Can You Hear the Shape of a Manifold?

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The URL for these Beamer Slides: *"Can You Hear the Shape of a Manifold?"*

`http://www.math.utah.edu/treiberg/HearManifoldSlides.pdf`

Some references about spectrum.

- Isaac Chavel, *Eigenvalues in Riemannian Geometry,* Academic Press, Orlando, 1984.
- Marcel Berger, Paul Gauduchon & Edmond Mazet, *Le Spectre d'une Variété Riemannienne,* Springer Lecture Notes in Mathematics 194, Springer-Verlag, Berlin, 1971.
- Peter Li & Andrejs Treibergs, Applications of eigenvalue techniques to geometry, in *Contemporary Geometry*, H. Wu, ed., Plenum (1991) 22–52.

Leon Green had already asked in 1960 if there are isospectral manifolds and Milnor's result was available in 1964. Study of what information was extractible from the spectrum was begun by Minakshisundaram & Plejel (1949), Mc Kean & Singer (1967), Patodi (1971).

In 1966, Mark Kac popularized the question for planar domains. Knowing only that a few geometric quantities could be determined about a drum from its vibration frequencies, *eg.* Weyl's Formula (1911), he boldly asked if the drum itself could be determined.

Figure: Mark Kac 1914–1984

Conjecture (Kac [1966])

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Viewing Ω as a membrane with fixed boundary, let $spec_D(\Omega)$ denote the set of its frequencies of vibration. Then $spec_D(\Omega)$ determines Ω up to rigid motion and reflection.

4. Outline.

- Differentiable Manifolds.
- Laplacian and spectrum.
- Rayleigh Quotient – Variational characterization – Basic properties.
- Weyl's Asymptotic Formula
- Milnor's Example
- Gordon-Webb-Wolpert Example

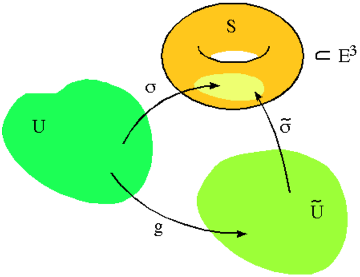

Figure: Coordinate Charts for Surface in $\mathbb{E}^3$

A differentiable manifold is a connected topological space that is locally Euclidean. Every point has a neighborhood endowed with a curvilinear coordinate system. The coordinates behave consistently on overlapping coordinate charts so that Calculus works.
For example, a smooth surface in $\mathbf{R}^3$ has the structure of a manifold.

Another example is the rectangular n-torus $\mathbf{T}^n$. Imagine gluing oppposite edges of the box (or fundamental region)

$$\mathcal{F} = [0, a_1] \times [0, a_2] \times \cdots \times [0, a_n]$$

with periodic boundary conditions. Equivalently, $\mathbf{T}^n$ is the product of circles, or the quotient of $\mathbb{R}^n$ by a lattice.

$$\mathbf{T}^n = \mathbb{S}^1_{a_1} \times \mathbb{S}^1_{a_2} \times \cdots \times \mathbb{S}^1_{a_n} = \mathcal{F}/\sim = \mathbb{R}^n/\Gamma$$

where $\mathbb{S}^1_a$ is a circle of length a,

$$\Gamma = a_1\mathbb{Z} \times a_2\mathbb{Z} \times \cdots \times a_n\mathbb{Z}$$

is a rectangular lattice and the identifications are $x \sim x + j$ for all $x \in \mathbb{R}^n$ and all $j \in \Gamma$.

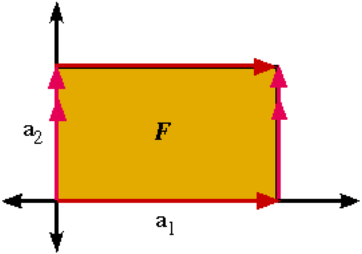

Figure: Fund. Domain of $\mathbf{T}^2$

7. Intrinsic Geometry. The Riemannian nmetric.

An additional structure on the manifold is a Riemannian metric which gives lengths and angles of vectors. It is given by a symmetric positive definite matrix function $\mathcal{G} = [g_{ij}(x)]$ in each coordinate patch in such a way to be consistently defined patch to patch. If $V = (v^1, v^2, \ldots, v^n)$ is a vector in local coordinates $x = (x_1, \ldots, x_n)$ on a manifold, then its length at x is

$$|v|_{\mathcal{G}} = \sqrt{\sum_{i,j=1}^{n} g_{ij}(x)\, v^i\, v^j}.$$

The length of a continuously differentiable curve $\gamma \in \mathcal{C}^1([a,b], M)$ is

$$\mathrm{L}(\gamma) = \int_a^b |\dot{\gamma}|_{\mathcal{G}}\, dt.$$

e.g., for a surface $S \subset \mathbb{R}^3$, the metric is the restriction of the background Euclidean metric $\mathcal{G} = (dx^2 + dy^2 + dz^2)|_S$. For the torus $\mathbf{T}^n$ we may take the flat $\mathbb{R}^n$ metric $\mathcal{G} = dx_1^2 + \cdots + dx_n^2$ so $g_{ij} = \delta_{ij}$.

The Riemannian metric induces a distance function on M. If $P, Q \in M$,

$$\mathrm{d}(P,Q) = \inf \left\{ \mathrm{L}(\gamma) : \begin{array}{c} \gamma : [\alpha,\beta] \to M \text{ is piecewise } C^1, \\ \gamma(\alpha) = P,\ \gamma(\beta) = Q \end{array} \right\}$$

Theorem

(S, d) is a metric space.

We shall assume M is compact so (M, d) is a complete metric space.

Integration is done with the volume form, which in local coordinates is

$$d\,\mathrm{V} = \sqrt{g(x)}\,dx_1\,dx_2 \cdots dx_n.$$

where $g(x) = \det(g_{ij}(x))$

Gradient, divergence and Laplacian are defined so that the usual Green's formulas continue to hold on the manifold. If $V(x) = (v^1(x), \ldots, v^n(x))$ is a $\mathcal{C}^1$ vector field in local coordinates $x = (x_1, \ldots, x_n)$ on a Riemannian manifold and $u \in \mathcal{C}^2(M)$, then using the inverse matrix $g^{ij} = [g_{ij}]^{-1}$,

$$\operatorname{grad} u = \left(\cdots, \sum_{j=1}^{n} g^{ij} \frac{\partial}{\partial x_j} u, \cdots \right)$$

$$\operatorname{div} V = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{g} v^i \right)$$

$$\Delta\, u = \operatorname{div} \operatorname{grad} u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_i} u \right)$$

Gradient, divergence and Laplacian are defined so that the usual Green's formulas continue to hold on the manifold. If $V(x) = (v^1(x), \ldots, v^n(x))$ is a $\mathcal{C}^1$ vector field in local coordinates $x = (x_1, \ldots, x_n)$ on a Riemannian manifold and $u \in \mathcal{C}^2(M)$, then using the inverse matrix $g^{ij} = [g_{ij}]^{-1}$,

When $M = \mathbf{T}^n$ is the flat torus,

$$\operatorname{grad} u = \left(\cdots, \sum_{j=1}^n g^{ij} \frac{\partial}{\partial x_j} u, \cdots \right) = \left(\cdots, \frac{\partial u}{\partial x_i}, \cdots \right);$$

$$\operatorname{div} V = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(\sqrt{g} v^i \right) = \sum_{j=1}^n \frac{\partial v^j}{\partial x_j};$$

$$\Delta u = \operatorname{div} \operatorname{grad} u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_i} u \right) = \sum_{j=1}^n \frac{\partial^2 u}{\partial {x_j}^2}.$$

Suppose that a manifold vibrates according to the wave equation. What frequencies are heard? Let ρ be the density and τ be the tension. Then the amount of a small transverse vibration is given by $v(x,t)$ where $x \in M$ and $t \geq 0$,

$$\frac{\partial^2 v}{\partial t^2} = \frac{\tau}{\rho} \Delta v.$$

We seek solutions of the form $v(x,t) = T(t)u(x)$. Thus

$$T''(t)u(x) = \frac{\tau}{\rho} T(t) \Delta u(x).$$

We can separate variables. The only way a t-expression equals an x-expression is if both equal $\lambda = \text{const}$.

$$\frac{\rho T''(t)}{\tau T(t)} = -\lambda = \frac{\Delta u(x)}{u(x)}$$

which results in two equations

$$\Delta u + \lambda u = 0,$$
$$\rho T'' + \lambda \tau T = 0.$$

Whenever λ is a number and there is a non-identically vanishing $u \in \mathcal{C}^2(M)$ such that

$$\Delta\, u + \lambda u = 0 \qquad (1)$$

we call λ the eigenvalue and u the corresponding eigenfunction. The collection of all eigenvalues $\operatorname{spec}(M)$ is the spectrum of the manifold.

Since Δ is self-adjoint, eigenvalues are real. Let u be an eigenfunction corresponding to λ. Multiplying by u and integrating by parts,

$$\lambda \int_M u^2 = -\int_M u\, \Delta\, u = \int_M |\operatorname{grad} u|^2 \ge 0. \qquad (2)$$

Thus eigenvalues are nonnegative. If $\lambda = 0$ then (2) implies $|\operatorname{grad} u| = 0$ so $u = \text{const}$.

12. Frequencies.

When $\lambda > 0$, the time equation

$$\rho T'' + \lambda \tau T = 0$$

has the solution

$$T(t) = A \cos\left(\sqrt{\frac{\tau \lambda t}{\rho}}\right) + B \sin\left(\sqrt{\frac{\tau \lambda t}{\rho}}\right).$$

Thus the time dependence is sinusoidal. Its frequency is

$$\frac{1}{2\pi}\sqrt{\frac{\tau \lambda}{\rho}}$$

cycles per unit time. The frequency increases with the eigenvalue λ and tension τ and decreases with density ρ.
The lowest frequency corresponds to smallest positive eigenvalue $\lambda_1 > 0$. Thus λ_1 is called the fundametal eigenvalue.

Theorem

Let M^n be a smooth compact manifold.

1. *Let λ be an eigenvalue and u its corresponding eigenfunction. Then $u \in C^\infty(M)$.*
2. *For all $\lambda \in spec(M)$, the eigenspace $\mathcal{E}_\lambda = \{u : \Delta\, u + \lambda u = 0\}$ is finite dimensional. Its dimension is called the multiplicity m_λ.*
3. *The zero eigenspace is one dimensional $m_0 = 1$.*
4. *The set of eigenvalues is discrete and tends to infinity. The eigenvalues can be ordered*

$$spec(M) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty\}$$

5. *Let u_i denote the λ_i eigenfunction. If $\lambda_i \ne \lambda_j$ then u_i and u_j are orthogonal. By adjusting bases in the eigenspaces $\mathcal{E}_\lambda$ we may assume $\{u_0, u_1, u_2, \dots\}$ is a complete orthonormal basis in $\mathcal{L}^2(M)$.*

Proof Sketch. To see orthogonality (5), suppose $\lambda_i \neq \lambda_j$ and u_i and u_j are corresponding eigenfunctions. Then

$$(\lambda_i - \lambda_j) \int_M u_i u_j = \int_M -(\Delta u_i) u_j + u_i \Delta u_j = 0$$

by Green's formula. □

Since eigenfunction u_j satisfy on (M, g)

$$\Delta u_j + \lambda_j u_j = 0, \tag{3}$$

eigenvalues scale like $\dfrac{1}{\mathrm{distance}^2}$. So if we scale the lengths of curves by a factor s on the manifold by multiplying the metric, $s^2 g$, then the eigenvalue becomes

$$\lambda_j(M, s^2 g) = \frac{\lambda_j(M, g)}{s^2}.$$

"Bigger manifolds make lower tones."

Since $\int_M u_0 u_1 = 0$, the first eigenfunction is orthogonal to constants. We seek functions v, orthogonal to constants, that have fixed $\int_M v^2 = 1$ and minimize the energy $\int_M |\operatorname{grad} v|^2$. Equivalently, we minimimize the Rayleigh Quotient.

$$\lambda_1 = \inf_{\substack{v \in H^1(M), \\ \int_M v = 0, \\ v \not\equiv 0}} \frac{\displaystyle\int_M |\operatorname{grad} v|^2}{\displaystyle\int_M v^2} = \inf_v \mathcal{R}(v)$$

By Calculus of Variations, the minimizer satisfies $\Delta u + \lambda u = 0$ so is an eigenfunction and by integrating the PDE, its corresponding Lagrange multiplier is $\lambda = \lambda_1$ because it is the smallest possible positive constant.

Thus the first eigenvalue has a variational characterization: u_1 minimizes the $\mathcal{R}(v)$ and it gives $\mathcal{R}(u_1) = \lambda_1$.

16. Example: Rectangular torus $\mathbf{T}^n$.

We seek (λ, u) so that $\Delta\, u + \lambda u = 0$ in $\mathbb{R}^n$ with periodic boundary conditions $u(x + j) = u(x)$ for all $x \in \mathbb{R}^n$ and $j \in \Gamma$.
It turns out by separating variables that a complete set of eigenfunctions may be taken the form

$$u(x_1, \ldots, x_n) = X_1(x_1) \cdots X_n(x_n)$$

where $X_i'' + c_i\, X_i = 0$ for some constant c_i and X_i is a_i-periodic. Thus

$$X_i(x_i) = A \cos\left(\frac{2\pi j_i x_i}{a_i}\right) + B \sin\left(\frac{2\pi j_i x_i}{a_i}\right)$$

where $j_i \in \mathbb{Z}$ is an integer. If $j_i = 0$ the eigenspace has multiplicity one, otherwise it has multiplicity two.

Inserting such solutions into $\Delta u + \lambda u = 0$, we find that the eigenvalues are

$$\lambda = 4\pi^2 \left(\frac{j_1^2}{a_1^2} + \frac{j_2^2}{a_2^2} + \cdots + \frac{j_n^2}{a_n^2} \right)$$

where $j_i \in \mathbb{Z}$ for all i.

The counting function gives the number of eigenvalues less than s counted with multiplicity

$$N_M(s) = \sharp \{\lambda \in \operatorname{spec}(M) : \lambda \le s\} = \sum_{\lambda \le s} m_\lambda$$

For the flat rectangular torus, this is the number of integer points in $\mathbb{Z}^n$ within an ellipsiod. (Counting positive and negative integers accounts for the multiplicity two eigenspaces.)

$$N_{\mathbf{T}^n}(s) = \sharp \left\{ j \in \mathbb{Z}^n : \frac{j_1^2}{a_1^2} + \frac{j_2^2}{a_2^2} + \cdots + \frac{j_n^2}{a_n^2} \le \frac{s}{4\pi^2} \right\}$$

18. $\mathbb{S}^1$.

A complete set of eigenfunctions of $\mathbb{S}_a{}^1$, the circle of length a are generated by

$$f(\theta) = A\cos\left(\frac{2\pi j\theta}{a}\right) + B\sin\left(\frac{2\pi j\theta}{a}\right)$$

so

$$\mathrm{spec}(\mathbb{S}_a^1) = \left\{\frac{4\pi^2}{a^2}j^2 : j \in \mathbb{Z}\right\}$$

- Play: *StandingWaveLoop_long.mpg*

The sphere is the hypersurface $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ with the induced metric. Using spherical coordinates $\theta \in \mathbb{S}^n$ and $r \geq 0$, the Laplacian $\Delta_{\mathbb{R}^{n+1}}$ in $\mathbb{R}^{n+1}$ may be expressed in terms of the spherical Laplacian Δ_θ

$$\Delta_{\mathbb{R}^{n+1}} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\,\Delta_\theta\,.$$

A homogeneous functions of degree d satisfies $u(r\theta) = r^d u(\theta)$ for all θ and $r \geq 0$. It turns out that harmonic homogeneous polynomials restrict to a complete set of eigenfunctions of the sphere. Indeed if $\Delta_{\mathbb{R}^{n+1}}\, u = 0$ and u is homogeneous of degree d, then

$$0 = \Delta_{\mathbb{R}^{n+1}}\, u = d(d-1)r^{d-2}u + ndr^{d-2}u + r^{d-2}\,\Delta_\theta\, u.$$

Thus on the sphere, $r = 1$ so

$$0 = \Delta_\theta\, u + d(d+n-1)u.$$

Thus on the sphere $\mathbb{S}^n$, for $d = 0, 1, 2, \ldots$,

$$\lambda_d = d(d + n - 1).$$

The dimension of the harmonic polynomials of degree d gives the multiplicity

$$m_d = \binom{n+d}{d} - \binom{n+d-2}{d-2}.$$

For example if $n = 1$ then $m_0 = 1$ and $m_d = 2$ for $d \geq 1$ corresponding to Fourier series. For example $\Re e(z^d)$ is a harmonic polynomial that restricts to $u(\theta) = \cos(d\theta)$ on $\mathbb{S}^1$.

If $n = 2$ then $m_d = 2d + 1$. For example, the coordinate function $u(x_1, x_2, x_3) = x_1$ is harmonic homogeneous of degree one that restricts to an eigenfunction with $\lambda_1 = 2$. Its multiplicity is three, corresponding to the three coordinates.

$$\mathrm{spec}(\mathbb{S}^2) = \{0, 2, 2, 2, 6, 6, 6, 6, 6, 6, 12, \dots, 12, 20, \dots\}$$

On $\mathbb{S}^2$, the counting function is

$$N(s) = \sum_{\lambda_d \leq s} m_d = \sum_{d(d+1) \leq s} (2d + 1)$$

22. Hear $\mathbb{S}^2$ and other manifolds.

Dennis DeTurck created a record of manifold sounds. his recordings are on line at

$$\texttt{http://www.toroidalsnark.net/som.html}$$

Figure: Hermann Weyl 1885–1955

Weyl's formula says you can hear the dimension and the volume of a manifold.

Theorem (Weyl [1911])

Let M be a closed, compact, connected manifold whose eigenvalues repeated with multiplicity are

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty.$$

Let $N(s)$ be the number of eigenvalues, counted with multiplicity, $\le s$. Then

$$N(s) \sim \frac{|B_1^n|\, V(M)\, s^{n/2}}{(2\pi)^n} \qquad \text{as } s \to \infty.$$

Equivalently

$$\lim_{k\to\infty} \frac{(2\pi)^n k}{|B_1^n|(\lambda_k)^{n/2}} = V(M).$$

We shall give the argument in case $\mathbf{T}^n = \mathbb{R}^n/\Gamma$ where $\Gamma = \{(a_1 k_1, \ldots, a_n k_n) : k_1, \ldots, k_n \in \mathbb{Z}\}$. In this case $\mathrm{V}(\mathbf{T}^n) = a_1 \cdots a_n$. The counting function

$$\begin{aligned} N_{\mathbf{T}^n}(s) &= \sharp \left\{ j \in \mathbb{Z}^n : \frac{j_1^2}{a_1^2} + \frac{j_2^2}{a_2^2} + \cdots + \frac{j_n^2}{a_n^2} \leq \frac{s}{4\pi^2} \right\} \\ &= \sharp \left\{ \gamma \in \Gamma^* : 4\pi^2 |\gamma|^2 \leq s \right\} \end{aligned}$$

where

$$\Gamma^* = \left\{ \left(\frac{j_1}{a_1}, \frac{j_2}{a_2}, \ldots, \frac{j_n}{a_n} \right) : j_1, \ldots, j_n \in \mathbb{Z} \right\}$$

is the dual lattice.

$N_{\mathbf{T}^n}(s)$ is the number of Γ^* points in the sphere

$$\mathcal{U}(s) = \left\{ x \in \mathbb{R}^n : |x| \leq \frac{\sqrt{s}}{2\pi} \right\}$$

Imagine a closed unit rectangle with center $\gamma \in \Gamma^*$,

$$\mathcal{Q}(\gamma) = \left[\gamma_1 - \frac{1}{2a_1}, \gamma_1 + \frac{1}{2a_1}\right] \times \cdots \times \left[\gamma_n - \frac{1}{2a_n}, \gamma_n + \frac{1}{2a_n}\right].$$

Let be the union of rectangles around γ's in $\mathcal{U}$.

$$\mathcal{P} = \bigcup_{\gamma \in \mathcal{U} \cap \Gamma^*} \mathcal{Q}(\gamma).$$

The volume $\mathrm{V}(\mathcal{P}) = N_{\mathbf{T}^n}(s)/(a_1 \cdots a_n)$. $\mathcal{P}$ is contained in a larger sphere and contains a smaller sphere. $\mathcal{Q}(\gamma)$ is contained in a ball of radius $R = \frac{1}{2}\left|\left(\frac{1}{a_1}, \dots, \frac{1}{a_1}\right)\right|$. By the triangle inequality in $\mathbb{R}^n$,

$$|(x_1, \dots, x_1)| - R \le \left|\left(x_1 \pm \frac{1}{2a_1}, \dots, x_n \pm \frac{1}{2a_n}\right)\right| \le |(x_1, \dots, x_n)| + R$$

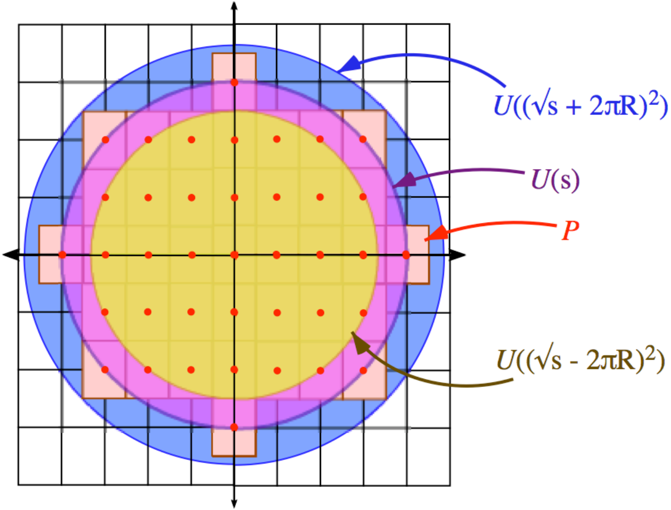

Figure: Lattice points within $\mathcal{U}(s)$, polyhedron $\mathcal{P}$ and its surrounding spheres.

It follows that $\mathcal{P}$ is contained in a larger sphere and contains a smaller sphere

$$\mathcal{U}\Big((\sqrt{s}-2\pi R)^2\Big) \subset \mathcal{P} \subset \mathcal{U}\Big((\sqrt{s}+2\pi R)^2\Big).$$

Taking volumes

$$\frac{|B_1^n|\,(\sqrt{s}-2\pi R)^n}{(2\pi)^n} \leq \frac{N_{\mathbf{T}^n}(s)}{a_1\cdots a_n} \leq \frac{|B_1^n|\,(\sqrt{s}+2\pi R)^n}{(2\pi)^n}.$$

which implies

$$N_{\mathbf{T}^n}(s) \sim \frac{\mathrm{V}(\mathbf{T}^n)|B_1^n|\,s^{n/2}}{(2\pi)^n} \qquad \text{as } n \to \infty. \quad \square$$

28. Proof of Weyl's formula for $\mathbb{S}^2$.

Since $N_{\mathbb{S}^2}(s) = \sum_{d=0}^{d_m} 2d + 1 = 1 + 3 + 5 + \cdots + (2d_m + 1) = (d_m + 1)^2$, where $d_m = \max\{d \in \mathbb{Z} : d(d+1) \leq s\}$ then

$$\left(d_m + \tfrac{1}{2}\right)^2 = d_m^2 + d_m + \tfrac{1}{4} \leq s + \tfrac{1}{4},$$

we get

$$-\tfrac{1}{2} + \sqrt{s + \tfrac{1}{4}} \leq d_m + 1 \leq \tfrac{1}{2} + \sqrt{s + \tfrac{1}{4}}$$

so

$$\left(-\tfrac{1}{2} + \sqrt{s + \tfrac{1}{4}}\right)^2 \leq N_{\mathbb{S}^2}(s) \leq \left(\tfrac{1}{2} + \sqrt{s + \tfrac{1}{4}}\right)^2.$$

It follows that

$$N_{\mathbb{S}^2}(s) \sim \frac{(4\pi)\,\pi\, s}{(2\pi)^2} = \frac{\mathrm{V}(\mathbb{S}^2)\,|B_1^2|\,s}{(2\pi)^2} \qquad \text{as } s \to \infty. \quad \square$$

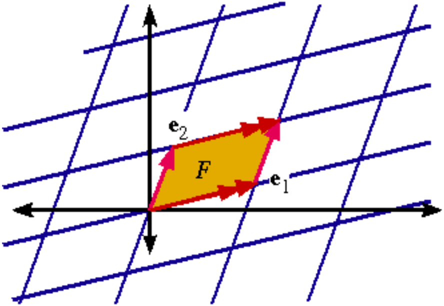

Figure: Fundamental Domain for general $\mathbf{T}^2$.

Let $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis for $\mathbb{R}^n$. A lattice

$$\Gamma = \{j_1\mathbf{e}_1 + \cdots + j_n\mathbf{e}_n : j_1, \ldots, j_n \in \mathbb{Z}\}.$$

Then a general flat torus is given by $\mathbf{T}^n = \mathbb{R}^n/\Gamma$. Its fundamental domain is a parallelopipied $\mathcal{F}$ spanned by $\mathcal{B}$.

The dual lattice is $\Gamma^* = \{y \in \mathbb{R}^2 : x \bullet y \in \mathbb{Z} \text{ for all } x \in \Gamma.\}$ The dual lattice is generated by $\{\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*\}$, a basis of $\mathbb{R}^n$ defined by the equations

$$\mathbf{e}_i \bullet \mathbf{e}_j^* = \delta_{ij} \qquad \text{for all } i, j.$$

For any $\gamma \in \Gamma^*$ the functions

$$u_\gamma(x) = e^{2\pi i x \bullet \gamma}$$

are Γ periodic: $u_\gamma(x + j) = y_\gamma(x)$ for all $x \in \mathbb{R}^n$ and $j \in \Gamma$. Hence they descend to $\mathbf{T}^n$. They turn out to provide a complete set of eigenfunctions

$$\Delta\, u_\gamma + 4\pi^2 |\gamma|^2 u_\gamma = 0.$$

Thus

$$\operatorname{spec}(\mathbf{T}^n) = \{4\pi^2 |\gamma| : \gamma \in \Gamma^*\}.$$

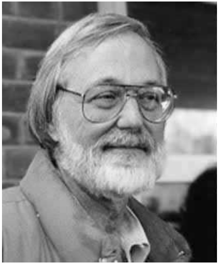

Figure: John Milnor 1931–

Theorem (Milnor [1966])

There are two lattices in $\Gamma, \tilde{\Gamma} \subset \mathbb{R}^{16}$ *such that the torii* $\mathbf{T}^{16}(\Gamma)$ *and* $\mathbf{T}^{16}(\tilde{\Gamma})$ *are isospectral*

$$spec\Big(\mathbf{T}^{16}(\Gamma)\Big) = spec\Big(\mathbf{T}^{16}(\tilde{\Gamma})\Big).$$

but are not isometric.

32. Milnor's construction.

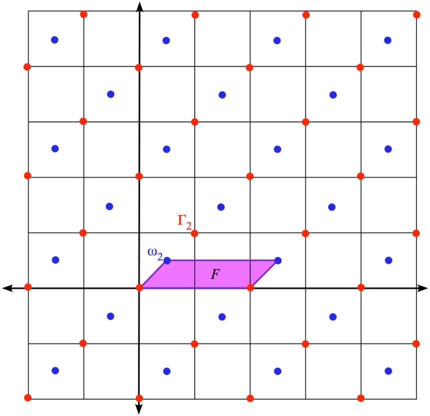

Figure: Milnor's Lattice Γ(2).

Let $\Upsilon = \{(j_1, \ldots, j_n) \in \mathbb{Z}^n : \sum_{i=1}^n j_i \in 2\mathbb{Z}\}$ and $\omega_n = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$.
Let $\Gamma(n)$ be the lattice generated by Υ and ω_n.
The torii are built from $\Gamma = \Gamma(16)$ and $\tilde{\Gamma} = \Gamma(8) \oplus \Gamma(8)$.

Since $\Gamma(n)$ is index 2 in Γ_2 so the volume

$$V\big(\mathbf{T}^n(\Gamma(n))\big) = \tfrac{1}{2}\,V\big(\mathbf{T}^n(\Gamma_2)\big) = V\big(\mathbf{T}^n(\mathbb{Z}^n)\big) = 1.$$

Lemma

Suppose $n \in 8\mathbb{N}$. If $y \in \Gamma(n)$ then $|y|^2 \in 2\mathbb{Z}$.

In case $y \in \Gamma_2$ so $\sum_{i=1}^n y_i \in 2\mathbb{Z}$. Hence

$$|y|^2 = \sum_{i=1}^n y_i^2 = \left(\sum_{i=1}^n y_i\right)^2 - 2\sum_{i<j} y_i y_j \in 2\mathbb{Z}.$$

In case $y \in \mathbb{Z}\omega_n$ then $y = h\omega_n$ so $y_i = \frac{h}{2}$. Thus

$$|y|^2 = h^2 \sum_{i=1}^n \tfrac{1}{4} = \tfrac{n}{4} h^2 \in 2\mathbb{Z}.$$

In case $y = x + h\omega$ with $x \in \Gamma_2$ then since $x \bullet \omega_n = \frac{1}{2}\sum_{i=1}^n x_i \in \mathbb{Z}$,

$$|y|^2 = |x|^2 + 2h\, x \bullet \omega_n + h^2 |\omega_n|^2 \in 2\mathbb{Z}. \quad \square$$

Lemma

Suppose $n \in 4\mathbb{Z}$. Then $\Gamma(n) = \Gamma^*(n)$.

Proof. "$\subset$": If $y, y' \in \Gamma(n)$ to show $y \bullet y' \in \mathbb{Z}$. Put $y = x + k\omega n$ and $y' = x' + k\omega_n$ with $x, x' \in \Gamma_2$. Then

$$y \bullet y' = x \bullet x' + k'\, x \bullet \omega_n + k\, x' \bullet \omega_n + kk'|\omega_n|^2 \in \mathbb{Z}.$$

"$\supset$": Follows from volumes being equal:

$$\mathrm{V}(\Gamma^*(n)) = \frac{1}{\mathrm{V}(\Gamma(n))} = \frac{1}{1} = 1. \quad \square$$

Lemma

$\mathbf{T}^{16}(\Gamma)$ *is not isometric to* $\mathbf{T}^{16}(\tilde{\Gamma})$.

$\Gamma(8)$ is generated by a set of vectors all of length $\sqrt{2}$:

$$\{\mathbf{e}_1 - \mathbf{e}_8,\, \mathbf{e}_2 - \mathbf{e}_8,\, \ldots, \mathbf{e}_7 - \mathbf{e}_8,\, \mathbf{e}_1 + \mathbf{e}_2,\, \mathbf{e}_3 + \mathbf{e}_4,\, \mathbf{e}_5 + \mathbf{e}_6,\, \omega_8\}$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_8\}$ is standard basis of $\mathbb{R}^8$. Hence so is $\tilde{\Gamma} = \Gamma(8) \oplus \Gamma(8)$.

But $\Gamma = \Gamma(16)$ cannot be. $x \in \Gamma(16)$ has form $\sum_{i=1}^{16} j_i\, \mathbf{e}_i$ or $\sum_{i=1}^{16} \left(j_i + \frac{1}{2}\right) \mathbf{e}_i$ with all $j_i \in \mathbb{Z}$ and $\sum_{i=1}^{16} j_i \in 2\mathbb{Z}$. There must be elements of the second type which have length at least 2:

$$\sum_{i=1}^{16} \left(j_i + \tfrac{1}{2}\right)^2 = \tfrac{1}{4} \sum_{i=1}^{16} (2j_i + 1)^2 \geq 4.$$

Thus $\mathbf{T}^{16}(\Gamma)$ cannot be isometric to $\mathbf{T}^{16}(\tilde{\Gamma})$. □

Lemma

Suppose both Γ and Γ' are $n = 8$, 12, 16 or 20 dimensional lattices. If both satisfy $\Gamma = \Gamma^*$ and $|y|^2 \in 2\mathbb{Z}$ for all $y \in \Gamma$ then the torii are isospectral: $spec(\mathbf{T}^n(\Gamma)) = spec(\mathbf{T}^n(\Gamma'))$.

Encode the spectrum in the partition function to show that $\Theta_\Gamma = \Theta_{\Gamma'}$:

$$\Theta_\Gamma(t) = \sum_{y \in \Gamma^*} e^{-\pi |y|^2 t}.$$

If f decays fast at infinity, its Fourier Transform is

$$\tilde{f}(x) = \int_{\mathbb{R}^n} f(y)\, e^{2\pi i\, x \bullet y} dy.$$

For example, $(e^{-\pi |x|^2})^\sim = e^{-\pi |y|^2}$.

Using the Poisson Summation Formula,

$$\sum_{x\in\Lambda} f(x) = \frac{1}{\mathrm{V}(\Gamma)} \sum_{y\in\Lambda^*} \tilde{f}(y)$$

applied to the lattice $\Lambda = \sqrt{t}\Gamma$ and using $|y|^2$ even and $\Gamma = \Gamma^*$,

$$\sum_{x\in\Gamma} e^{-\pi|x|^2 t} = \frac{1}{\mathrm{V}(\Gamma)} \sum_{y\in\Gamma^*} e^{-\pi|y|^2/t} = \frac{1}{t^{n/2}} \sum_{y\in\Gamma} e^{-\pi|y|^2/t}$$

which implies

$$\Theta_\Gamma(t) = t^{-n/2}\Theta_\Gamma(1/t). \tag{4}$$

Also, since $|x|^2 \in 2\mathbb{Z}$,

$$\Theta_\Gamma(t+i) = \sum_{x\in\Gamma} e^{-\pi|x|^2(t+i)} = \sum_{x\in\Gamma} e^{-\pi|x|^2 t} = \Theta_\Gamma(t). \tag{5}$$

(4) and (5) and say that Θ_Γ is a modular form of weight $\frac{n}{4} = 2, 3, 4$ or 5. But it is known from analytic number theory that the space of such modular forms is one-dimensional! Hence since both agree at infinity

$$\Theta_\Gamma = \Theta_{\Gamma'},$$

so the torii $\mathbf{T}^n(\Gamma)$ and $\mathbf{T}^n(\Gamma')$ are isospectral. □

38. Gordon, Webb and Wolpert's isospectral drums.

Figure: David Webb and Carolyn Gordon with some isospectral domains.

Theorem (Gordon, Webb & Wolpert [1991])

There are two nonisometric domains $\Omega, \tilde{\Omega} \subset \mathbb{R}^2$ that are isospectral
$spec_D(\Omega) = spec_D(\Omega')$.

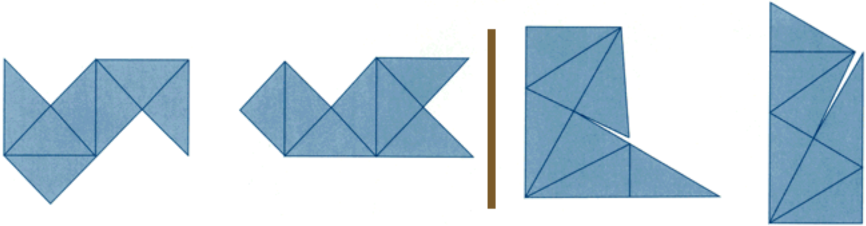

Figure: Two other pairs of isospectral drums.

40. Milnor's construction.

Thanks!

Let C be a piecewise $\mathcal{C}^1$ curve or network (a union of $\mathcal{C}^1$ curves.) Given a line L in the plane, let $n(L \cap C)$ be the number of intersection points. If C contains a linear segment and if L agrees with that segment, $n(C \cap L) = \infty$. For any such C, however, the set of lines for which $n = \infty$ has dK-measure zero.

Figure: Mark Kac 1914–1984

Theorem (Poincaré Formula for lines [1896])

Let C be a piecewise $\mathcal{C}^1$ curve in the plane. Then the measure of unoriented lines meeting C, counted with multiplicity, is given by

$$2\,L(C) = \int_{\{L : L \cap C \neq \emptyset\}} n(C \cap L)\, dK(L).$$