Elastic Rings and Nanotubes

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Nanotubes–

Find the compression modulus of elastic rings under hydrostatic pressure. Application of a geometric variational problem for bending energy to materials science.

"Elastic Rings and Nanotubes" Beamer Slides may be found at the URL

http://www.math.utah.edu/∼treiberg/ElasticNano2010.pdf

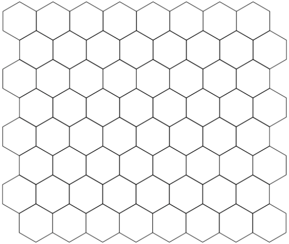
This is joint work with Feng Liu, Materials Science & Engineering, University of Utah. Summary in

J. Zang, A. Treibergs, Y. Han & F. Liu, Geometric Constant Defining Shape Transitions of Carbon Nanotubes under Pressure, Phys. Rev. Lett. **92**, 105501 (2004)

Mathematical details are available in the manuscript

F. Liu & A. Treibergs, On the compression of elastic tubes, University of Utah preprint, (2006)

Single walled carbon nanotubes (SWNT's) were first noticed in the 1990's in the soot of electrical discharge from graphite electrodes. They have a diameter around 1 nm and a length 10^6 nm.



They are "rolled up" from sheets of graphene. Graphite consits of layers of graphene. Graphene is composed of carbon atoms with hybridized $Sp3$ carbon bonds, which means that the bonds to each atom are planar and 120° apart. The atoms of graphene are located at vertices of a uniform hexagonal grid whose edges have length about .14 nm.

Graphene may be rolled up in several ways, depending on which atoms are identified. If $\mathbf{a}_1$ ande $\mathbf{a}_2$ are generating vectors of the hex lattice, then the chiral vector indicates which atom is identified in the rollup.

$$C_b(n, m) = n\mathbf{a}_1 + m\mathbf{a}_2.$$

C_b rolls up into a circle and $C_b^{\perp}$ is the axis direction.

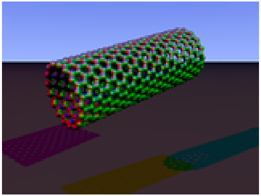
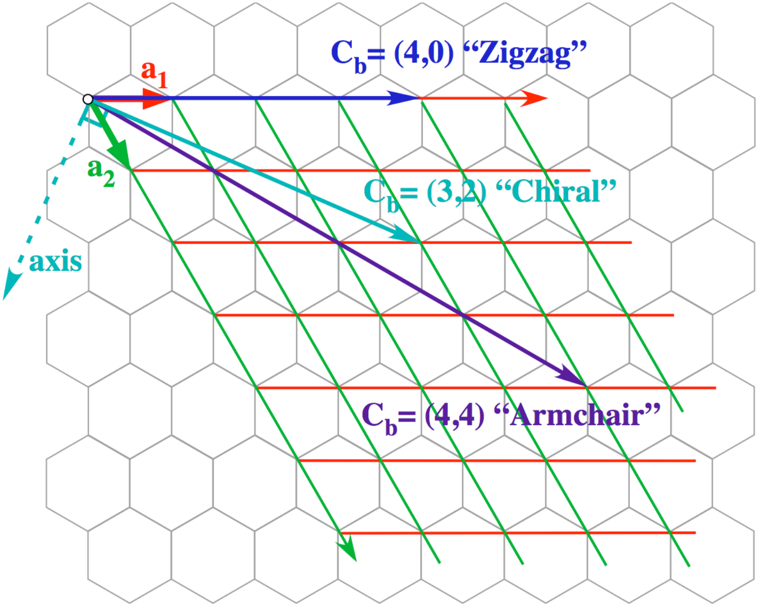


Figure: SWNT with chiral vector $C_b(n, 0)$ is called "Zigzag."

Chiral vector $C_b(n,m)$ determines geometry of SWNT



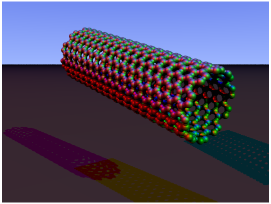


Figure: SWNT with chiral vector $C_b(n, n)$ is called "Armchair."

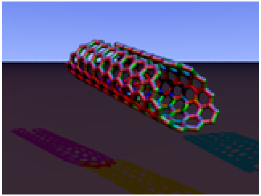


Figure: SWNT with other chiral vectors called "Chiral."

Researchers led by Feng Liu have done quantum transport calculations using molecular dynamics simulations to determine equilibrium shapes, mechanical and electrical properties of SWNTs under constant pressure. The tube undergoes a metal to semiconductor transition with corresponding decrease in conductance.

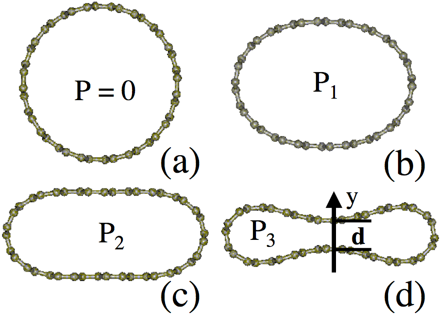


Figure: Cross sections of armchair nanotube as hydrostatic pressure is increased.

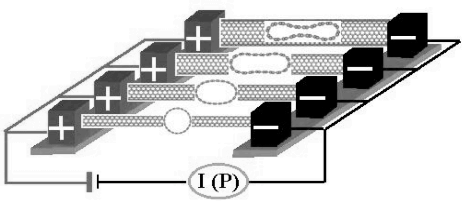


Figure: Pressure sensor design

How hard is the tube? Find the *modulus of compression*

$$\mu = \frac{d\,\mathrm{pressure}}{d \ln \mathrm{Area}}$$

Approximate using a continuum model, the *section of an elastic tube.* Answer can be found using calculus. (And MAPLE!)

The cross section of the tube is to be regarded as an inextensible elastic rod in the plane which is subject to a constant normal hydrostatic pressure $\mathcal{P}$ along its outer boundary. The section is assumed to have a uniform wall thickness h_0 and elastic properties. The centerline of the wall is given by a smooth embedded closed curve in the plane $\Gamma \subset \mathbf{R}^2$ which bounds a compact region Ω whose boundary has given length L_0 and which encloses a given area A_0. Among such curves we seek one, Γ_0, that minimizes the energy

$$\mathcal{E}(\Gamma) = \frac{\mathcal{B}}{2} \int_\Gamma (K - K_0)^2 \, ds + \mathcal{P} \left(\mathrm{Area}(\Omega) - A_0 \right),$$

where $\mathcal{B} = \dfrac{E h_0^3}{12(1-\nu^2)}$ is the flexural rigidity modulus of the section, E is Young's modulus, ν is Poisson's ratio, K denotes the curvature of the curve and K_0 is the undeformed curvature $\left(= \dfrac{2\pi}{L_0}\right.$ for the circle.)

Equivalent geometric model

Because the total curvature of a curve $\int_\Gamma K\, ds = 2\pi$, this is equivalent to the problem of minimizing the *bending energy* $\mathrm{E}(\Gamma) = \int_\Gamma K^2\, ds$, among curves of fixed length L_0 that enclose a fixed area $A_0 = \mathrm{Area}(\Omega)$.

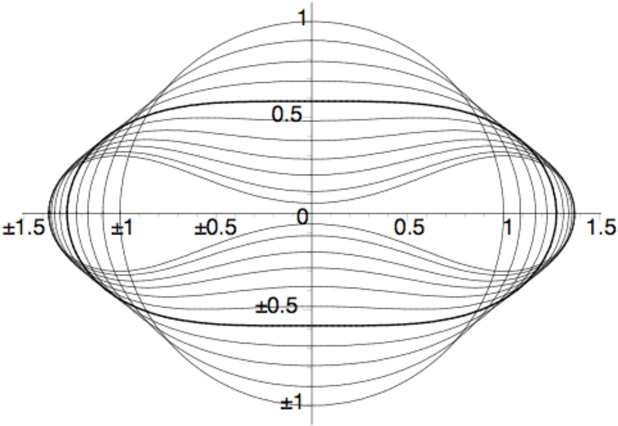


Figure: Elastica with length 2π

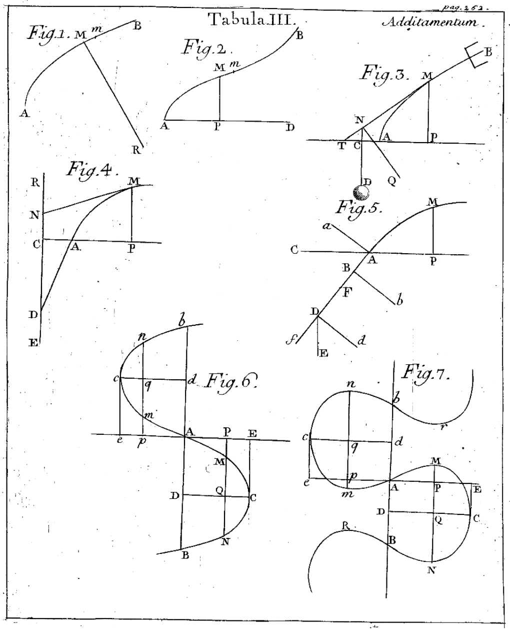


Figure 9: Euler's elastica figures, Tabula III.

Euler Elastica -

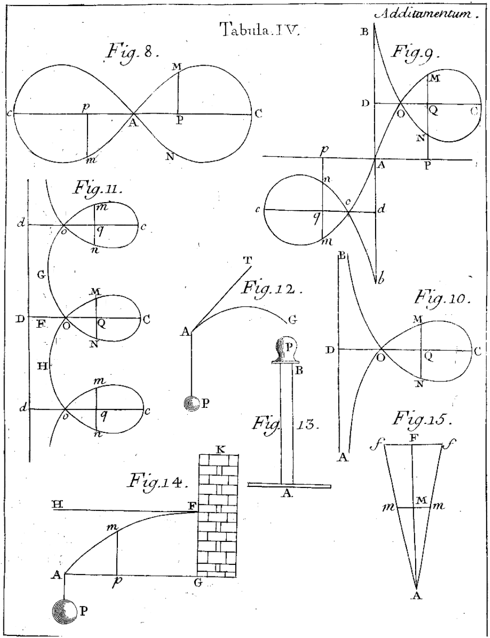
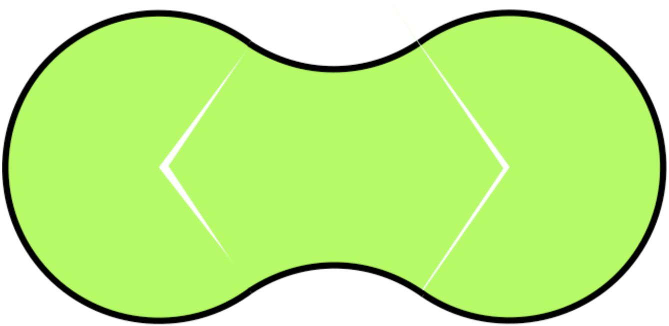


Figure 10: Euler's elastica figures, Tabula IV.

In work with Ralph Howard, we showed that the problem of minimizing the *area* among curves of fixed length $2\pi \leq L_0 \leq \frac{14}{3}\pi$ that satisfy

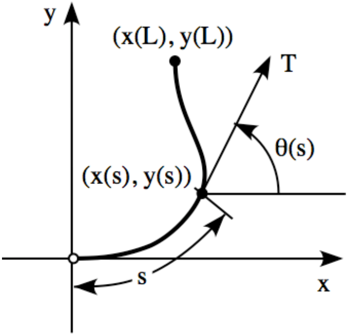
$$\sup_{\Gamma} |K| \leq 1$$

is solved by the *bang-bang* (piecewise unit circle arc) curve



Formulate Variational Problem for Quarter of Curve

$s =$ arclength
$\theta(s) =$ angle from x-axis to tangent vector (indicatrix) $T(s) = (\cos\theta(s), \sin\theta(s))$



Assuming minimizer has x,y-reflection symmetry (can be proved). Seek $\theta(s) \in C^1([0, L])$ where $4L = L_0$.

$$X(s) = (x(s), y(s)) = \int_0^s T(\sigma)\, d\sigma$$

$\theta(0) = 0$ and $\theta(L) = \frac{\pi}{2}$.
If $\theta(s)$ is minimizer, $\gamma = X([0, L])$.
Reflect to get closed curve Γ.
Assume Γ embedded.
Let $\hat{\gamma}$ denote the closed curve γ followed by the line segment from $X(L)$ to $(0, y(L))$ followed by the line segment back to $(0, 0)$.

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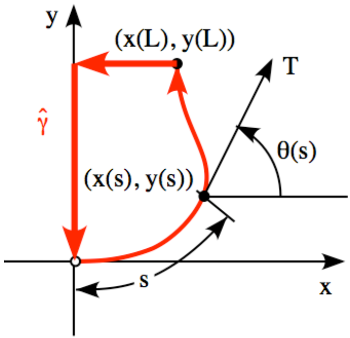

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Reflect to get closed curve Γ.
Assume Γ embedded.
Let $\hat{\gamma}$ denote the closed curve γ followed by the line segment from $X(L)$ to $(0, y(L))$ followed by the line segment back to $(0, 0)$.

Functional in terms of $\theta(s)$

$$X(s) = (x(s), y(s)) = \int_0^s (\cos\theta(\sigma), \sin\theta(\sigma))\ d\sigma$$

By Green's theorem, because $dy = 0$ on the horizontal segment and $x = 0$ on the vertical segment,

$$\tfrac{1}{4}\mathrm{Area}(\Gamma) = \tfrac{1}{4}\int_\Gamma x\,dy = \oint_{\hat{\gamma}} x\,dy = \int_\gamma x\,dy$$

Since we are looking to minimize E subject to $\mathrm{Area}(\theta) \le A_0/4 = A$, the Lagrange Multiplier $\lambda = 8\mathcal{P}/\mathcal{B}$ is nothing more than scaled pressure. $\lambda \ge 0$ since it takes energy to squeeze the curve. At minimum, $4\,\delta E = -\lambda\,\delta\mathrm{Area}$. Lagrange Functional is thus

$$\begin{aligned}
\mathcal{L}[\gamma] &= 4\int_\gamma K(s)^2\,ds - \lambda\left\{A - \int_\gamma x\,dy\right\} \\
&= 4\int_0^L \dot{\theta}(s)^2\,ds - \lambda\left\{A - \int_0^L \int_0^s \cos\theta(\sigma)\,d\sigma\ \sin\theta(s)\,ds\right\}.
\end{aligned}$$

A minimizer $\theta(s)$ satisfies $\theta(0) = 0$ and $\theta(L) = 2/\pi$. Variation $\theta + \epsilon v$ where $v \in C^1([0, L])$ with $v(0) = v(L) = 0$. Then

$$0 = \delta\mathcal{L} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{L} = 8 \int_0^L \dot{\theta}\dot{v}\, ds$$

$$- \lambda \int_0^L \left\{ \int_0^s v(\sigma) \sin\theta(\sigma)\, d\sigma \; \sin\theta(s) - \int_0^s \cos\theta(\sigma)\, d\sigma \; \cos\theta(s) v(s) \right\} ds$$

Integrating by parts, and reversing the order of integration in the second integral,

$$\delta\mathcal{L} = -8 \int_0^L \ddot{\theta} v\, ds$$

$$-\lambda \left\{ \int_0^L \int_\sigma^L \sin\theta(s)\, ds\, v(\sigma) \sin\theta(\sigma)\, d\sigma - \int_0^L \int_0^s \cos\theta(\sigma)\, d\sigma \; \cos\theta(s) v(s)\, ds \right\}.$$

Switching names of the integration variables in the second term yields

$$\delta\mathcal{L} =$$

$$-\int_0^L \left[8\ddot{\theta}(s) + \lambda \left\{ \int_s^L \sin\theta(\sigma)\,d\sigma\, \sin\theta(s) - \int_0^s \cos\theta(\sigma)\,d\sigma\, \cos\theta(s) \right\} \right] v(s)\,ds.$$

Since $v \in C_0^1([0, L])$ was arbitrary, the minimizer satisfies the integro-differential equation

$$8\ddot{\theta}(s) + \lambda \left\{ \int_s^L \sin\theta(\sigma)\,d\sigma\, \sin\theta(s) - \int_0^s \cos\theta(\sigma)\,d\sigma\, \cos\theta(s) \right\} = 0$$

which is for appropriate λ, the L^2 gradient of E on the manifold of closed curves with given L_0 and A_0

$$-\nabla\mathrm{E}[\theta] = 0.$$

Theorem (Yingzhong Wen 1996)

Let $\theta_0(s) \in C^\infty(\mathbf{S}^1)$ *be an immersed closed curve in the plane of length* L_0 *with total turning index* $\eta \in \mathbb{Z}$. *(*$2\pi\eta = \theta(L_0) - \theta(0)$*.)*

- *Then there is a unique long time solution* $\theta(s,t)$ *on* $\mathbf{S}^1 \times [0,\infty)$ *of turning index* η *of the curve strtaightening flow*

$$\begin{cases} \dfrac{\partial\theta}{\partial t} = -\nabla \mathrm{E}[\theta], \\ \theta(s,0) = \theta_0(s). \end{cases}$$

- *The flow preserves convex curves.*
- *If* $\eta \neq 0$ *then the flow converges exponentially to a* η*-fold covered circle of total length* L_0 *as* $t \to \infty$*.*

Thus if $\lambda = 0$ we must have $\theta(s) = \frac{\pi s}{2L}$ and γ is a circle of radius $\frac{L}{\pi}$. If not a circle then $\lambda > 0$. To see the DE, assume $\dot{\theta} \neq 0$, differentiate

$$\begin{aligned}
\theta''' &= \frac{\lambda}{8}\left\{\sin\theta(s)\,\sin\theta(s) + \cos\theta(s)\,\cos\theta(s)\right\} \\
&\quad -\frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\,\cos\theta(s) + \int_0^s \cos\theta(\sigma)\,d\sigma\,\sin\theta(s)\right\}\theta'(s) \\
&= \frac{\lambda}{8} - \frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\,\cos\theta(s) + \int_0^s \cos\theta(\sigma)\,d\sigma\,\sin\theta(s)\right\}\theta'(s)
\end{aligned}$$

$$\begin{aligned}
\theta'''' &= \frac{\lambda}{8}\left\{\sin\theta(s)\,\cos\theta(s) - \cos\theta(s)\,\sin\theta(s)\right\}\theta'(s) \\
&\quad +\frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\,\sin\theta(s) - \int_0^s \cos\theta(\sigma)\,d\sigma\,\cos\theta(s)\right\}(\theta'(s))^2 \\
&\quad -\frac{\lambda}{8}\left\{\int_s^L \sin\theta(\sigma)\,d\sigma\,\cos\theta(s) + \int_0^s \cos\theta(\sigma)\,d\sigma\,\sin\theta(s)\right\}\theta''(s)
\end{aligned}$$

Euler Lagrange Equation

From which we get

$$\theta''''\theta' = -\theta''\left(\theta'\right)^3 + \left[\theta''' - \frac{\lambda}{8}\right]\theta''(s).$$

This differential equation may be integrated as follows:

$$\frac{\theta''''\theta' - \theta'''\theta''}{(\theta')^2} = \left[\frac{\theta'''}{\theta'}\right]' = -\theta'\theta'' - \frac{\lambda\theta''}{8(\theta')^2} = \left[-\frac{1}{2}(\theta')^2 + \frac{\lambda}{8\theta'}\right]'$$

so there is a constant c_1 so that

$$\theta''' = c_1\theta' - \frac{1}{2}(\theta')^3 + \frac{\lambda}{8}. \tag{1}$$

In other words, the curvature $K = \theta'$ satisfies

$$K'' = c_1 K + \frac{\lambda}{8} - \frac{1}{2}K^3.$$

Buckling occurs when there is sufficient pressure

Consider L_0-periodic variations of the circle with $K = k_0$ constant.

$$c_1 = \frac{k_0^2}{2} - \frac{\lambda}{8k_0}$$

Linearizing around the $K = k_0$, the variation of curvature satisfies

$$\ddot{w} + \left(k_0^2 + \frac{\lambda}{8k_0}\right) w = 0$$

The m-th eigenvalue of the circle is $m^2 k_0^2$. The first corresponds to translation of the circle and $\lambda = 0$. The shape does not deform until the second eigenvalue, the $m = 2$ mode. Hence the tube buckles when

$$k_0^2 + \frac{\lambda}{8k_0} = m^2 k_0^2$$

or (since $m \geq 2$) the pressure exceeds $\lambda = 24k_0^3$.
The noncircular elastica occur post-buckling.

Multiply by K' and integrate. For some constant H,

$$(K')^2 = c_1 K^2 + H + \frac{\lambda K - K^4}{4} = F(K).$$

The Euler-Lagrange equations have the following immediate consequence. The area is given using (1)

$$\begin{aligned}
A &= \frac{1}{2}\int_\gamma x\,dy - (y - y(L))\,dx \\
&= \frac{1}{2}\int_0^L \left\{ \sin\theta(s)\int_0^s \cos\theta(\sigma)\,d\sigma - \cos\theta(s)\int_s^L \sin\theta(\sigma)\,d\sigma \right\} ds \\
&= \frac{4}{\lambda}\int_0^L \left\{ \frac{\frac{\lambda}{8} - \theta'''}{\theta'} \right\} ds \\
&= \frac{2}{\lambda}\int_0^L K^2\,ds - \frac{4c_1 L}{\lambda}.
\end{aligned}$$

Since the curve closes, the curvature is a L_0-periodic function which satisfies the nonlinear spring DE. The curvature continues as an even function at $\{0, L\}$. BC on θ imply $K'(0) = K'(L) = 0$. As we have differentiated twice, the solutions of the ODE have two extra constants of integration which have to satisfy IE.

Buckling occurs in the $n = 2$ mode so the optimal curves will have four vertices: the curves will be elliptical or peanut shaped, the endpoints of the quarter curves will be the minima and maxima of the curvature around the curve, and these will be the only critical points of curvature. Since the minimum K may be negative, as in peanut shaped regions, the embeddedness implies on $[0, L]$,

$$\begin{array}{rcll} K(0) & = & K_1 & \text{is the maximum of the curvature, and} \\ K(L) & = & K_2 & \text{is the minimum of curvature around the curve.} \\ & K & & \text{is strictly decreasing on } [0, L] \end{array}$$

Thus K may be used to replace s as an integration variable.

One degree of freedom is homothety. Scale $\tilde{X} = cX$ then

$$\tilde{K} = c^{-1}K, \quad d\tilde{K}/d\tilde{s} = c^{-2}K', \quad \tilde{c}_1 = c^{-2}c_1, \quad \tilde{H} = c^{-4}H, \quad \tilde{\lambda} = c^{-3}\lambda.$$

$\lambda > 0$ for noncircular regions.

As K and K' vary, they satisfy the first integral

$$(K')^2 = c_1 K^2 + H + \tfrac{\lambda K - K^4}{4} = F(K)$$

$0 = F(K_1) = F(K_2)$. Thus, given K_1, K_2 we can solve for c_1 and H,

$$\begin{aligned} c_1 &= \tfrac{1}{4}\left(K_1^2 + K_2^2 - \tfrac{\lambda}{K_1+K_2}\right), \\ H &= -\tfrac{K_1 K_2}{4}\left(K_1 K_2 + \tfrac{\lambda}{K_1+K_2}\right), \end{aligned}$$

provided $K_2 \neq -K_1$.

$$(K')^2 = c_1 K^2 + H + \frac{\lambda K - K^4}{4} = F(K)$$

$4F(K) = Q_1(K)Q_2(K)$ factors into quadratic polynomials, where

$$\begin{aligned} Q_1 &= (K_1 - K)(K - K_2); \\ Q_2 &= K^2 + (K_1 + K_2)K + K_1 K_2 + \frac{\lambda}{K_1 + K_2} \end{aligned}$$

Assume that $F(K)$ is positive in the interval $K_2 < K < K_1$.
Since the possible homotheties and translations (shifts like $K(s + c)$) have been eliminated, the remaining indeterminacy comes from the angle Θ changes by exactly $\pi/2$ over γ. Thus given K_2, we solve for K_1 so that

$$\frac{\pi}{2} = \Theta(L) = \int_0^{L_0} K(s)\, ds = \int_{K_2}^{K_1} \frac{K\, dK}{\sqrt{F(K)}}$$

If there are 2ℓ vertices in the solution curve, then

$$\frac{\pi}{\ell} = \int_0^{L_0} K(s)\, ds = \int_{K_2}^{K_1} \frac{K\, dK}{\sqrt{F(K)}} = \Theta(K_1, K_2, \lambda).$$

In fact, this integral can be reduced to a complete elliptic integral. Similarly

$$L = \int_0^{L_0} ds = \int_{K_2}^{K_1} \frac{dK}{\sqrt{F(K)}} = \Lambda(K_1, K_2, \lambda)$$

is also a complete elliptic integral.

$$L = \int_{K_2}^{K_1} \frac{dK}{\sqrt{F(K)}} = \Lambda(K_1, K_2, \lambda)$$

Change variables of form $T = \frac{\alpha K - \beta}{\gamma K + \delta}$ to get

$$L = c_2 \int_{-1}^{1} \frac{dT}{\sqrt{(1-T^2)(1-m^2T^2)}} = 2c_2\mathcal{K}(m)$$

where $\mathcal{K}(m)$ is the complete elliptic integral of the first kind.

$$c_2 = \frac{4}{(K_1 - K_2)\sqrt{\mu_2}}, \quad m = \sqrt{\frac{\mu_2}{\mu_1}}, \qquad S = K_1 + K_2, \quad P = K_1 K_2,$$

and α, β, γ, δ are functions of μ_1, μ_2 where

$$\mu_1, \mu_2 = \frac{S^3 + 4PS + 2\lambda \pm 2\sqrt{(\lambda + 2K_1S^2)(\lambda + 2K_2S^2)}}{S(K_1 - K_2)^2}. \qquad (2)$$

$4F(K) = Q_1(K)Q_2(K)$. Choose μ_i so that $Q_2 - \mu Q_1$ is a perfect square. The discriminant vanishes since roots are equal, so can solve for μ (2).

$$\Delta = D^2(\mu+1)^2 - 4S^2\mu - 4(\mu+1)\frac{\lambda}{S} = 0$$

Say, $\mu_1 > \mu_2$. The factors are

$$\begin{aligned} Q_2 - \mu_1 Q_1 &= F_1^2 = (\alpha K - \beta)^2 \\ Q_2 - \mu_2 Q_1 &= F_2^2 = (\eta K + \delta)^2. \end{aligned}$$

We can now solve for the factors as sums of squares.

$$Q_1 = \frac{F_1^2 - F_2^2}{\mu_2 - \mu_1}, \qquad Q_2 = \frac{\mu_2 F_1^2 - \mu_1 F_2^2}{\mu_2 - \mu_1}$$

The idea is to change variables in the integral according to

$$T = \frac{F_1}{F_2} = \frac{\alpha K - \beta}{\eta K + \delta}, \qquad K = \frac{\beta + \delta T}{\alpha - \eta T}, \qquad \frac{dT}{dK} = \frac{\alpha\delta + \beta\eta}{(\eta K + \delta)^2}.$$

The function T is increasing. Since $Q_1(K_1) = Q_1(K_2) = 0$ it follows that $T = 1$ when $K = K_1$ and $T = -1$ when $K = K_2$. Moreover,

$$Q_1 Q_2 = \frac{(F_1^2 - F_2^2)(\mu_2 F_1^2 - \mu_1 F_2^2)}{(\mu_2 - \mu_1)^2} = \frac{(T^2 - 1)(\mu_2 T^2 - \mu_1)F_2^4}{(\mu_2 - \mu_1)^2}$$

Therefore, the integral becomes

$$L = \frac{2(\mu_1 - \mu_2)}{(\alpha\delta + \beta\eta)\sqrt{\mu_1}} \int_{-1}^{1} \frac{dT}{\sqrt{(1 - T^2)(1 - \frac{\mu_2}{\mu_1} T^2)}} = \frac{4(\mu_1 - \mu_2)}{(\alpha\delta + \beta\eta)\sqrt{\mu_1}} \mathcal{K}(m)$$

where $m = \sqrt{\mu_2/\mu_1}$ is imaginary.

Solution of M. Levy (1883) and G. Carrier (1945)

For fixed L and ℓ, we may solve

$$\begin{aligned} L &= \Lambda(K_1, K_2, \lambda) \\ \frac{\pi}{\ell} &= \Theta(K_1, K_2, \lambda) \end{aligned}$$

for $K_1(\lambda, \ell)$, $K_2(\lambda, \ell)$ and then the incomplete elliptic integral for the arclength as function of the curvature

$$s = \int_{K_2}^{K} \frac{dK}{\sqrt{F(K)}}$$

may be inverted to give a solution of the curvature equation in the form

$$K = \frac{\beta - \delta\,\mathrm{cn}(\zeta s; m)}{\alpha + \eta\,\mathrm{cn}(\zeta s; m)}$$

where α, β, γ, δ, m and ζ depend on λ and ℓ.

Unphysical solution for the $\ell = 3$ mode

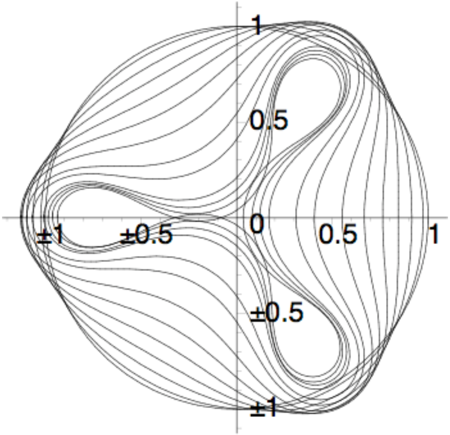

Figure: Mode $n = 3$ elastica for various pressures and length $L = \pi/2$.

Computation of the compression modulus

We saw how to express the quarter area

$$\begin{aligned} A &= \frac{2}{\lambda}\int_0^L K^2\,ds - \frac{4c_1 L}{\lambda} \\ &= \frac{2}{\lambda}\int_{K_2}^{K_1} \frac{K^2\,dK}{\sqrt{F(K)}} - \frac{4c_1 L}{\lambda} \\ &= \mathrm{A}(K_1, K_2, \lambda) \end{aligned}$$

which is a complete elliptic integral. Thus, A depends on λ and its compression modulus is gotten by differentiating (implicitly)

$$\mu = \frac{d\lambda}{d\,\log A}$$

Pressure vs. Area for elastica

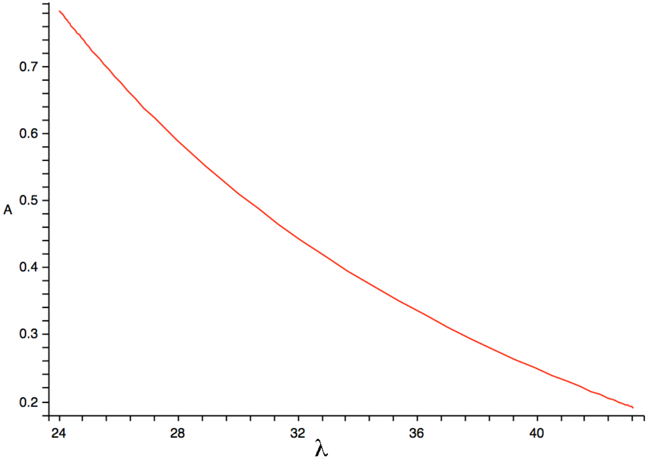

Figure: Elastica pressure λ vs. quarter area A.

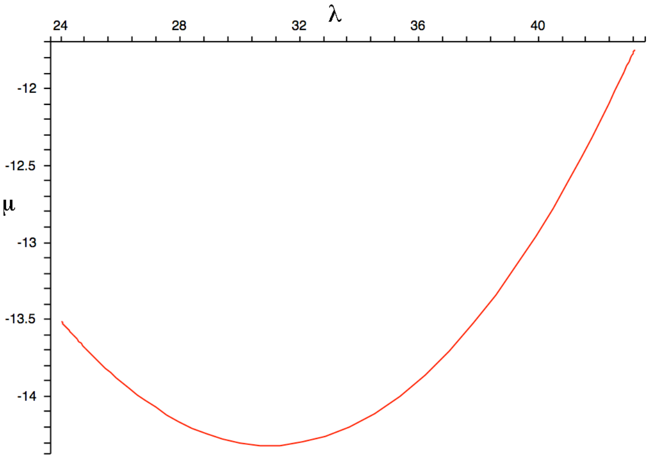

Figure: Modulus $\mathrm{d}\lambda/\mathrm{d}\ln A$ versus λ.

The parameters can all be expressed in terms of a basic set $(\lambda, \mu_1, \mu_2, c_2)$ where c_2 determines the shift $K(s + c_2)$. In these parameters,

$$
\begin{aligned}
L &= \frac{4(1-\mu_1\mu_2)^{\frac{1}{3}}(1+\mu_1)^{\frac{1}{6}}(1+\mu_2)^{\frac{1}{6}}}{\lambda^{\frac{1}{3}}\sqrt{\mu_1}}\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\\
\Theta &= \frac{4(\mu_1-\mu_2)}{(1+\mu_1)^{\frac{1}{2}}(1+\mu_2)^{\frac{1}{2}}\sqrt{\mu_1}}\Pi\left(\frac{1+\mu_2}{1+\mu_1}, \sqrt{\frac{\mu_2}{\mu_1}}\right)\\
&\quad - \frac{2(1+\mu_1)^{\frac{1}{2}}(1-\mu_2)}{(1+\mu_2)^{\frac{1}{2}}\sqrt{\mu_1}}\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\\
A &= \frac{8\mu_1\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right) - 4\left(\mu_1\mu_2 + 2\mu_1 + 1\right)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)}{\lambda^{\frac{2}{3}}(1-\mu_1\mu_2)^{\frac{1}{3}}(1+\mu_1)^{\frac{1}{6}}(1+\mu_2)^{\frac{1}{6}}\sqrt{\mu_1}}.
\end{aligned}
$$

With L and Θ fixed, eliminating μ_1, μ_2 yields relation of λ to A.

Elliptic integrals of the first, second and third kinds

$$\mathcal{K}(m) = \int_0^1 \frac{dT}{\sqrt{(1-T^2)(1-m^2T^2)}}$$

$$\mathcal{E}(m) = \int_0^1 \frac{\sqrt{1-m^2T^2}}{\sqrt{1-T^2}}\,dT$$

$$\Pi(n,m) = \int_0^1 \frac{dT}{(1-nT^2)\sqrt{(1-T^2)(1-m^2T^2)}}$$

The explicit formulæ allow differentiation to obtain explicit rates of change. For example let us compute the pressure modulus of area $\mathrm{d}\lambda/\mathrm{d}\ln \mathrm{A}$. Then there is a mapping $F(\mu_1, \mu_2, \lambda) = (\Theta(\mu_1, \mu_2), L(\mu_1, \mu_2, \lambda))$ implicitly defines (μ_1, μ_2) in terms of λ so the result follows from differentiating

$$\frac{d \ln A}{d\lambda} = \frac{\partial \ln A}{\partial \mu_1}\frac{\partial \mu_1}{\partial \lambda} + \frac{\partial \ln A}{\partial \mu_2}\frac{\partial \mu_2}{\partial \lambda} + \frac{\partial \ln A}{\partial \lambda}$$

Since Θ and L are constant, differentiating F, we find

$$\begin{aligned} 0 &= \frac{\partial \ln \Theta}{\partial \lambda} &&= \frac{\partial \ln \Theta}{\partial \mu_1}\frac{\partial \mu_1}{\partial \lambda} + \frac{\partial \ln \Theta}{\partial \mu_2}\frac{\partial \mu_2}{\partial \lambda} \\ 0 &= \frac{\partial \ln L}{\partial \lambda} &&= \frac{\partial \ln L}{\partial \mu_1}\frac{\partial \mu_1}{\partial \lambda} + \frac{\partial \ln L}{\partial \mu_2}\frac{\partial \mu_2}{\partial \lambda} + \frac{\partial \ln L}{\partial \lambda} \end{aligned}$$

So by Cramer's rule,

$$\begin{pmatrix} \frac{\partial \mu_1}{\partial \lambda} \\ \\ \frac{\partial \mu_2}{\partial \lambda} \end{pmatrix} = - \begin{pmatrix} \frac{\partial \ln \Theta}{\partial \mu_1} & \frac{\partial \ln \Theta}{\partial \mu_2} \\ \\ \frac{\partial \ln L}{\partial \mu_1} & \frac{\partial \ln L}{\partial \mu_2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \\ \frac{\partial \ln L}{\partial \lambda} \end{pmatrix}$$

$$= \frac{\frac{\partial \ln L}{\partial \lambda}}{\frac{\partial \ln \Theta}{\partial \mu_1}\frac{\partial \ln L}{\partial \mu_2} - \frac{\partial \ln \Theta}{\partial \mu_2}\frac{\partial \ln L}{\partial \mu_1}} \begin{pmatrix} \frac{\partial \ln \Theta}{\partial \mu_2} \\ \\ -\frac{\partial \ln \Theta}{\partial \mu_1} \end{pmatrix}$$

which means that

$$\frac{d \ln A}{d\lambda} = \frac{\dfrac{\partial \ln L}{\partial \lambda}\left(\dfrac{\partial \ln A}{\partial \mu_1}\dfrac{\partial \ln \Theta}{\partial \mu_2} - \dfrac{\partial \ln A}{\partial \mu_2}\dfrac{\partial \ln \Theta}{\partial \mu_1}\right)}{\dfrac{\partial \ln \Theta}{\partial \mu_1}\dfrac{\partial \ln L}{\partial \mu_2} - \dfrac{\partial \ln \Theta}{\partial \mu_2}\dfrac{\partial \ln L}{\partial \mu_1}} + \frac{\partial \ln A}{\partial \lambda}.$$

By using derivatives of elliptic functions, *e.g.*,

$$\frac{d\mathcal{K}(m)}{d\, m^2} = \frac{\mathcal{E}(m)}{2m^2(1-m^2)} - \frac{\mathcal{K}(m)}{2m^2}$$

we compute the six derivatives and plug into the modulus formula, *e.g.*,

$$\begin{aligned}\ln L = \ln 4 &+ \frac{\ln(1-\mu_1\mu_2)}{3} + \frac{\ln(1+\mu_1)}{6} \\ &+ \frac{\ln(1+\mu_2)}{6} - \frac{\ln\lambda}{3} - \frac{\ln\mu_1}{2} + \ln\left(\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)\right)\end{aligned}$$

$$\frac{\partial \ln L}{d\,\mu_1} = \frac{1-2\mu_2-3\mu_1\mu_2}{6(1-\mu_1\mu_2)(1+\mu_1)} - \frac{\mathcal{E}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)}{2(\mu_1-\mu_2)\mathcal{K}\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)}$$

etc.

Thanks!