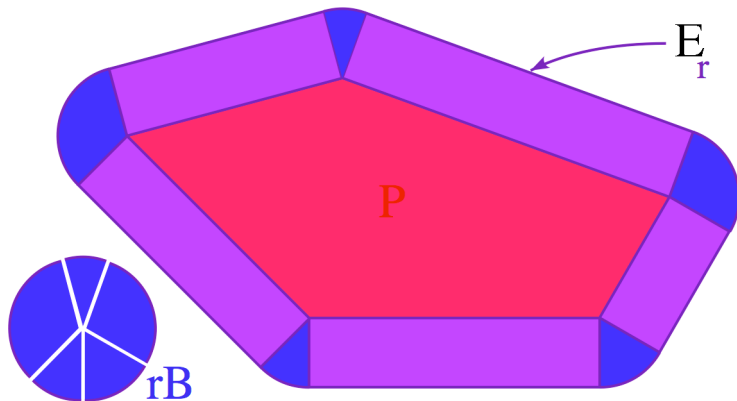


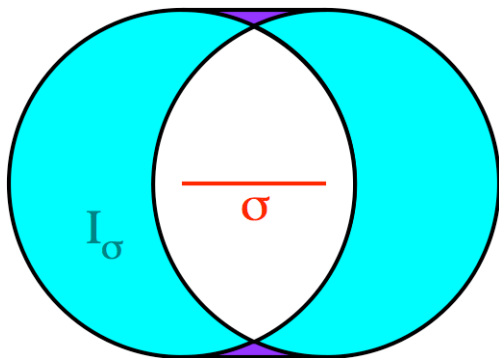
Figure: E_r of polygon has area $A(P_n) + L(\partial P_n)r + \pi r^2$.

Thus

$$2A + 2Lr + 2\pi r^2 \leq I.$$

55. Measure of points touching a single boundary segment.

It follows from $R_{\text{in}} < r < R_{\text{out}}$ that B_r touches \mathcal{P}_n if and only if it touches $\partial\mathcal{P}_n$. Let σ be one of the boundary line segments. Let $n_\sigma(x, y)$ be the number of points in $\sigma \cap \partial B_r(x, y)$. The set I_σ of centers of circles touching σ is the union of circles with centers on σ .



$$\int_{\sigma \cap \partial B_r(x, y) \neq \emptyset} n_\sigma \, dx \, dy = 4r L(\sigma).$$

Hence

$$I = \int_{E_r} n \, dx \, dy = \sum_{\sigma} \int_{\sigma \cap B_r(x,y) \neq \emptyset} n_{\sigma} \, dx \, dy = \sum_{\sigma} 4r L(\sigma) = 4r L(\partial \mathcal{P}_n).$$

Putting both computations together

$$2A + 2Lr + 2\pi r^2 \leq I \leq 4rL$$

or (9),

$$A + \pi r^2 \leq rL. \quad \square$$

57. Support Function.

The distance of the tangent line to the origin is called the **support function**

$$p = -X \cdot N = -X \cdot \frac{\mathcal{R}X_u}{|X_u|}.$$

Hence, the area is

$$A = -\frac{1}{2} \int_{\Gamma} X \cdot \mathcal{R}X_u \, du = \frac{1}{2} \int_{\Gamma} p \, ds$$

The length is

$$L = \int_{\Gamma} T \cdot T \, ds = \int_{\Gamma} T \cdot X_s \, ds = - \int_{\Gamma} T_s \cdot X \, ds = - \int_{\Gamma} \kappa N \cdot X \, ds = \int_{\Gamma} \kappa p \, ds$$



Figure: M. Gage

Theorem (Gage's Inequality [1983])

Let Ω be a C^2 convex plane domain whose boundary Γ has length L and whose area is A . Then

$$\frac{\pi L}{A} \leq \int_{\Gamma} \kappa^2 ds. \quad (10)$$

Proof. By Schwarz's Inequality,

$$L^2 = \left(\int_{\Gamma} \kappa p ds \right)^2 \leq \int_{\Gamma} \kappa^2 ds \int_{\Gamma} p^2 ds$$

(10) follows if Ω can be moved so

$$\int_{\Gamma} p^2 ds \leq \frac{AL}{\pi}. \quad (11)$$

Proof. First we assume Ω is symmetric about the origin. Then

$$R_{\text{in}} \leq p \leq R_{\text{out}}.$$

Integrate Bonnesen's Inequality

$$\begin{aligned} L \int_{\Gamma} p \, ds &\geq A \int_{\Gamma} ds + \pi \int_{\Gamma} p^2 \, ds \\ 2AL &\geq AL + \pi \int_{\Gamma} p^2 \, ds \end{aligned}$$

which is (11) for symmetric domains.

Now assume Ω is any convex domain. We claim that Ω can be bisected by a line that divides the area into equal parts and cuts both boundary points at parallel tangents.

The claim depends on the Intermediate Value Theorem. Let $g(s)$ be the continuous function such that $s < g(s) < s + L$ which gives the place along Γ such that the line through the points $X(s)$ and $X(g(s))$ bisects the area. Hence $g(g(s)) = s + L$. Consider the continuous function

$$h(s) = [T(s) \times T(g(s))] \cdot Z$$

where Z is the positively oriented normal to the plane. $h(s) = 0$ iff $T(s) = -T(g(s))$.

Observe $h(0) = -h(g(0))$. If $h(0) = 0$ then $s_0 = 0$ determines the line. Otherwise, by IVT, there is an $s_0 \in (0, g(0))$ where $h(s_0) = 0$ and s_0 determines the line.

Let L be the line segment from $X(s_0)$ to $X(g(s_0))$. Move the curve so the midpoint of L is the origin.

Let γ_1 and γ_2 be the sides of Γ split by L .

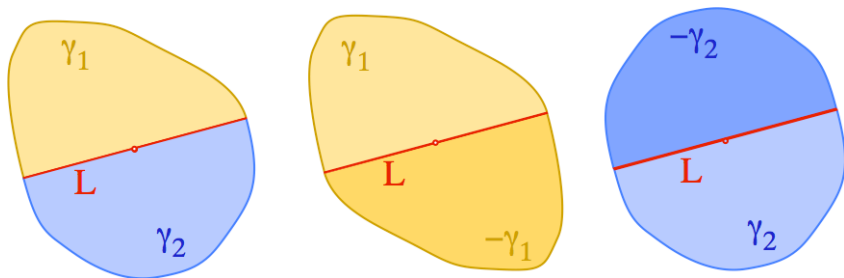


Figure: Cut and Reglue along Line L that Halves the Area and Touches the Curve at Parallel Tangents.

Erasing γ_2 for the moment, reflect γ_1 through the origin to form a closed, convex curve, that is symmetric with respect to the origin. Note that the tangents at the endpoints of γ_1 must be parallel for $\gamma_1 \cup (-\gamma_1)$ to be convex. We apply (11)

$$2 \int_{\gamma_1} p^2 ds \leq \frac{2AL_1}{\pi}.$$

where $2L_1$ is the length of $\gamma_1 \cup (-\gamma_1)$. Similarly with $\gamma_2 \cup (-\gamma_2)$,

$$2 \int_{\gamma_2} p^2 ds \leq \frac{2AL_2}{\pi}.$$

Adding these two inequalities yields

$$2 \int_{\Gamma} p^2 ds \leq \frac{2AL_1 + 2LA_2}{\pi} = \frac{2AL}{\pi}$$

which is (11) for general convex domains.



63. Convex Curves get Rounder as they Flow.

For any closed curve, the **isoperimetric inequality** says that the boundary cannot be shorter than that of a circle with the same area

$$L^2 \geq 4\pi A.$$

Theorem (Gage [1983])

Let Γ_t be a \mathcal{C}^2 family of curves flowing by curvature $X_t = X_{ss}$ for $0 \leq t < T$ starting from a convex Γ_0 . Then

- ① The isoperimetric ratio decreases: $\frac{d}{dt} \left(\frac{L^2}{4\pi A} \right) \leq 0$.
- ② If the flow survives until $T = \frac{1}{2\pi} A_0$ (it does by the Gage and Hamilton Theorem) and $A \rightarrow 0$ as $t \rightarrow T$, then

$$\frac{L^2}{4\pi A} \rightarrow 1 \quad \text{as } t \rightarrow T.$$

Hence, the curves become circular in the sense that

$$\frac{(R_{out}(t) - R_{in}(t))^2}{4R_{out}^2} \leq \frac{L^2}{4\pi A} - 1 \rightarrow 0$$

(1) is a computation with the first variation formulas and Gage's Inequality

$$\begin{aligned}\frac{d}{dt} \left(\frac{L^2}{4\pi A} \right) &= \frac{2ALL' - L^2 A'}{4\pi A^2} \\ &= \frac{L}{2\pi A} \left(- \int_{\Gamma_t} \kappa^2 ds + \frac{\pi L}{A} \right) \\ &\leq 0.\end{aligned}$$

(2) is similar, using a stronger form of Gage's Inequality.

(3) follows from the Strong Isoperimetric Inequality of Bonnesen and $A \leq \pi R_{\text{out}}^2$

$$\frac{\pi^2 (R_{\text{out}}(t) - R_{\text{in}}(t))^2}{4\pi^2 R_{\text{out}}^2} \leq \frac{L^2 - 4\pi A}{4\pi A} = \frac{L^2}{4\pi A} - 1.$$

65. Strong Isoperimetric Inequality of Bonnesen.

Theorem (Strong Isoperimetric Inequality of Bonnesen)

Let Ω be a convex planar domain with boundary length L and area A . Let R_{in} and R_{out} denote the inradius and circumradius of the Ω . Then

$$L^2 - 4\pi A \geq \pi^2(R_{out} - R_{in})^2. \quad (12)$$

Proof. Consider the quadratic function $f(s) = \pi s^2 - Ls + A$. By Bonnesen's inequality, $f(s) \leq 0$ for all $R_{in} \leq s \leq R_{out}$. Hence these numbers are located between the zeros of $f(s)$, namely

$$R_{out} \leq \frac{L + \sqrt{L^2 - 4\pi A}}{2\pi} \\ \frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} \leq R_{in}.$$

Subtracting these inequalities gives

$$R_{out} - R_{in} \leq \frac{\sqrt{L^2 - 4\pi A}}{\pi},$$

which is (12). □

Theorem (M. Gage & R. Hamilton [1986], M. Grayson [1987])

Suppose that $\Gamma_0 \in \mathcal{C}^2$ is an embedded curve in the plane with bounded curvature and encloses area A_0 . Then there is a unique smooth family of embedded curves independent of parametrization Γ_t that satisfies $X_t = X_{ss}$ on $[0, T_\infty)$, where $T_\infty = \frac{1}{2\pi}A_0$. The solution has the following properties:

- ① *If Γ_0 is nonconvex, there is a time $t_1 \in (0, T_\infty)$ where Γ_{t_1} is convex.*
- ② *If Γ_{t_1} is ever convex but possibly with straight line segments, then Γ_t will be strictly convex for $t > t_1$.*
- ③ *The flow shrinks to a round point in the sense that*
 - $\frac{R_{in}}{R_{out}} \rightarrow 1$ as $t \rightarrow T_\infty$;
 - $\frac{\min_{\Gamma_t} \kappa}{\max_{\Gamma_t} \kappa} \rightarrow 1$ as $t \rightarrow T_\infty$;
 - *Curvature stays bounded on compact intervals $[0, T_\infty - \epsilon]$;*
 - $\max_{\Gamma_t} |\partial_\theta^\alpha \kappa| \rightarrow 0$ as $t \rightarrow T_\infty$ for all $\alpha \geq 1$, where $T = (\cos \theta, \sin \theta)$.

Theorem

Let Γ_t be a \mathcal{C}^2 family of curves flowing by curvature $X_t = X_{ss}$ for $t \in [0, T)$. Then

- ① If $g(t, u) \in \mathcal{C}^2$ is any function, then $g_{st} = g_{ts} + \kappa^2 g_s$;
- ② $T_t = \kappa_s N$ and $N_t = -\kappa_s T$;
- ③ $\kappa_t = \kappa_{ss} + \kappa^3$.

Compute (1) using $g_{tu} = g_{ut}$ and $X_{tu} = X_{ut}$ (but $g_{st} \neq g_{ts}$):

$$\begin{aligned}
 g_{st} &= \left(\frac{g_u}{|X_u|} \right)_t = \frac{g_{tu}}{|X_u|} - \frac{X_u \cdot X_{tu}}{|X_u|^3} g_u \\
 &= g_{ts} - T \cdot (X_t)_s g_s \\
 &= g_{ts} - T \cdot (\kappa N)_s g_s \\
 &= g_{ts} - T \cdot (\kappa_s N - \kappa^2 T) g_s \\
 &= g_{ts} + \kappa^2 g_s.
 \end{aligned}$$

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To see (2a), use (1) and $X_t = \kappa N$,

$$T_t = X_{st} = X_{ts} + \kappa^2 X_s = (\kappa N)_s + \kappa^2 T = \kappa_s N - \kappa^2 T + \kappa^2 T = \kappa_s N.$$

To see (2b), as the rotation \mathcal{R} doesn't depend on t ,

$$N_t = (\mathcal{R}T)_t = \mathcal{R}(T_t) = \mathcal{R}(\kappa_s N) = -\kappa_s T.$$

Finally, to see (3), differentiate the defining equation for curvature $T_s = \kappa N$, using (2b)

$$T_{st} = (\kappa N)_t = \kappa_t N + \kappa N_t = \kappa_t N - \kappa \kappa_s T$$

Then using (1) and (2a)

$$T_{st} = T_{ts} + \kappa^2 T_s = (\kappa_s N)_s + \kappa^3 N = \kappa_{ss} N - \kappa \kappa_s T + \kappa^3 N.$$

Equating gives (3)

$$\kappa_t = \kappa_{ss} + \kappa^3.$$

Corollary

Let Γ_t be a \mathcal{C}^2 family of curves flowing by curvature $X_t = X_{ss}$ for $t \in [0, T)$. Suppose that Γ_0 is convex. Then Γ_t is convex for all $t \in [0, T)$.

Proof idea. The curve is convex iff it's curvature is nonnegative. Hence $\kappa(0, s) \geq 0$ for all s . Apply the **maximum principle** to κ which satisfies

$$\kappa_t = \kappa_{ss} + \kappa^3.$$

If $t_0 \in [0, T)$ is the first time κ is zero, say at s_0 , then we have

$$\kappa(t_0, s_0) = 0 \quad \text{and} \quad \kappa(t_0, s) \geq 0 \quad \text{for all } s.$$

Thus $\kappa_{ss}(t_0, s_0) \geq 0$ and

$$\kappa_t(t_0, s_0) = \kappa_{ss}(t_0, s_0) + \kappa^3(t_0, s_0) \geq 0 + 0$$

so the curvature is increasing and cannot dip below the x-axis. □

Theorem (G. Huisken, [1984].)

Suppose that $M_0^n \subset \mathcal{R}^{n+1}$ is a \mathcal{C}^2 embedded closed convex hypersurface in Euclidean Space with bounded curvature. Then there is a unique smooth family of embedded hypersurfaces, independent of parametrization, M_t , that satisfy $X_t = \Delta X = nHN$ on a maximal interval $[0, T_\infty)$. The solution has the following properties:

- ① M_t will be strictly convex for $t > 0$.
- ② The flow shrinks to a round point in the sense that
 - $\frac{R_{in}}{R_{out}} \rightarrow 1$ as $t \rightarrow T_\infty$;
 - $\frac{\min_{\Gamma_t} \kappa_1}{\max_{\Gamma_t} \kappa_n} \rightarrow 1$ as $t \rightarrow T_\infty$, where the $\kappa_1 \leq \dots \leq \kappa_n$ are principal curvatures of M_t (eigenvalues of the second fundamental form h_{ij} .);
 - The principal curvatures stay bounded on compact intervals $[0, T_\infty - \epsilon]$;
 - $\max_{\Gamma_t} |\partial_\theta^\alpha h_{ij}| \rightarrow 0$ as $t \rightarrow T_\infty$ for all $\alpha \geq 1$, where $\theta = N \in \mathbb{S}^n$.

Theorem (R. Hamilton, [1982].)

Suppose that (M_0^n, g_0) be a smooth compact Riemannian Manifold (thus g_0 is a metric with bounded sectional curvature). Then there is a unique smooth family of metrics g_t , that satisfy

$$g_t = -2 \operatorname{Ric}(g)$$

on a maximal interval $[0, T_\infty)$. The solution has the following properties:

- ① *If (M^3, g_0) has positive sectional curvature, then it will have positive curvature for $t > 0$. (Same for Ricci Curvature.)*
- ② *The flow shrinks to a manifold of constant positive sectional curvature in the sense that*
 - $\frac{\min_{\Gamma_t} \operatorname{Sect}(g)}{\max_{\Gamma_t} \operatorname{Sect}(g)} \rightarrow 1$ as $t \rightarrow T_\infty$, where the $\operatorname{Sect}(g)$ is the sectional curvature of g ;
 - *The sectional curvatures stay bounded on compact intervals $[0, T_\infty - \epsilon]$;*
 - $\max_{\Gamma_t} (T - t) |\nabla^\alpha \operatorname{Riem}(g)| \rightarrow 0$ as $t \rightarrow T_\infty$ for all $\alpha \geq 1$.

Thanks!

