On the Realizability of Electric Fields in Conducting Materials

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“Which electric fields are realizable in conducting materials?”


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3. Outline.

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4. Conductivity $\iff$ Electric Field. Electric Field $\iff$ Conductivity?

- Let $Y = [0, 1]^d$ be the unit cube in $\mathbb{R}^d$ and $\sigma \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d \times d})$ be symmetric, uniformly elliptic conductivity. Assume $\sigma$ is $Y$-periodic:

$$\sigma(x + k) = \sigma(x) \quad \text{for all } x \in \mathbb{R}^d \text{ and } k \in \mathbb{Z}^d.$$ 

For all $\lambda \in \mathbb{R}^d - \{0\}$ there is $u^\lambda \in \mathcal{H}^1_{\text{loc}}(\mathbb{R}^d)$, unique up to constant multiple, such that $u(x) - \lambda \cdot x$ is $Y$-periodic and

$$\text{div}(\sigma \nabla u^\lambda) = 0 \quad (1)$$

The effective conductivity of the periodic material is then $\sigma^*$ given by averaging over a cell

$$\sigma^* \lambda = \sigma^* \langle \nabla u^\lambda \rangle = \langle \sigma \nabla u^\lambda \rangle$$

where $\nabla u^\lambda$ is the electric field and $J = \sigma \nabla u^\lambda$ is the current field.

- We reverse the question: given a periodic electric field $\nabla u$, is it possible to find a symmetric periodic positive definite conductivity $\sigma$ that satisfies the conductivity equation (1)? In other words, which electric fields are realizable?
Consider the case that conductivity $\sigma = sI$ is isotropic.

**Theorem (I 1)**

Assume that $u \in C^2(\mathbb{R}^d)$ satisfies $\nabla u \neq 0$. Then $\nabla u$ is locally isotropically realizable.

Let $x_0 \in D$. Writing $\sigma = e^z$ the conductivity equation $\text{div}(e^z \nabla u) = 0$ becomes a first order PDE for the unknown $z(x)$,

$$\nabla u(x) \cdot \nabla z = -\Delta u(x) \quad (2)$$

The usual method of characteristics gives the solution. Since $\nabla u$ is a characteristic direction, if $\mathcal{H}$ is a hypersurface through $x_0$, transverse to $\nabla u(x_0)$ and $z_0(h)$ a function on $\mathcal{H}$, then the solution may be given by using the PDE to propagate the solution off of $\mathcal{H}$. Let $X(t, x)$ be the gradient flow of $\nabla u$, satisfying the characteristic ODE for $(t, h) \in I \times G$ in some neighborhood $G$ of $x_0$ and some $I = (-\varepsilon, \varepsilon)$ where $\varepsilon > 0$,

$$\frac{\partial}{\partial t} X(t, h) = \nabla u(X(t, h)), \quad \text{for } (t, h) \in I \times G$$

$$X(0, h) = h$$
Then \( z \) satisfies an ODE along the trajectories since

\[
\frac{\partial}{\partial t} z(X(t, h)) = \nabla z(X(t, h)) \cdot \frac{\partial}{\partial t} X(t, h) = -\Delta u(X(t, h))
\]

If also the initial condition holds

\[
z(0, h) = z_0(h) \quad \text{for } h \in G \cap \mathcal{H}
\]

then the solution is

\[
\zeta(t, h) = z_0(h) - \int_0^t \Delta u(X(\tau, h)) \, d\tau
\]

Finally, the mapping \( \Psi : (t, h) \mapsto X(t, h) \) is a local \( \mathcal{C}^1 \) diffeomorphism from \( I \times (G \cap \mathcal{H}) \) to a neighborhood of \( x_0 \) since the Jacobian \( d\Psi(0, x_0) \) is invertible because \( \nabla u(x_0) \) is transverse to \( \mathcal{H} \). Writing its inverse \( (t, h) = \Phi(x) \), a solution of (2) near \( x_0 \) is

\[
z(x) = \zeta(\Phi(x)).
\]
We say that the hypersurface $\mathcal{H}$ is a **global section** for the flow of $\nabla u$ if the trajectory of the gradient flow starting from any point $y \in \mathbb{R}^d$ meets $\mathcal{H}$ transversally in exactly one point.

**Theorem (II 2)**

Assume that $u \in C^2(\mathbb{R}^d)$ satisfies $\nabla u \neq 0$ and that $\nabla u$ has a global section $\mathcal{H}$. Then $\nabla u$ is isotropically globally realizable.

Note that if $\nabla u$ is periodic then $z$ may not be periodic.
Example (1)

Let \( u(x, y) = x - \cos(2\pi y) \), and \( Y = [0, a] \times [0, 1] \), where \( a > 0 \).

\[
\nabla u = e_1 + 2\pi \sin(2\pi y)e_2, \quad \Delta u = 4\pi^2 \cos(2\pi y).
\]

On the section \( x = x_1 \) the initial condition is \( X(0, x) = x \) and the gradient flow decouples

\[
\frac{\partial}{\partial t} X_1 = 1 \\
\frac{\partial}{\partial t} X_2 = 2\pi \sin(2\pi X_2(t, x))
\]

It can be integrated: for \( x = (x_1, x_2) \)

\[
X(t, x) = \left( x_1 + t, n + \frac{1}{\pi} \arctan\left( e^{4\pi^2 t} \tan(\pi x_2) \right) \right), \quad \text{if} \ x_2 \in (n - \frac{1}{2}, n + \frac{1}{2})
\]

\[
X(t, x) = \left( x_1 + t, n + \frac{1}{2} \right), \quad \text{if} \ x_2 = n + \frac{1}{2}
\]
Also

\[
\frac{\partial}{\partial t} z = \nabla z \cdot \frac{\partial}{\partial t} X = -\Delta u(X(t,x)) = \frac{4\pi^2 e^{8\pi^2 t} \tan^2(\pi x_2) - 4\pi^2}{e^{8\pi^2 t} \tan^2(\pi x_2) + 1}
\]

If \( z \) vanishes at \( x_1 = 0 \), this can be integrated to yield

\[
\sigma = e^z = \begin{cases} 
\frac{1 + \tan^2(\pi x_2)}{e^{4\pi^2 x_1} + e^{-4\pi^2 x_1} \tan^2(\pi x_2)}, & \text{if } x_2 \not\in \frac{1}{2} + \mathbb{Z}; \\
e^{4\pi^2 x_1}, & \text{if } x_2 \in \frac{1}{2} + \mathbb{Z};
\end{cases}
\]

(3)

We see it is not periodic in \( Y \).
Figure: Trajectories of the Gradient Flow for Example 1.
Example (2)

Let the characteristic function of periodic intervals be given by
\( \chi(t) = 1 \) if \( 0 \leq \lfloor t \rfloor \leq \frac{1}{2} \) (fractional part) and 0 otherwise. Then

\[
    u(x, y) = y - x + \int_0^x \chi(t) \, dt
\]

is Lipschitz continuous and

\[
    \nabla u = \chi e_2 + (1 - \chi)(e_2 - e_1) \quad \text{a.e. in } \mathbb{R}^2,
\]

For this \( \nabla u \) there is no positive function \( \sigma \in L^\infty(\mathbb{R}^2) \) such that \( \sigma \nabla u \) is divergence free.

\( \nabla u \) has discontinuities on the lines \( x_1 = k/2 \) for some \( k \in \mathbb{Z} \). Let
\( Q = (-r, r)^2 \) for some \( r \in (0, \frac{1}{2}) \). If there were positive \( \sigma \in L^\infty(Q) \) such that \( \sigma \nabla u \) is divergence free, then there is a stream function \( \nu \in H^1 \) satisfying \( \nabla \nu = R \sigma \nabla u \), which is unique up to additive constant and is Lipschitz continuous.
\[ \nabla v = R\sigma \nabla u \text{ implies} \]

\[ 0 = \nabla u \cdot \nabla v = (e_2 - e_1) \cdot \nabla v \quad \text{in} \ ( -r, 0 ) \times ( -r, r ) \]

hence \( v(x, y) = f(x + y) \) for some Lipschitz function \( f \) in \([-2r, r]\). On the other hand

\[ 0 = \nabla u \cdot \nabla v = e_2 \cdot \nabla v \quad \text{in} \ ( 0, r ) \times ( -r, r ) \]

Hence \( v(x, y) = g(x) \) for some Lipschitz function \( g \) in \([0, r]\).

By continuity on the line \( x_1 = 0 \), \( f(y) = g(0) \). Hence \( f \) is constant on \([-r, r]\) implying \( v \) is too. Thus

\[
\begin{align*}
\nabla v &= 0 \text{ a.e. in} \ ( -r, 0 ) \times ( 0, r ) \text{ and} \\
\sigma \nabla u &= \sigma (e_2 - e_1) \neq 0 \text{ a.e. in} \ ( -r, 0 ) \times ( 0, r )
\end{align*}
\]

which contradicts the equality \( \nabla v = R\sigma \nabla u \text{ a.e.} \). Thus \( \nabla u \) is not isotropically realizable in neighborhoods near the lines \( x = k/2, \ k \in \mathbb{Z} \).
13. On the Vanishing of Realized Fields

**Theorem (Ⅱ 3)**

Let $Y \in \mathbb{R}^d$ be a closed parallelepiped. Assume that $u \in C^1(\mathbb{R}^d)$ satisfies

- $\nabla u$ is $Y$-periodic and the cell average $\langle \nabla u \rangle \neq 0$.
- $\nabla u$ is realized as an electric field associated with a smooth periodic conductivity.

Then

1. if $d = 2$ then $\nabla u \neq 0$ in all of $\mathbb{R}^2$;
2. if $d = 3$ then there is an example where $\nabla u(y_0) = 0$ for some point $y_0 \in \mathbb{R}^3$.

(2) One example is given by Ancona[2002], another may be constructed from the periodic chain mail of Briane, Milton and Nesi[2004].
14. On the Vanishing of Realized Fields -

**Theorem (Alessandrini & Nesi (2001))**

Let $Y \subset \mathbb{R}^2$ be a parallelogram, $\sigma \in L^\infty$ be uniformly positive definite, symmetric and $Y$-periodic. For a symmetric matrix $A$ with $\det A > 0$ consider $U \in W^{2,2}_{loc}(\omega, \mathbb{R}^2)$ such that $U - Ax$ is a $Y$-periodic and satisfies

$$\text{Div}(\sigma DU) = 0$$

and the cell average $\langle \det(DU) \rangle > 0$. Then

$$\det(DU) > 0 \quad \text{a. e. in } \mathbb{R}^2.$$ 

*In the isotropic case $u$ is a scalar, $\langle \nabla u \rangle \neq 0$ implies $\nabla u \neq 0$ in $\mathbb{R}^2$.***
Figure: Periodic chain mail of Briane, Milton and Nesi consisting of linked toroidal rings of highly conductive material.
Rings have $\sigma \gg 1$. There is a matrix field such that $\langle DU \rangle = I$, $\langle \det(DU) \rangle = 1$ but $\det(DU) < 0$ in green region. Hence there is $\lambda \in \mathbb{R}^3 - \{0\}$ such that $\nabla (u \cdot \lambda)$ vanishes in $\mathbb{R}^3$. 

Figure: Section of periodic chain mail.
17. Realizability for Periodic Fields.

**Theorem (I 4)**

Let $Y \subset \mathbb{R}^d$ be a compact parallelopiped and $d \geq 2$. Let $u \in C^3(\mathbb{R}^d)$ such that $\nabla u$ is $Y$-periodic,

$$\nabla u \neq 0 \text{ in } \mathbb{R}^d \text{ and the cell average } \langle \nabla u \rangle \neq 0.$$  

Then $\nabla u$ is globally isotropically realizable.

Since $\nabla u$ is nonvanishing and periodic, $0 < c_1 \leq |\nabla u(x)| \leq c_2$ for all $x$ and the function $f(t) = u(X(t, x_0))$ satisfies

$$f'(t) = \nabla u(X(t, x - 0)) \cdot \frac{\partial X}{\partial t}(t, x_0) = |\nabla u(X(t, x_0))|^2 \in [c_1^2, c_2^2].$$

Thus

$$\lim_{t \to \infty} f(t) = \infty \text{ and } \lim_{t \to -\infty} f(t) = -\infty$$

and there is a unique $\tau(x) \in \mathbb{R}$ such that $f(\tau(x)) = 0$. By differentiable dependence and the implicit function theorem $\tau \in C^2(\mathbb{R}^d)$. 
Hence the level set \( \{ x \in \mathbb{R}^d : \tau(x) = 0 \} \) is a \( C^1 \) global section. Put

\[
w(x) = \int_0^{\tau(x)} \Delta u(X(s, x)) \, ds \quad \text{for } x \in \mathbb{R}^d
\]

By the change of variables formula \( r = s + t \),

\[
w(X(t, x)) = \int_0^{\tau(x) - t} \Delta u(X(s + t, x)) \, ds = \int_t^{\tau(x)} \Delta u(X(r, x)) \, dr
\]

so

\[
\frac{\partial}{\partial t} w(X(t, x)) = \nabla w(X(t, x)) \cdot \nabla u(X(t, x)) = -\Delta u(X(t, x))
\]

For the conductivity \( \sigma = e^{w(x)} \) we have at \( t = 0 \)

\[
\text{div} (\sigma \nabla (u)) = e^w (\nabla w \cdot \nabla v + \Delta u) = 0.
\]
19. Conductivity Might Not be Periodic for Smooth Electric Field.

**Theorem (II.5. For Example 1, no periodic isotropic \( \sigma \) is possible.)**

For \( u(x, y) = x - \cos(2\pi y) \), the \( Y = [0, a] \times [0, 1] \)-periodic electric field \( \nabla u \) does not admit a continuous non-vanishing \( Y \)-periodic isotropic conductivity \( \sigma \) that makes \( \sigma \nabla u \) divergence free.

Note \( \nabla u = e_1 + 2\pi \sin(2\pi y)e_2 \). Assume there is a \( Y \)-periodic function \( \sigma \) such that \( \sigma \nabla u \) is divergence free. Let \( Q = [0, a] \times [-r, r] \) for some \( 0 < r < \frac{1}{2} \). Then, using Green’s Theorem,

\[
0 = \int_Q \text{div}(\sigma \nabla u) \, dx \, dy = \oint_{\partial Q} (\sigma u_x) \, dy - (\sigma u_y) \, dx \\
= \int_{-r}^{r} \left[ \sigma(a, y)u_x(a, y) - \sigma(0, y)u_x(0, y) \right] \, dy \\
+ \int_{0}^{a} \left[ \sigma(x, r)u_y(x, r) - \sigma(x, -r)u_y(x, -r) \right] \, dx \\
= 0 + 2\pi \sin(2\pi r) \int_{0}^{a} \left[ \sigma(x, r) + \sigma(x, -r) \right] \, dx > 0 \]
Let $Y \subset \mathbb{R}^d$ be a compact parallelopiped and $d \geq 2$. Let $u \in C^3(\mathbb{R}^d)$ such that $\nabla u$ is $Y$-periodic, $\nabla u \neq 0$ in $\mathbb{R}^d$ and the cell average $\langle \nabla u \rangle \neq 0$. Assume that there is $C < \infty$ such that for all $x \in \mathbb{R}^d$,

$$\left| \int_0^{\tau(x)} \Delta u(X(t,x)) \, dt \right| \leq C$$

(4)

where $\tau(x)$ is the unique time such that $u(\tau(x), x) = 0$ as in the proof of Theorem $\text{II} \, 4$. Then $\nabla u$ is isotropically realizable with $Y$-periodic conductivity $\sigma, \sigma^{-1} \in \mathcal{L}_Y^\infty(\mathbb{R}^d)$.

Conversely, if $\nabla u$ is isotropically realizable with $Y$-periodic conductivity $\sigma \in C^1_Y(\mathbb{R}^d)$, then (4) holds.
Example (1, cont. Assumptions of Theorem II 6 do not hold.)

Let $u(x, y) = x - \cos(2\pi y)$, and $Y = [0, a] \times [0, 1]$, where $a > 0$.

\[
\nabla u = (1, 2\pi \sin(2\pi x_2)) \\
\Delta u = 4\pi^2 \cos(2\pi x_2)
\]

Put $p_0 = (x_1, 0)$. Thus, $X(t, p_0) = (x_1 + t, 0)$ so

\[
w(p_0) = \int_0^{\tau(p_0)} \Delta u(X(t, p_0)) \, dt = 4\pi^2 \cos(0)\tau(p_0).
\]

But by the definition of $\tau$,

\[
0 = u(X(\tau(p_0), p_0)) = x_1 + \tau(p_0) - \cos(0)
\]

so that

\[
w(p_0) = 4\pi^2(1 - x_1)
\]

which is not bounded.
Assume there is a positive periodic \( \sigma = e^w \in C^1_Y(\mathbb{R}^d) \) such that \( \text{div}(\sigma \nabla u) = 0 \). Then \( \nabla u \cdot \nabla w + \Delta u = 0 \) in \( \mathbb{R}^d \). Hence

\[
\int_0^{\tau(x)} \Delta u(X(t, x)) \, dt = -\int_0^{\tau(x)} \nabla w(X(t, x)) \cdot \nabla u(X(t, x)) \, dt
\]

\[
= -\int_0^{\tau(x)} \nabla w(X(t, x)) \cdot \frac{\partial}{\partial t} X(t, x) \, dt
\]

\[
= w(X(0, x)) - w(X(\tau(x), x))
\]

\[
= x - w(X(\tau(x), x))
\]

which is bounded by assumption. Hence (4) follows.
For simplicity, assume $Y = [0, 1]^d$. For $x \in \mathbb{R}^d$ define

$$\sigma_0(x) = \exp \left( \int_0^{\tau(x)} \Delta u(X(t, x)) \, dt \right)$$

and for $n \in \mathbb{N}$, average over the $2n+1$ integer vectors in $[-n, n]^d$

$$\sigma_n(x) = \frac{1}{(2n + 1)^d} \sum_{k \in \mathbb{Z}^d \cap [-n, n]^d} \sigma_0(x + k)$$

By (4), $\sigma_n$ is bounded in $L^\infty(\mathbb{R}^d)$. Hence a subsequence $\sigma_{n'}$ converges weak-$*$ to $\sigma_\infty$ in $L^\infty(\mathbb{R}^d)$. 
For any $k \in \mathbb{Z}^d$

\[
\left| (2n + 1)^d \sigma_n(x + k) - (2n + 1)^d \sigma_n(x) \right|
\]

\[
= \left| \sum_{|j-k|_{\infty} \leq n} \sigma_n(x + j) - \sum_{|j|_{\infty} \leq n} \sigma_n(x + j) \right|
\]

\[
\leq \sum_{|j|_{\infty} \leq n + |k|_{\infty}} \sigma_n(x + k) + \sum_{|j|_{\infty} > n} \sigma_n(x + k)
\]

\[
\leq 2e^C \left( (2n + 2k + 1)^d - (2n + 1)^d \right) \leq C_2(C, d, k)n^{d-1}
\]

Letting $n' \to \infty$ implies that $\sigma_\infty(x + k) = \sigma_\infty(x)$ a.e. in $\mathbb{R}^d$ and for any $k$. Thus $\sigma_\infty \in \mathcal{L}^\infty_\gamma(\mathbb{R}^d)$. Since $\sigma_0$ is bounded below by $e^{-C}$, $\sigma_\infty^{-1} \in \mathcal{L}^\infty_\gamma(\mathbb{R}^d)$.
As $\nabla u \in C^2_Y(\mathbb{R}^d)$, it is realized by the conductivity $\sigma_0$. Periodicity implies that also $\text{div}(\sigma_n \nabla u) = 0$ in $\mathbb{R}^d$. From weak-* convergence, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have

$$0 = \lim_{n' \to \infty} \int_{\mathbb{R}^d} \sigma_{n'} \nabla u \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^d} \sigma_\infty \nabla u \cdot \nabla \varphi \, dx$$

Hence $\text{div}(\sigma_\infty \nabla u) = 0$ in $\mathcal{D}'(\mathbb{R}^d)$ so that $\nabla u$ is isotropically realized by the $Y$-periodic conductivity $\sigma_\infty$. $\square$
26. Anisotropic Realizability.

**Theorem (A1)**

Let $Y \subset \mathbb{R}^2$ be a closed parallelogram. Let $u \in C^1(\mathbb{R}^2)$ such that $\nabla u \neq 0$ is $Y$-periodic in $\mathbb{R}^2$ and the cell average $\langle \nabla u \rangle \neq 0$. Then necessary and sufficient that $\nabla u$ be realizable by a continuous, $Y$-periodic, symmetric positive definite matrix-valued conductivity $\sigma$ is that there is a function $v \in C^1(\mathbb{R}^2)$ such that $\nabla v$ is $Y$-periodic in $\mathbb{R}^2$ and the cell average $\langle \nabla v \rangle \neq 0$ such that

$$R \nabla u \cdot \nabla v = \det(\nabla u, \nabla v) > 0 \quad \text{everywhere in } \mathbb{R}^2.$$  \hfill (5)

where $R$ is rotation by a right angle.

Theorem A1 continues to hold under the weaker assumptions that $\nabla u$ is $Y$-periodic, $\nabla u \in L^2(Y)$, $\nabla u \neq 0$ a.e. in $\mathbb{R}^2$ and $\langle \nabla u \rangle \neq 0$. In this case, the $Y$-periodic conductivity $\sigma$ defined only a.e. by the formula below and does not remain bounded in $Y$. However $\sigma \nabla u$ is divergence free in the sense of distributions.
Assume there is such \( v \). (5) says that \( \nabla v \) is nonvanishing. Then define

\[
\sigma = \frac{1}{|\nabla u|^4} \left( \begin{array}{cc}
\frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\
-\frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial x_1}
\end{array} \right)^T \left( \begin{array}{cc}
R \nabla u \cdot \nabla v & -\nabla u \cdot \nabla v \\
-\nabla u \cdot \nabla v & \frac{|\nabla u \cdot \nabla v|^2 + 1}{R \nabla u \cdot \nabla v}
\end{array} \right) \left( \\
-\frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial x_1}
\right)
\]

which is a continuous, symmetric positive definite matrix function.

\( \sigma \nabla u = -R \nabla v \) in \( \mathbb{R}^2 \) so it is divergence free.

Now assume there is \( u \) and a continuous positive definite symmetric \( \sigma \).

Let \( v \in C^1(\mathbb{R}^2) \) be the stream function which satisfies \( \nabla v = -R \nabla u \).

Hence \( \nabla v \) is \( \Gamma \)-periodic and

\[
R \nabla u \cdot \nabla v = -\nabla u \cdot R \nabla v = \sigma \nabla u \cdot \nabla u
\]

Allesandrin & Nesi’s result implies \( \nabla u \) is nonvanishing, which implies (5). By the div-curl lemma,

\[
\langle R \nabla u \cdot \nabla v \rangle = R \langle \nabla u \rangle \cdot \langle \nabla v \rangle = \langle \sigma \nabla u \cdot \nabla u \rangle > 0
\]

so \( \langle \nabla v \rangle > 0 \) also.
Example (1)

Let $u(x, y) = x - \cos(2\pi y)$, and $Y = [0, a] \times [0, 1]$, where $a > 0$. Then $\nabla u$ is anisotropically realizable.

$$\nabla u = e_1 + 2\pi \sin(2\pi x_2)e_2.$$ Take $v(x) = x_2$

We find

$$R\nabla u \cdot v = (-2\pi \sin(2\pi x_2)e_1 + e_2) \cdot e_2 = 1$$

so Theorem A.1 applies: for $\delta = 1 + 4\pi^2 \sin^2(2\pi x_2)$, let

$$\sigma = \frac{1}{\delta^2} \begin{pmatrix} \delta^2 + \delta - 1 & -2\pi \sin(2\pi x_2) \\ -2\pi \sin(2\pi x_2) & 1 \end{pmatrix}$$

Now $\sigma \nabla u = e_1$ which is divergence free.
Example (2)

\[ u(x) = x_2 - x_1 + \int_0^{x_1} \chi(t) \, dt \] where \( \chi = 1 \) if \( 0 \leq [t] \leq \frac{1}{2} \) and 0 otherwise. Then \( \nabla u \) is anisotropically realizable.

\[ \nabla u = \chi e_2 + (1 - \chi)(e_2 - e_1) \] a.e. in \( \mathbb{R}^2 \) satisfies the weaker assumptions. For a.e. \( x \in \mathbb{R}^2 \), define

\[ v(x) = -x_2 - \int_0^{x_1} \chi(t) \, dt, \quad \nabla v = -\chi(e_1 + e_2) - (1 - \chi)e_2 \]

so that a.e. in \( \mathbb{R}^2 \), \( -\nabla u \cdot \nabla v = R\nabla u \cdot \nabla v = 1 \).

Then formula (5) yields the rank one laminate conductivity a.e. in \( \mathbb{R}^2 \),

\[ \sigma = \chi \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} + \frac{1 - \chi}{4} \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix} \]

Hence a.e. in \( \mathbb{R}^2 \), \( \sigma \nabla u = \chi(-e_1 + e_2) + (1 - \chi)e_2 \) which is divergence free in \( \mathcal{D}'(\mathbb{R}^2) \).
30. Matrix Field Realizability

Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ be open. If $U \in \mathcal{H}^1(\Omega, \mathbb{R}^d)$ then the matrix electric field $DU$ is said to be realizable if there is a symmetric positive definite matrix-valued function $\sigma \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^{d \times d})$ such that

$$\text{Div}(\sigma DU) = 0$$

**Theorem (M1)**

Let $d \geq 2$ and $Y \subset \mathbb{R}^d$ be a closed parallelopiped. Let $U \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ such that $DU$ is $Y$-periodic.

1. Assume also $\det(\langle DU \rangle DU) > 0$ in $\mathbb{R}^d$ and $\det(\langle DU \rangle) \neq 0$. Then $DU$ is a realizable matrix electric field with continuous conductivity.
2. If $d = 2$, $\det(\langle DU \rangle) \neq 0$ and the matrix electric field realized by a $C^1$ conductivity, then $\det(\langle DU \rangle DU) > 0$.
3. If $d = 3$ there exists a smooth $Y$-periodic matrix field $DU$ such that $\det(\langle DU \rangle) \neq 0$ and an associated smooth periodic conductivity $\sigma$ such that $\det(DU)$ takes both positive and negative values in $\mathbb{R}^3$. 
(i.) For $Y$-periodic $U \in C^1$ such that $\det(\langle DU \rangle DU) \neq 0$ we define

$$\sigma = \det(\langle DU \rangle DU) (DU^{-1})^T DU^{-1} = \det(\langle DU \rangle) \text{Cof}(DU) DU^{-1}$$

where Cof is the cofactor matrix. $\sigma$ is $Y$-periodic, continuous, symmetric and positive definite. Also by Piola’s identity, as a distribution,

$$\text{Div}(\text{Cof } DU) = 1 \quad \text{in } \mathcal{D}'(\mathbb{R}^d)$$

Hence $\sigma DU$ is divergence free and $DU$ is realizable with associated conductivity $\sigma$.

(ii.) Follows from a theorem of Alessandrini and Nesi.

(iii.) Example is constructed from periodic chain mail constructed by Briane, Milton and Nesi.
The result (i.) may be generalized:

**Corollary (M 2)**

Let $d \geq 2$ and $Y \subset \mathbb{R}^2$ be a closed parallelepiped. Let $U \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ with $Y$-periodic $DU$, $\det(\langle DU \rangle DU) > 0$ in $\mathbb{R}^d$ and $\det (\langle DU \rangle) \neq 0$. Then the matrix electric field $DU$ is realized by a family of continuous conductivities $\sigma_\varphi$ parameterized by convex functions $\varphi \in C^2(\mathbb{R}^d)$, those whose Hessian matrices $D^2 \varphi$ are positive definite everywhere in $\mathbb{R}^d$.

Define

$$\sigma_\varphi = \det(\langle DU \rangle) \ \text{Cof}(D(\nabla \varphi \circ U) \ DU^{-1})$$

$\sigma_\varphi DU$ is divergence free by Piola’s identity. We also have

$$\text{Cof}(D(\nabla \varphi \circ U)) = \text{Cof}(DU \ D^2 \varphi \circ U) = \text{Cof}(DU) \ \text{Cof}(D^2 \varphi \circ U)$$

so that $\sigma_\varphi$ satisfies

$$\sigma_\varphi = \det(\langle DU \rangle DU) (DU^{-1})^T \ \text{Cof}(D^2 \varphi \circ U) \ DU^{-1}$$

Since $D^2 \varphi$ is symmetric positive definite, so is its cofactor matrix. Thus $\sigma_\varphi$ is an admissible, continuous with $\sigma_\varphi DU$ divergence free in $\mathbb{R}^d$. \qed
Let $d, n \in \mathbb{N}$. A rank-$n$ laminate in $\mathbb{R}^d$ is a multiscale microstructure defined at $n$ ordered scales $\varepsilon_n \ll \cdots \ll \varepsilon_1$ depending on a small positive parameter $\varepsilon \to 0$ and in multiple directions in $\mathbb{R}^d\setminus\{0\}$, by the following process.

**Figure:** A rank-two laminate with directions $\xi_1 = e_1$ and $\xi_{1,2} = e_2$. 
At the smallest scale $\varepsilon_n$ there is a set of $m_n$ rank-one laminates, the $i$th one of which, for $i = 1, \ldots, m_n$, is composed of an $\varepsilon_n$ periodic repetition in the $\xi_{i,n}$ direction of homogeneous layers with constant positive definite conductivity matrices $\sigma_{i,n}^h$, $h \in l_{i,n}$.

At the scale $\varepsilon_k$ there is a set of $m_k$ laminates, the $i$th one of which, for $i = 1, \ldots, m_k$, is composed of an $\varepsilon_k$-periodic repetition in the $\xi_{i,k} \in \mathbb{R}^d \setminus \{0\}$ direction of homogeneous layers and/or a selection of the $m_{k+1}$ laminates which are obtained at stage $k + 1$ with constant positive definite conductivity matrices $\sigma_{i,k}^h$ and/or $\sigma_{i,j}^h$, resp., for $j = k + 1, \ldots, n$, $h \in l_{i,j}$.

At the scale $\varepsilon_1$ there is a single laminate ($m_1 = 1$) which is composed of an $\varepsilon_1$-periodic repetition in the $\xi_1 \in \mathbb{R}^d \setminus \{0\}$ direction of homogeneous layers and/or a selection of the $m_2$ laminates which are obtained at scale $\varepsilon_2$ with constant positive definite conductivity matrices $\sigma_{i,1}^h$ and/or $\sigma_{i,j}^h$, resp., for $j = 2, \ldots, n$, $h \in l_{i,j}$.

The laminate conductivity at stage $k = 1, \ldots, n$ is denoted by $L_k^\varepsilon(\hat{\sigma})$ where $\hat{\sigma}$ is the whole set of constant laminate conductivities.
Briane and Milton showed that there is a set \( \hat{P} \) of constant \( d \times d \) matrices such that \( P_\varepsilon = L^\varepsilon_n(\hat{P}) \) is a corrector (or a matrix electric field) associated to the conductivity \( \sigma_\varepsilon = L^\varepsilon_n(\hat{\sigma}) \) in the sense of Murat-Tartar:

\[
\begin{align*}
P_\varepsilon & \rightharpoonup I \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^{d \times d}), \\
\text{Curl}(P_\varepsilon) & \rightarrow 0 \quad \text{strongly in } H^{-1}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^{d \times d}), \\
\text{Div}(\sigma_\varepsilon P_\varepsilon) & \text{ is compact in } H^{-1}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d).
\end{align*}
\]

The weak limit of \( \sigma_\varepsilon P_\varepsilon \) in \( L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^{d \times d}) \) is then the homogenized limiting conductivity of the laminate. The three conditions (6) satisfied by \( P_\varepsilon \) extend in the laminate case to the three respective conditions

\[
\begin{align*}
\langle DU \rangle & = I, \\
\text{Curl}(DU) & = 0, \\
\text{Div}(\sigma DU) & = 0.
\end{align*}
\]

satisfied by any electric field \( DU \) in the periodic case.
Theorem (I 1)

Let $d, n \in \mathbb{N}$. Consider the rank-$n$ laminate multiscale field $L^n_\varepsilon(\hat{P})$ built from a finite set $\hat{P}$ of $d \times d$ matrices satisfying

\[ P_\varepsilon \rightharpoonup I \quad \text{weakly in } \mathcal{L}^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^{d \times d}), \]
\[ \text{Curl}(P_\varepsilon) \to 0 \quad \text{strongly in } \mathcal{H}^{-1}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^{d \times d}), \] (8)

Then necessary and sufficient that the field be realized, i.e., $\text{Div}(\sigma_\varepsilon P_\varepsilon)$ is compact in $\mathcal{H}^{-1}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$ for some rank-$n$ laminate conductivity $L^n_\varepsilon(\hat{\sigma})$ is that $\det(L^n_\varepsilon(\hat{P})) > 0$ a.e. in $\mathbb{R}^d$, or equivalently, that the determinant of each matrix in $\hat{P}$ is positive.

Determinant positivity follows from a theorem of Briane, Milton and Nesi. Conversely, suppose there is a laminate field $P_\varepsilon = L^n_\varepsilon(\hat{p})$ satisfying (8) and $\det(P_\varepsilon) > 0$ a.e.
As in the matrix field case consider the rank-$n$ conductivity defined by

\[ \sigma_\varepsilon = \det(P_\varepsilon) (P_\varepsilon^{-1})^T P_\varepsilon^{-1} = L_n^\varepsilon(\hat{\sigma}), \]

where \( \hat{\sigma} = \{ \det(P) (P^{-1})^T P^{-1} : P \in \hat{P} \} \). Then compactness is equivalent to the compactness of

\[ \text{Div}(\text{Cof}(P_\varepsilon)). \]

Contrary to the periodic case, \( \text{Cof}(P_\varepsilon) \) is not divergence free as a distribution. But using the homogenization procedure for laminates of Briane, by the quasi-affinity of cofactors for gradients compactness holds if the matrices \( P \) and \( Q \) of two neighboring layers in a direction \( \xi \) of the laminate satisfy the jump condition for the divergence

\[ (\text{Cof}(P) - \text{Cof}(Q))^T \xi = 0. \] (9)
More precisely, at the given scale $\varepsilon_k$ of the laminate the matrix $P$ or $Q$ is either a matrix in $\widehat{P}$ or the average of rank-one laminates obtained at the smallest scales $\varepsilon_{k+1}, \ldots, \varepsilon_n$

In the first case the matrix $P$ is the constant value of the field in a homogeneous layer of the rank-$n$ laminate.

In the second case, the average of the cofactors of the matrices involved in those rank one laminations is equal to the cofactors matrix of the average, $\text{Cof}(P)$, by virtue of the quasi-affinity of the cofactors applied iteratively to the rank-one connected matrices in the rank-one laminate.

Therefore, it remains to prove (9) for any matrices $P$ and $Q$ with positive determinant satisfying the condition that controls the jumps in the convergence of $\text{Curl}(P_{\varepsilon}) \to 0$ in (8).
For any matrices $P$ and $Q$ with positive determinant satisfying the condition we must show

$$P - Q = \xi \otimes \eta, \quad \text{for some } \eta \in \mathbb{R}^d.$$

So by multiplicativity of cofactor matrices we have

$$(\text{Cof}(P) - \text{Cof}(Q))^T \xi = \text{Cof}(Q)^T \left[ \text{Cof} \left( I + (\xi \otimes \eta)Q^{-1} \right)^T - I \right]$$

$$= \text{Cof}(Q)^T \left[ \text{Cof} \left( I + \xi \otimes \lambda \right)^T - I \right]$$

where $\lambda = (Q^{-1})^T \eta$. Moreover if $\xi \cdot \eta \neq -1$ we have

$$\text{Cof}(I + \xi \otimes \lambda)^T = \det(I + \xi \otimes \lambda)(I + \xi \otimes \lambda)^{-1} = (I + \xi \cdot \lambda)I - \xi \otimes \lambda,$$

which extends to the case $\xi \cdot \eta = -1$ by continuity. Hence

$$(\text{Cof}(P) - \text{Cof}(Q))^T = \text{Cof}(Q)(((\xi \cdot \lambda)I - \xi \otimes \lambda),$$

which implies (9) since $(\xi \otimes \lambda)\xi = (\xi \cdot \lambda)\xi$. 

Thanks!