Eigenvalues of Spherical Triangles and a Brownian Pursuit Problem

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Joint work with **Jesse Ratzkin,** University of Cape Town, South Africa.

The URL for Beamer Slides for my October 14 talk,
"Eigenvalues of spherical triangles and a Brownian pursuit problem"

`http://www.math.utah.edu/~treiberg/EigenvalCapture.pdf`

References.

- J. Ratzkin & A Treibergs, A capture problem in Brownian motion and eigenvalues of spherical domains, Transactions AMS 361 (2009) 391–404.
- —, A Payne - Weinberger eigenvalue estimate fro wedge domains in the sphere, Proceedings AMS 137 (2009) 2299–2309.
- J. Ratzkin, Eigenvalues of Euclidean wedge domains in higher dimensions.

3. Outline.

- Eigenvalues
- Capture problem.
- Reduction to geometric eigenvalue problem.
- Eigenvalue basic properties
- Eigenvalue computation for simple domains.
- Recap proof in known cases.
- Domain perturbation and proof in remaining case.

Analytic arguments up to finding a few roots of exact expressions involving special functions via computer algebra system MAPLE.

- Numerical Computation.

Run on department's mainframe.

A number $\lambda \in \mathbb{C}$ is an eigenvalue of a nice domain $\mathcal{D}_n \subset \mathbb{R}^n$ (or in $\mathcal{M}^n$) if there is a nonzero eigenfunction $U \in C\left(\overline{\mathcal{D}_n}\right) \cap C^2\left(\mathcal{D}_n\right)$ satisfying

$$\begin{cases} \Delta U + \lambda U = 0 & \text{for } x \in \mathcal{D}_n \\ U = 0 & \text{if } x \in \partial\mathcal{D}_n. \end{cases} \tag{1}$$

where, *e.g.*, the Laplacian on $\mathbb{R}^n$ is

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

Compact domains have discrete spectrum
$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \lambda_4 \le \cdots \to \infty$, where each eigenvalue has finite multiplicity.

Corresponding to λ_j are eigenfunctions $U_j \in C\left(\overline{\mathcal{D}_n}\right) \cap C^2\left(\mathcal{D}_n\right)$ which may be chosen orthonormal with respect to L^2.

On a manifold, if $\mathcal{G} = [g_{ij}(x)]$ is the Riemannian metric, then gradient, divergence and Laplacian are defined so that the usual Green's formulas continue to hold on the manifold. If $V(x) = (v^1(x), \ldots, v^n(x))$ is a $\mathcal{C}^1$ vector field in local coordinates $x = (x_1, \ldots, x_n)$ on a Riemannian manifold and $u \in \mathcal{C}^2(M)$, then using the inverse matrix $g^{ij} = [g_{ij}]^{-1}$,

$$\operatorname{grad} u = \left(\cdots, \sum_{j=1}^{n} g^{ij} \frac{\partial}{\partial x_j} u, \cdots \right)$$

$$\operatorname{div} V = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{g} v^i \right)$$

$$\Delta u = \operatorname{div} \operatorname{grad} u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_i} u \right)$$

On a manifold, if $\mathcal{G} = [g_{ij}(x)]$ is the Riemannian metric, then gradient, divergence and Laplacian are defined so that the usual Green's formulas continue to hold on the manifold. If $V(x) = (v^1(x), \ldots, v^n(x))$ is a $\mathcal{C}^1$ vector field in local coordinates $x = (x_1, \ldots, x_n)$ on a Riemannian manifold and $u \in \mathcal{C}^2(M)$, then using the inverse matrix $g^{ij} = [g_{ij}]^{-1}$,

When M is Euclidean with $g_{ij} = \delta_{ij}$

$$\operatorname{grad} u = \left(\cdots, \sum_{j=1}^{n} g^{ij} \frac{\partial}{\partial x_j} u, \cdots \right) = \left(\cdots, \frac{\partial\, u}{\partial\, x_i}, \cdots \right);$$

$$\operatorname{div} V = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{g} v^i \right) = \sum_{j=1}^{n} \frac{\partial\, v^j}{\partial\, x_j};$$

$$\Delta\, u = \operatorname{div} \operatorname{grad} u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_i} u \right) = \sum_{j=1}^{n} \frac{\partial^2\, u}{\partial\, {x_j}^2}.$$

Suppose that a domain vibrates according to the wave equation. What frequencies are heard? Let ρ be the density and τ be the tension. Then the amount of a small transverse vibration is given by $v(x,t)$ where $x \in M$ and $t \geq 0$,

$$\frac{\partial^2 v}{\partial t^2} = \frac{\tau}{\rho} \Delta v.$$

We seek solutions of the form $v(x,t) = T(t)u(x)$. Thus

$$T''(t)u(x) = \frac{\tau}{\rho} T(t) \Delta u(x).$$

We can separate variables. The only way a t-expression equals an x-expression is if both equal $\lambda = \text{const}$.

$$\frac{\rho T''(t)}{\tau T(t)} = -\lambda = \frac{\Delta u(x)}{u(x)}$$

which results in two equations

$$\Delta u + \lambda u = 0,$$
$$\rho T'' + \lambda \tau T = 0.$$

When $\lambda > 0$, the time equation

$$\rho T'' + \lambda \tau T = 0$$

has the solution

$$T(t) = A\cos\left(\sqrt{\frac{\tau\lambda}{\rho}}t\right) + B\sin\left(\sqrt{\frac{\tau\lambda}{\rho}}t\right).$$

Thus the time dependence is sinusoidal. Its frequency is

$$\frac{1}{2\pi}\sqrt{\frac{\tau\lambda}{\rho}}$$

cycles per unit time. The frequency increases with the eigenvalue λ and tension τ and decreases with density ρ.
The lowest frequency corresponds to smallest positive eigenvalue $\lambda_1 > 0$.
Thus λ_1 is called the fundametal eigenvalue.

Theorem

Let Ω be a piecewise $\mathcal{C}^1$ domain in a smooth manifold.

1. *Let λ be an eigenvalue and u its corresponding eigenfunction. Then $u \in C_0^\infty(\Omega)$.*
2. *For all $\lambda \in spec(\Omega)$, the eigenspace $\mathcal{E}_\lambda = \{u : \Delta\, u + \lambda u = 0\}$ is finite dimensional. Its dimension is called the multiplicity m_λ.*
3. *The λ_1 eigenspace is one dimensional $m_1 = 1$.*
4. *The set of Dirichlet eigenvalues is discrete and tends to infinity. The eigenvalues can be ordered*

$$spec(\Omega) = \{0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty\}$$

5. *Let u_i denote the λ_i eigenfunction. If $\lambda_i \ne \lambda_j$ then u_i and u_j are orthogonal. By adjusting bases in the eigenspaces $\mathcal{E}_\lambda$ we may assume $\{u_1, u_2, \ldots\}$ is a complete orthonormal basis in $\mathcal{L}^2(\Omega)$.*

Proof Sketch. To see orthogonality (5), suppose $\lambda_i \neq \lambda_j$ and u_i and u_j are corresponding eigenfunctions. Then

$$(\lambda_i - \lambda_j) \int_M u_i u_j = \int_M -(\Delta\, u_i) u_j + u_i\, \Delta\, u_j = 0$$

by Green's formula. □

Since eigenfunction u_j satisfy on (M, g)

$$\Delta\, u_j + \lambda_j u_j \ = 0, \tag{2}$$

eigenvalues scale like $\dfrac{1}{\mathrm{distance}^2}$. So if we scale the lengths of curves by a factor s on the manifold by multiplying the metric, $s^2 g$, then the eigenvalue becomes

$$\lambda_j(M, s^2 g) = \frac{\lambda_j(M, g)}{s^2}.$$

"Bigger tambourines have lower tones."

10. Eigenvalues of a rectangle.

For example in the rectangle $R = [0, a] \times [0, b] \subset \mathbb{R}^2$, the functions

$$u(x, y) = \sin\left(\frac{\pi k x}{a}\right) \sin\left(\frac{\pi \ell y}{b}\right)$$

with $k, \ell \in \mathbb{N}$ satisfy $\Delta u + \lambda u = 0$ with

$$\lambda = \pi^2 \left(\frac{k^2}{a^2} + \frac{\ell^2}{b^2}\right).$$

These turn out to be all the eigenfunctions. So $\lambda_1 = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right)$.

Note that if the area is fixed $ab = A$ then λ_1 is minimized when R is a square and $a = b$.

11. $\mathbb{S}^1$.

A complete set of eigenfunctions of $\mathbb{S}_a{}^1$, the circle of length a are generated by

$$f(\theta) = A\cos\left(\frac{2\pi j\theta}{a}\right) + B\sin\left(\frac{2\pi j\theta}{a}\right)$$

so

$$\mathrm{spec}(\mathbb{S}^1_a) = \left\{\frac{4\pi^2}{a^2}j^2 : j \in \mathbb{Z}\right\}$$

12. Example: Unit sphere $\mathbb{S}^n$.

The sphere is the hypersurface $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ with the induced metric. Using spherical coordinates $\theta \in \mathbb{S}^n$ and $r \geq 0$, the Laplacian $\Delta_{\mathbb{R}^{n+1}}$ in $\mathbb{R}^{n+1}$ may be expressed in terms of the spherical Laplacian Δ_θ

$$\Delta_{\mathbb{R}^{n+1}} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\,\Delta_\theta\,.$$

A homogeneous functions of degree d satisfies $u(r\theta) = r^d u(\theta)$ for all θ and $r \geq 0$. It turns out that harmonic homogeneous polynomials restrict to a complete set of eigenfunctions of the sphere. Indeed if $\Delta_{\mathbb{R}^{n+1}}\, u = 0$ and u is homogeneous of degree d, then

$$0 = \Delta_{\mathbb{R}^{n+1}}\, u = d(d-1)r^{d-2}u + ndr^{d-2}u + r^{d-2}\,\Delta_\theta\, u.$$

Thus on the sphere, $r = 1$ so

$$0 = \Delta_\theta\, u + d(d+n-1)u.$$

Thus on the sphere $\mathbb{S}^n$, for $d = 0, 1, 2, \ldots$,

$$\lambda_d = d(d + n - 1).$$

The dimension of the harmonic polynomials of degree d gives the multiplicity

$$m_d = \binom{n+d}{d} - \binom{n+d-2}{d-2}.$$

For example if $n = 1$ then $m_0 = 1$ and $m_d = 2$ for $d \geq 1$ corresponding to Fourier series. For example $\Re e(z^d)$ is a harmonic polynomial that restricts to $u(\theta) = \cos(d\theta)$ on $\mathbb{S}^1$.

If $n = 2$ then $m_d = 2d + 1$. For example, the coordinate function $u(x_1, x_2, x_3) = x_1$ is harmonic homogeneous of degree one that restricts to an eigenfunction with $\lambda_1 = 2$. Its multiplicity is three, corresponding to the three coordinates.

$$\mathrm{spec}(\mathbb{S}^2) = \{0, 2, 2, 2, 6, 6, 6, 6, 6, 6, 12, \ldots, 12, 20, \ldots\}$$

Since eigenfunction U_j satisfy

$$\begin{cases} \Delta_n U_j + \lambda_j U_j = 0 & \text{for } x \in \mathcal{D}_n \\ U_j = 0 & \text{if } x \in \partial\mathcal{D}_n. \end{cases} \qquad (3)$$

they scale like $\dfrac{1}{\mathrm{distance}^2}$. So for $D_n \subset \mathbb{R}^n$, $\lambda_n(sD_n) = \dfrac{\lambda_n(D_n)}{s^2}$.

The first eigenvalue has a variational characterization. U_1 minimizes the Rayleigh Quotient

$$\lambda_1 = \inf_{\substack{u \in H_0^1(\mathcal{D}_n),\\ u \not\equiv 0}} \frac{\displaystyle\int_{\mathcal{D}_n} |du|^2}{\displaystyle\int_{\mathcal{D}_n} u^2} := \inf_u \mathcal{R}(u)$$

16. Properties of eigenvalues that follow from Rayleigh Quotient

- UPPER BOUND PRINCIPLE:
  If $0 \not\equiv f \in H_0^1(\Omega)$ then $\lambda_1(\Omega) \leq \mathcal{R}(f)$.
- NODAL DOMAIN PRINCIPLE:
  If U satisfies $\Delta_n U + \mu U = 0$ in $\mathcal{D}_n$, $U = 0$ on $\partial\mathcal{D}_n$ and $\Omega \subset \mathcal{D}_n$ is a nodal domain (component of $U^{-1}((0,\infty))$) then $u_1 = U$ is a first eigenfunction of Ω and $\mu = \lambda_1(\Omega)$.
- MONOTONICITY PRINCIPLE:
  If $\Omega_1 \subset \Omega_2$ then $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$.

(Hence $U_1 > 0$ in $\mathcal{D}_n$.)

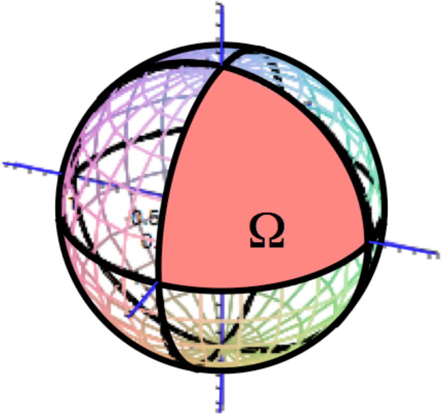


Figure: First Octant Triangle $\Omega \subset \mathbb{S}^2$

The domain Ω is a nodal domain of the cubic spherical harmonic

$$h(x, y, z) = xyz.$$

Thus

$$\lambda_1(\Omega) = 12.$$

18. Symmetrization lowers the first eigenvalue.

- FABER-KRAHN/SPERNER INEQUALITY:
  For nice $\Omega \subset \mathbb{S}^n$ or $\Omega \subset \mathbb{R}^n$. If $|B_{R^*}| = |\Omega|$ then $\lambda_1(B_{R^*}) \le \lambda_1(\Omega)$. "$=$" implies Ω is isometric to B_{R^*} Rayleigh (1877) for analytic disk near round disk. Faber-Krahn (1923) for $\Omega \subset \mathbb{R}^n$. Sperner (1955) for $\Omega \subset \mathbb{S}^n$.

  Proved by symmetrization argument and Isoperimetric Inequality. Let $u > 0$ be the first eigenfunction of Ω. Let u^* be the spherical rearrangement, *i.e.*, $u^*(x) = u^*(|x|)$ is defined on $\Omega^* = B_{R^*}$ such that

  $$|\{x \in \Omega : u(x) > t)\}| = |\{x \in \Omega^* : u^*(x) > t)\}| \qquad \text{for all } t > 0.$$

  Then

  $$\int_\Omega u^2 = \int_{\Omega^*} (u^*)^2 \qquad \text{but} \qquad \int_\Omega |\mathrm{d}u|^2 \ge \int_{\Omega^*} |\mathrm{d}u^*|^2$$

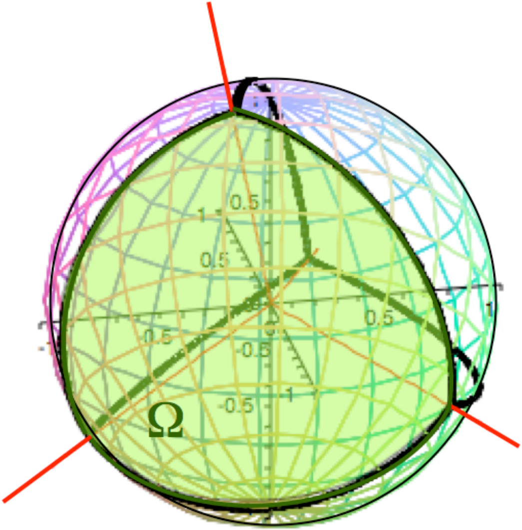


Figure: Tetrahedral Triangle $\Omega \subset \mathbb{S}^2$

The domain Ω is not a nodal domain. Odd reflection does not produce an eigenfunction on $\mathbb{S}^2$. It does lift to an eigenfunction of the two-fold cover with branching at the vertices.

Numerical computation

$$\lambda_1(\Omega) \approx 5.159\ldots$$

Let Ω^* be spherical cap with same area $|\Omega| = |\Omega^*|$. Then

$$\lambda_1(\Omega^*) = 4.93604187$$

20. The capture problem

Let $X_1(t), \ldots, X_n(t)$ be n predators, $X_0(t)$ the prey, all doing independent standard Brownian motions on $\mathbb{R}$.

Suppose the predators start to the left of the prey:

$$X_j(0) < X_0(0) \ \text{ all } \ j = 1, ..., n.$$

The capture time is defined to be

$$\tau_n = \inf\{t > 0 : \exists j : X_j(t) \geq X_0(t)\}$$

Conjecture (Bramson, Griffeath 1991)

$$\mathbb{E}\tau_n = \infty \textit{ for } n = 1, 2, 3 \quad \textit{and} \quad \mathbb{E}\tau_n < \infty \textit{ for } n \geq 4.$$

Bramson & Griffeath gave a proof for $n \leq 3$ & did extensive simulation.

Theorem (H. Kesten 1992)

$$\mathbb{E}\tau_n < \infty \text{ for } n \gg 1.$$

Theorem (W. Li & Q. M. Shao, 2001)

$$\mathbb{E}\tau_n < \infty \text{ for } n \geq 5.$$

Theorem (J. Ratzkin & T.)

$$\mathbb{E}\tau_4 < \infty.$$

Then

$$\mathbf{X}(t) = \big(X_0(t), \ldots, X_n(t)\big) \in \mathbb{R}^{n+1}$$

is an $(n+1)$-dimensional Brownian Motion in the cone

$$\mathcal{C}_{n+1} = \{(X_0, \ldots, X_n) \in \mathbb{R}^{n+1} : X_0 > X_i \text{ all } i = 1, \ldots, n\}.$$

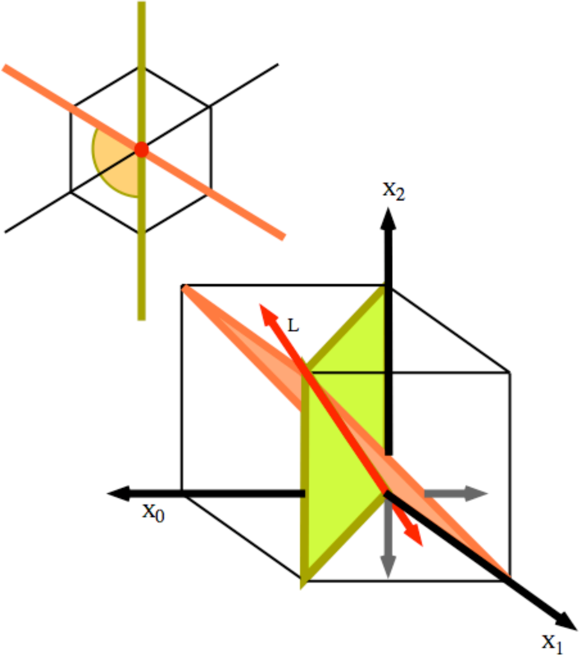
Its spherical angle is

$$\mathcal{D}_n = \mathcal{C}_{n+1} \cap \mathbb{S}^n.$$

Initial data $\mathbf{X}(0) = \mathbf{b} \in (\mathcal{C}_{n+1})^{\circ}$. Capture time becomes

$$\tau_n(\mathbf{b}) = \inf\{t > 0 : \mathbf{X}(t) \notin \mathcal{C}_{n+1}\}$$

23. Two predator cone $\mathcal{C}_3$ and angle $\mathcal{D}_2$



Theorem (De Blassie 1987)

$$\mathbb{P}_b(\tau_n > t) \sim C(b)t^{-a} \qquad \textit{as } t \to \infty$$

where $a = a(n)$ *depends on* $\lambda_1(\mathcal{D}_n)$*, the first Dirichlet eigenvalue, and* $\mathcal{D}_n = \mathcal{C}_{n+1} \cap \mathbb{S}^n$ *is the spherical angle of the cone.*

$$2a(n) = \left\{ \left(\frac{n-1}{2} \right)^2 + \lambda_1(\mathcal{D}_n) \right\}^{\frac{1}{2}} - \frac{n-1}{2} \tag{4}$$

Hence

$$\mathbb{E}\tau_n < \infty \quad \text{iff} \quad a(n) > 1 \quad \text{iff} \quad \lambda_1(\mathcal{D}_n) > 2n+2.$$

Theorem (De Blassie 1987)

$$\mathbb{P}_b(\tau_n > t) \sim C(b)t^{-a} \qquad as\ t \to \infty$$

Bramson & Griffeath gave a proof for $n \leq 3$. They found by extensive simulation

$$a(3) \cong .91 \qquad \text{and} \qquad a(4) \cong 1.032.$$

We prove

$$.90671950 < a(3) < .995648748 \qquad \text{and} \qquad a(4) > 1.00007318.$$

Moreover, our numerical calculation gives

$$a(3) \approx .9128... \qquad \text{and} \qquad a(4) \gtrsim 1.0057...$$

Example. 1. $\mathcal{C}_2 = \{(X_0, X_1) : X_0 > X_1\}$ is a halfplane so

$$\mathcal{D}_1 = \left\{(\cos\phi, \sin\phi) : -\frac{3}{4}\pi \leq \phi \leq \frac{1}{4}\pi\right\} \cong \left[-\frac{3}{4}\pi, \frac{1}{4}\pi\right]$$

so $\lambda_1(\mathcal{D}_1) = 1 \leq 4$ so $\mathbb{E}\tau_1 = \infty$.

27. Capture probabilities satisfy the heat equation.

Spitzer (1958) estimated probability in cones of $\mathbb{R}^2$.

Use Burkholder's (1977) PDE method. $u(\mathbf{x},t) = \mathbb{P}_{\mathbf{x}}(\tau_n > t)$ satisfies the heat equation.

$$\begin{aligned}
u_t &= \tfrac{1}{2}\Delta u & (\mathbf{x},t) &\in \mathcal{C}_{n+1} \times [0,\infty) \\
u(\mathbf{x},0) &= 1 & \mathbf{x} &\in \mathcal{C}_{n+1} \\
u(\mathbf{x},t) &= 0 & (\mathbf{x},t) &\in \partial\mathcal{C}_{n+1} \times (0,\infty)
\end{aligned}$$

Write cone in polar coordinates $r = |\mathbf{x}|$, $\theta = \frac{\mathbf{x}}{|\mathbf{x}|} \in \mathcal{D}_n$. Equation becomes

$$2u_t = u_{rr} + \frac{n}{r}u_r + \frac{1}{r^2}\Delta_n u$$

where Δ_n is the Laplacian on $\mathbb{S}^n$. Since there is self-similarity, look for solutions by separating variables $p(r, \theta, t) = R(\xi)U(\theta)$ where $\xi = \dfrac{r^2}{2t}$

$$\lambda_n(\mathcal{D}_n) = -\frac{\Delta_n U}{U} = \frac{4\xi^2 \ddot{R} + (4\xi^2 + 2(n+1)\xi)\dot{R}}{R}$$

Let $R(\xi) = \xi^a \rho(-\xi)$. Setting $\eta = -\xi$ the ρ satisfies the *Confluent Hypergeometric Equation:*

$$\eta \frac{\partial^2 \rho}{\partial \eta^2} + \left(2a + \frac{n+1}{2} - \eta\right) \frac{\partial \rho}{\partial \eta} - a\rho = 0$$

so

$$\rho(\xi) = {}_1\mathrm{F}_1\left(a; 2a + \frac{n+1}{2}; -\xi\right)$$

where

$${}_1\mathrm{F}_1(\alpha; \beta; z) = 1 + \frac{\alpha}{\beta}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\frac{z^3}{3!} + \cdots$$

Exit time from cone $\mathcal{C}_{n+1} \subset \mathbb{R}^{n+1}$. Argue formal series

$$P_{\mathbf{x}}(\tau_n > t) = \sum_{j=1}^{\infty} B_j \ {}_1\mathrm{F}_1\left(a_j, 2a_j + \frac{n+1}{2}, -\frac{|\mathbf{x}|^2}{2t}\right) \ U_j\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \ \left(\frac{|\mathbf{x}|^2}{2t}\right)^{a_j}$$

converges uniformly on $K \times [T, \infty)$, where $K \subset\subset \mathcal{D}_n$ and $T > 0$. Here

$$\begin{aligned} \Delta_n U_j + \lambda_j U_j &= 0 \qquad \text{for } \mathbf{x} \in \mathcal{D}_n \\ U_j &= 0 \qquad \text{if } \mathbf{x} \in \partial\mathcal{D}_n. \end{aligned}$$

and

$$2a_j(n) = \left[\left(\frac{n-1}{2}\right)^2 + \lambda_j(\mathcal{D}_n)\right]^{\frac{1}{2}} - \frac{n-1}{2}.$$

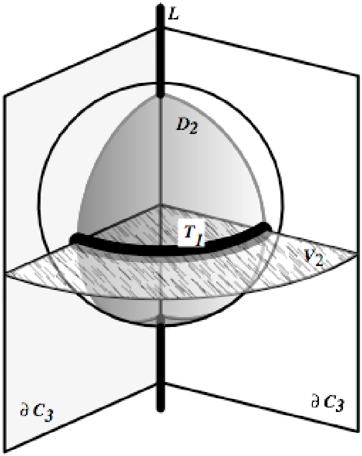
Decay rate is given by

$$2a_j(n) = \left[\left(\frac{n-1}{2}\right)^2 + \lambda_j(\mathcal{D}_n)\right]^{\frac{1}{2}} - \frac{n-1}{2}.$$

Corollary

$$P_{\mathbf{x}}(\tau_n > t) \sim B_1\, U_1\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)\, \left(\frac{|\mathbf{x}|^2}{2t}\right)^{a_1}.$$

Hence $\mathbb{E}\tau_n < \infty$ iff $a = a_1 > 1$ iff $\lambda_1(\mathcal{D}_n) > 2n+2.$



- Cone splits line $\mathcal{L}$ (all coordinates equal)

$$\mathcal{C}_{n+1} = \{\mathbf{X} \in \mathbb{R}^{n+1} : X_i < X_0, \forall i > 0\} = \mathcal{L} \oplus \mathcal{V}_n$$

where $\mathcal{L} = \mathbb{R}(1, 1, \ldots, 1)$

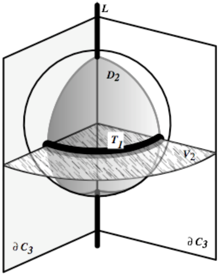
- Perpendicular part of the cone

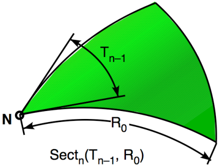
$$\mathcal{V}_n = \mathcal{C}_{n+1} \cap (1, 1, \ldots, 1)^{\perp}.$$

- Perp. part of cone angle

$$\mathcal{T}_{n-1} = \mathcal{V}_n \cap \mathbb{S}^{n-1} = \mathcal{V}_n \cap \mathcal{D}_n.$$

Dimension reduction: suffices to estimate $\mathcal{T}_{n-1}$.





- $\mathcal{T}_{n-1}$ is the face of the regular $(n+1)$-hedral tesselation in $\mathbb{S}^{n-1}$.
- At vertex $v \in \mathcal{T}_{n-1}$, spherical angle of $\mathcal{T}_{n-1}$ is $\mathcal{T}_{n-2} \subset T_v\mathbb{S}^{n-1}$.
- Let $\mathbf{N} \in \mathbb{S}^n \cap \mathcal{L}$ and regard $\mathbb{S}^{n-1} \subset T_{\mathbf{N}}\mathbb{S}^n$. In polar coordinates $\mathbf{x} = (r, \theta) \in \mathbb{S}^n$ where $\theta \in \mathbb{S}^{n-1}$ and $0 \le r \le \pi$ and $r = \text{dist}(\mathbf{x}, \mathbf{N})$. The R_0-truncated cone of any domain $T_{n-1} \subset \mathbb{S}^{n-1}$ is

$$\text{Sect}_n(T_{n-1}, R_0) = \{(r, \theta) \in \mathbb{S}^n : \theta \in T_{n-1} \text{ and } 0 \le r \le R_0\}.$$

- $\mathcal{D}_n = \text{Sect}_n(\mathcal{T}_{n-1}, \pi)$.

Let $\mathbf{N} \in \mathbb{S}^n$, $r = \mathrm{dist}_{\mathbb{S}^n}(\cdot, \mathbf{N})$ and $\theta \in \mathbb{S}^{n-1} \subset T_{\mathbf{N}}\mathbb{S}^n$.
The Laplacian on $u \in C^2(\mathbb{S}^n)$,

$$\Delta_n u = \frac{\partial^2 u}{\partial r^2} + (n-1)\cot r\,\frac{\partial u}{\partial r} + \csc^2 r\,\Delta_{n-1} u,$$

Lemma (Eigenvalues of domain in great sphere & of its suspension)

If $\mathcal{D}_n = Sect_n(\mathcal{T}_{n-1}, \pi)$ *then*

$$\lambda_1(\mathcal{D}_n) = \lambda_1(\mathcal{T}_{n-1}) - \frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} + \lambda_1(\mathcal{T}_{n-1})}.$$

In particular,

$$\mathbb{E}\tau_n < \infty \quad \text{iff} \quad \lambda_1(\mathcal{D}_n) > 2n+2 \quad \text{iff} \quad \lambda_1(\mathcal{T}_{n-1}) > 2n.$$

Example. Two policemen

If $n = 2$ then

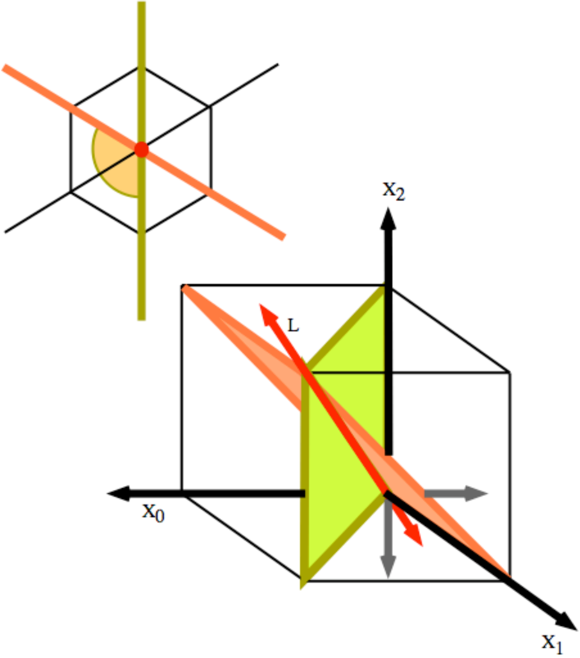
$$\mathcal{T}_1 \cong \left[0, \frac{2}{3}\pi\right] \cong \left\{e^{i\phi} : 0 \leq \phi \leq \frac{2}{3}\pi\right\} \subset \mathbb{S}^1$$

then the eigenfunction on $\mathcal{T}_1$ is

$$u_1 = \sin\left(\frac{3}{2}\theta\right) \quad \Longrightarrow \quad {u_1}'' + \frac{9}{4}u_1 = 0 \quad \Longrightarrow$$

$$\lambda_1 = \frac{9}{4} \leq 4 \quad \Longrightarrow \quad \mathbb{E}\tau_2 = \infty.$$

36. Two predator cone $\mathcal{C}_3$ and angle $\mathcal{D}_2$



Proof. Let $\mu = \lambda_1(\mathcal{D}_n)$. Put $u(r,\theta) = R(r)u(\theta)$, where $R(0) = R(\pi) = 0$ and $u(\theta) = 0$ whenever $\theta \in \partial\mathcal{T}_{n-1}$. Then

$$\frac{\sin^2 r\,\ddot{R} + (n-1)\sin r\cos r\,\dot{R} + \mu\sin^2 r\,R}{R} = \lambda = -\frac{\Delta_{n-1}u}{u},$$

so $\lambda = \lambda_1(\mathcal{T}_{n-1})$ and $u(\theta)$ is its first eigenfunction. Then

$$\sin^2 r\;\ddot{R} + (n-1)\sin r\cos r\,\dot{R} + \left(\mu\sin^2 r - \lambda\right)R = 0,$$

Hence

$$R(r) = \sin^m r \quad \text{where} \quad m = \frac{2-n}{2} + \sqrt{\frac{(2-n)^2}{4} + \lambda}$$

$$\implies \quad \mu = \lambda + m$$

Since μ is increasing in λ, we solve for λ when $\mu = 2n+2$. Answer: $\lambda = 2n$. □

In case $n = 3$ then $\mathcal{T}_2$ is a triangle. Let

$$\varphi(\mathbf{x}) = \sin\big(\mathrm{dist}(\mathbf{x}, \partial\mathcal{T}_2)\big).$$

Then by the upper bound principle,

$$\lambda_1(\mathcal{T}_2) \leq \frac{\displaystyle\int_{\mathcal{T}_2} |\mathrm{d}\varphi|^2}{\displaystyle\int_{\mathcal{T}_2} \varphi^2} = \frac{2\pi + \sqrt{3}}{\pi - \sqrt{3}} \approx 5.68641 \leq 6$$

$$\implies \quad \mathbb{E}\tau_3 = \infty.$$

Hence $a(3) < .995649$.

Theorem (W. Li & Q. M. Shao, 2001)

$$\mathbb{E}\tau_n < \infty \quad \text{for} \quad n \geq 5.$$

Proof Idea. Let $B_{cr}^{n-1} \subset \mathbb{S}^{n-1}$ satisfy $\lambda_1(B_{cr}^{n-1}) = 2n$. Let $B_{R^*}^{n-1}$ satisfy $|B_{R^*}^{n-1}| = |\mathcal{T}_{n-1}|$. Suppose that $B_{R^*}^{n-1} \subset B_{cr}^{n-1}$. By the Faber-Krahn/Sperner Inequality and the monotonicity principle,

$$\lambda_1(\mathcal{T}_{n-1}) > \lambda_1(B_{R^*}^{n-1}) \geq \lambda_1(B_{cr}^{n-1}) = 2n \quad \Longrightarrow \quad \mathbb{E}\tau_n < \infty.$$

Li & Shao show that $B_{R^*}^{n-1}$ is smaller than B_{cr}^{n-1} iff $n \geq 5$. Compare radii. R^* satisfies

$$|\mathcal{T}_{n-1}| = \frac{|\mathbb{S}^{n-1}|}{n+1} = |B_{R^*}^{n-1}| = |\mathbb{S}^{n-2}| \int_0^{R^*} \sin^{n-2} \rho \, d\rho.$$

Luckily, R_{cr} is easy! A harmonic function on $\mathbb{R}^n$ restricts to an eigenfunctions on $\mathbb{S}^{n-1}$: in polar coordinates $(r, \theta) \in \mathbb{R}^n$, the function

$$h(x_1, \ldots, x_n) = (n-1)x_1^2 - x_2^2 - \cdots - x_n^2$$

is homogeneous $h(r\theta) = r^2 h(\theta)$ and is harmonic $\Delta h = 0$ so

$$\begin{aligned} 0 &= h_{rr} + \frac{n-1}{r} h_r + \frac{1}{r^2} \Delta_{n-1} h \\ &= 2h + 2(n-1)h + \Delta_{n-1} h \\ &= \Delta_{n-1} h + 2nh. \end{aligned}$$

The nodal domain is a ball B_{cr}^{n-1} with $\lambda_1(B_{cr}^{n-1}) = 2n$. Its radius is $R_{cr} = \mathrm{Atn}\,\sqrt{n-1}$.

$$\mathbb{E}\tau_n < \infty \quad \Longleftrightarrow \quad \lambda_1(\mathcal{T}_{n-1}) > 2n.$$

For all n, $\lambda_1(\mathcal{T}_n) > \lambda_1(B^n_{R^*})$.

n	$\lvert\mathcal{T}_n\rvert = \lvert B^n_{R^*}\rvert$	R^*	$\lambda_1(B^n_{R^*})$	$R_{cr} =$ Atn $\left(\sqrt{n-1}\right)$
2	3.141592654	1.047197551	4.93604187	0.78539816
3	3.947841762	1.056569480	7.84104544	0.95531662
4	4.386490846	1.068200504	10.8876959	1.04719755
5	4.429468100	1.080033938	14.0396033	1.10714872

The $\mathcal{T}_{n-1}$ bulge in the middle. The diameter is the distance from a vertex ot $\mathcal{T}_{n-1}$ to the center of the opposite face.

$$\delta(n-1) = \mathrm{diam}(\mathcal{T}_{n-1}) = \arccos\left(-\sqrt{\tfrac{n-1}{2n}}\right).$$

Since the spherical angle at a vertex of $\mathcal{T}_n$ is $\mathcal{T}_{n-1}$, we can construct outer comparison domains inductively

$$\hat{\mathcal{T}}_1 = \mathcal{T}_1 = \left[0, \tfrac{2}{3}\pi\right]$$
$$\hat{\mathcal{T}}_n = \mathrm{Sect}_n\left(\hat{\mathcal{T}}_{n-1}, \delta(n)\right) \qquad \text{for } n \geq 2$$

By induction, and the monotomicity principle, for all n,

$$\mathcal{T}_n \subset \hat{\mathcal{T}}_n \quad \implies \lambda_1(\mathcal{T}_n) \geq \lambda_1(\hat{\mathcal{T}}_n).$$

Similarly, we construct inner comparison domains $\check{\mathcal{T}}_n$

$$\mathbb{E}\tau_n < \infty \iff \lambda_1(\mathcal{T}_{n-1}) > 2n \Longleftarrow \lambda_1(\hat{\mathcal{T}}_{n-1}) > 2n.$$

Computed using the Truncated Cone Lemma. ($\check{\mathcal{T}}_n \subset \mathcal{T}_n \subset \hat{\mathcal{T}}_n$.)

n	$\mathrm{vol}(\check{\mathcal{T}}_n)$	$\lambda_1(\check{\mathcal{T}}_n)$	$\mathrm{vol}(\hat{\mathcal{T}}_n)$	$\lambda_1(\hat{\mathcal{T}}_n)$
1	2.094395103	2.250000000	2.094395103	2.250000000
2	2.792526804	6.195617753	3.303594680	5.004635381
3	2.884035172	12.04009682	4.482940454	7.884040724
4	2.491806389	19.93880798	5.445852727	10.77018488
5	1.877352230	30.01419568	6.039182278	13.62031916

Lemma (Truncated Cone Eigenvalues.)

$T_{n-1} \subsetneq \mathbb{S}^{n-1}$ is nice, proper so $\lambda = \lambda_1(T_{n-1})$ and $0 < r < \pi$. Then

$$\lambda_1(Sect_n(T_{n-1}, r)) = \mu_1(n, \lambda_1(T_{n-1}), r)$$

where μ_1 is the least $\mu > 0$ so the solution $R(\rho)$ of the equation

$$\sin^2\rho\, \ddot{R} + (n-1)\sin\rho\,\cos\rho\,\dot{R} + \big(\mu\sin^2\rho - \lambda\big)R = 0.$$

is positive on $(0, r)$ and $R(r) = 0$. If $r \ge \frac{\pi}{2}$, it is the unique $\mu \in (m + \lambda, 3m + \lambda + n)$ such that ${}_2\mathrm{F}_1\big(\alpha_1, \beta_1; \gamma_1; \frac{1}{2}(1 - \cos r)\big) = 0$, where ${}_2\mathrm{F}_1(\alpha, \beta, \gamma, z)$ is the hypergeometric function, and

$$\alpha_1, \beta_1 = \frac{1+\sqrt{(n-2)^2+4\lambda}\pm\sqrt{(n-1)^2+4\mu}}{2},$$

$$\gamma_1 = \frac{2+\sqrt{(n-2)^2+4\lambda}}{2}.$$

Lemma (Spherical Cap Eigenvalues.)

For the ball $B_r^n \subset \mathbb{S}^n$ the first eigenvalue is given by $\lambda_1(B_r^n) = \mu_2(n, r)$ where μ_2 is the least $\mu > 0$ so a solution $R(\rho)$ of the equation

$$\sin^2\rho \; \ddot{R} + (n-1)\sin\rho\cos\rho \; \dot{R} + \mu \; \sin^2\rho \; R = 0.$$

is positive on $(0, r)$ and $R(r) = 0$. If $r \le \frac{\pi}{2}$, it can be computed as the unique value $\mu \in (0, n)$ such that

$${}_2\mathrm{F}_1\left(\alpha_2, \beta_2; \gamma_2; \tfrac{1}{2}(1 - \cos r)\right) = 0,$$

$$\alpha_2, \beta_2 = \tfrac{n-1 \pm \sqrt{(n-1)^2 + 4\mu}}{2}, \qquad \gamma_2 = \tfrac{n}{2}.$$

For the spherical cap B_r^n, the radial eigenfunction $u(\theta) \equiv 1$ and $\lambda = 0$.

Proof. Let $R(r) = \sin^m(r)\, u(r)$ on $[0, R_0]$ where

$$m = -\tfrac{n-2}{2} + \sqrt{\tfrac{(n-2)^2}{4} + \lambda_1(\mathcal{T}_{n-1})}.$$

$u(t) \neq 0$ for $t \in [0, R_0)$ but $u(R_0) = 0$. Substituting $u(r) = y\,(x)$ where $x = \frac{1}{2}(1 - \cos r)$ and writing "$'$" for $\frac{\partial}{\partial x}$ yields

$$x(1-x)\, y'' + \left(m + \tfrac{1}{2} n - (2m+n)x\right)\, y' - (\lambda + m - \mu)\, y = 0.$$

Solution is the hypergeometric function $y(x) = {}_2\mathrm{F}_1(\alpha, \beta; \gamma; x)$, taking

$$\begin{aligned} \alpha, \beta &= \tfrac{2m+n-1 \pm \sqrt{(2m+n-1)^2 - 4\lambda - 4m + 4\mu}}{2}, \\ \gamma &= \tfrac{2m+n}{2}, \end{aligned}$$

Thus $R(r) = \sin^m r\ {}_2\mathrm{F}_1\left(\alpha, \beta; \gamma; \frac{1}{2}(1 - \cos r)\right)$ and μ is chosen so that $R(R_0) = 0$. The eigenvalue of the ball is gotten by a similar analyis.

Gauß's ordinary hypergeometric function is given by

$$ {}_2F_1(\alpha, \beta; \gamma; z) = $$

$$ 1 + \frac{\alpha\beta}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!} + \cdots $$

Eigenfunctions of truncated cones on spheres can also be represented by other special functions.

These functions are regarded as known since they are canned in MAPLE. Finding a parameter that zeros an expression involving these functions is accomplished by simple root search.

$$\mathbb{E}\tau_4 < \infty \quad \Longleftrightarrow \quad \lambda_1(\mathcal{T}_3) > 8 = 2 \cdot 4.$$

For any domain $\mathcal{Q}_2 \subset \mathbb{S}^2$,

$$\lambda_1\big(\mathrm{Sect}_3(\mathcal{Q}_2, \delta(3))\big) > \ 8 \quad \Longleftrightarrow \quad \lambda_1(\mathcal{Q}_2) > \lambda_{cr} = \ 5.101267527.$$

Using our PW eigenvalue estimate, we show that $\lambda_1(\mathcal{T}_2) \geq 5.11641465$. so that $\lambda_1(\mathcal{T}_3) > \lambda_1(\hat{\mathcal{T}}_3) > 8$.

In [RT], we found a domain $\mathcal{Q}_2 \subset \mathbb{S}^2$ such that $\mathcal{T}_2 \subset \mathcal{Q}_2$ and $\lambda(\mathcal{Q}_2) = 5.102$. Thus

$$\mathcal{T}_3 \subseteq \hat{\mathcal{T}}_3 \subseteq \mathrm{Sect}_3(\mathcal{Q}_2, \delta(3)).$$

and

$$\lambda_1(\mathcal{T}_3) \geq \lambda_1(\hat{\mathcal{T}}_3) \geq \lambda_1(\mathrm{Sect}_3(\mathcal{Q}_2, \delta(3))) = 8.000878153.$$

49. Numerical calculation of $\lambda_1(\mathcal{T}_2)$. (Test if idea could work.)

First pulling $\mathcal{T}_2$ back to a rectangle in $\mathbb{R}^2$ by a conformal map and then using a sinc-collocation method, we find by

$$\text{Numerical result: } \lambda_1(\mathcal{T}_2) \approx 5.159\ldots \quad > \lambda_{cr} \qquad \text{YES!}$$

Thus the numerical values of the critical numbers are

$$\lambda_1(\mathcal{T}_3) > \lambda_1(\mathrm{Sect}_3(\mathcal{T}_2, \delta(3))) \approx 8.000878153$$

so

$$a(3) \approx .9128\ldots \qquad \text{and} \qquad a(4) \underset{\sim}{\rightarrow} > 1.0057\ldots$$

This provides a numerical verification of the conjecture $\mathbb{E}\tau_4 < \infty$.

Faber-Krahn type argument: apply isoperimetric inequality to level sets.

Theorem (Payne-Weinberger 1960)

Suppose that $\Omega \subset \mathbb{R}^2$ is a subdomain in the wedge $\mathcal{W} = \{(\rho, \theta) : 0 \le \rho,\ 0 \le \theta \le \pi/\alpha\}$, where $\alpha > 1$. Then

$$\lambda_1(\Omega) \ge \lambda_1(Sect_2([0, \pi/\alpha], r))$$

where r is chosen so that for $w = \rho^\alpha \sin \alpha\theta$,

$$\int_\Omega w^2\, da = \int_{Sect_2([0,\pi/\alpha],r)} w^2\, da$$

In fact

$$\lambda_1(\mathrm{Sect}_2([0, \pi/\alpha], r)) = \left\{ \frac{4\alpha(\alpha+1)}{\pi} \int_G w^2\, da \right\}^{-\frac{1}{\alpha+1}} j_\alpha^2$$

where j_α is the smallest zero of the Bessel function J_α.

51. Spherical setup.

(ρ,θ) are polar coordinates of $\mathbb{S}^2$ with metric

$$ds^2 = d\rho^2 + \sin^2\rho\, d\theta^2.$$

Sector in $\mathbb{S}^2$ of angle π/α, for $\alpha > 1$

$$\mathcal{W} = \{(\rho,\theta) : 0 \le \theta \le \pi/\alpha,\ 0 \le \rho < \pi\}$$

Let G be a domain such that $\overline{G} \subset \mathcal{W}$ is compact.

Truncated sector

$$\mathcal{S}(r) := \{(\rho,\theta) : 0 \le \theta \le \pi/\alpha, 0 \le \rho \le r\}$$

A positive harmonic function in $\mathcal{W}$, with zero boundary values

$$w = \tan^{\alpha}\Big(\frac{\rho}{2}\Big)\ \sin\alpha\theta$$

Theorem

For every subdomain G with compact $\overline{G} \subset \mathcal{W}$, we have the estimate

$$\lambda_1(G) \geq \lambda_1(\mathcal{S}(r^*)), \tag{5}$$

where r^* is chosen such that

$$\int_G w^2 \, da = \int_{\mathcal{S}(r^*)} w^2 \, da.$$

Equality holds if and only if G is the sector $\mathcal{S}(r^*)$.

Lemma

Let $\psi, \phi : [0,\omega) \to [0,\infty)$ be locally integrable functions with ψ nonnegative and ϕ nondecreasing. Let $\Phi(y) = \int_0^y \phi(t)\,dt$ and $\Psi(x) = \int_0^y \psi(s)\,ds$ be their primitives. Let $E \subset [0,\omega)$ be a bounded measurable set. Then

$$\Phi\left(\int_E \psi(x)\,dx\right) \le \int_E \phi(\Psi(x))\,\psi(x)\,dx. \tag{6}$$

For ϕ increasing, equality holds if and only if the measure of $E \cap [0,R]$ is R.

For example, if $0 \le r_1 \le r_2 \le r_3, \le \ldots, \le r_{2n}$, by choosing $\phi = py^{p-1}$ some $p > 1$ and $\psi(x) = 1$,

$$\left(\sum_{i=1}^{2n} (-1)^i\, r_i\right)^p \le \sum_{i=1}^{2n} (-1)^i\,(r_i)^p.$$

Change variables $y = \Psi(x)$. $dy = \psi(x)dx$. Let E' be the image of E under the map Ψ. Because ϕ is nondecreasing, for $y \geq 0$,

$$\phi\left(\int_0^y \chi_{E'}\,dy\right) \leq \phi(y).$$

For ϕ increasing, equality holds iff $\mu(E' \cap [0, y]) = y$. Multiply by $\chi_{E'}$ and integrate:

$$\begin{aligned}\Phi\left(\int_E \psi(x)dx\right) &= \Phi\left(\int_{E'} dy\right) \\ &= \int_0^\omega \phi\left(\int_0^y \chi_{E'}\,dt\right)\chi_{E'}\,dy \\ &\leq \int_0^\omega \phi(y)\chi_{E'}\,dy \\ &= \int_{E'} \phi(y)dy = \int_E \phi(\Psi(x))\psi(x)dx. \qquad \square\end{aligned}$$

Lemma

Let $G \subset \mathcal{W}$ be a domain with compact closure. Then there is a function $\Upsilon_\alpha = \mathcal{F} \circ Z^{-1}$ so that

$$\int_{\partial G} w^2 \, ds \geq \frac{\pi}{2\alpha} \Upsilon_\alpha \left(\frac{2\alpha}{\pi} \int_G w^2 \, da \right).$$

Here $\mathcal{F}(\rho) = \tan^{2\alpha}(\rho/2) \sin \rho$ and Z is given by

$$Z(r) = \int_0^r \tan^{2\alpha}\left(\frac{\rho}{2}\right) \sin \rho \, d\rho.$$

Equality holds if and only if G is a sector $\mathcal{S}(r)$.

Map the domain G into a domain $\tilde{G}$ in the upper halfplane using

$$x = f(\rho)\cos\alpha\theta, \qquad y = f(\rho)\sin\alpha\theta,$$

The Euclidean line element is $dx^2 + dy^2 = \dot{f}^2\,d\rho^2 + \alpha^2 f^2\,d\theta^2$. We claim for some f the map satisfies

$$\alpha^2 \tan^{4\alpha}\left(\tfrac{\rho}{2}\right)\,\sin^4\alpha\theta\,(d\rho^2 + \sin^2\rho\,d\theta^2) \geq y^4(dx^2 + dy^2).$$

For this to be true pointwise, we need the inequalities to hold

$$\alpha \tan^{2\alpha}\left(\tfrac{\rho}{2}\right) \geq f^2\,\dot{f} = \left(\frac{f^3}{3}\right)' \qquad (7)$$

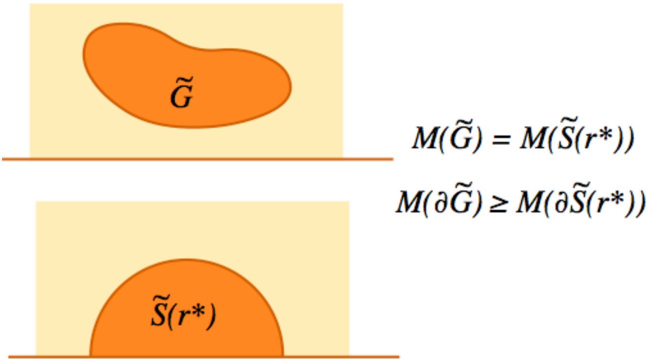
$$\sin\rho\,\tan^{2\alpha}\left(\tfrac{\rho}{2}\right) \geq f^3. \qquad (8)$$

Use equality in inequality (8) to define $f = \tan^{\frac{2\alpha}{3}}\left(\frac{\rho}{2}\right)\sin^{\frac{1}{3}}\rho$.
Differentiating,

$$f^2\,\dot{f} = \tfrac{1}{3}\tan^{2\alpha}\left(\tfrac{\rho}{2}\right)\left[2\alpha + \cos\rho\right],$$

which implies that the inequality (7) holds as well.

57. Pull back the variational problem for moments of inertia to $\mathcal{W}$.



Among $\tilde{G}$ in the upper halfspace $y > 0$, the calculus of variations problem

$$\text{minimize} \quad \int_{\partial\tilde{G}} y^2\, ds \quad \text{subject to} \quad \int_{\tilde{G}} y^2\, dxdy = \text{fixed}.$$

is solved by semicircles centered on the x-axis.

Inequality (7) implies

$$\alpha \int_{\partial G} w^2 \, ds \geq \int_{\partial \tilde{G}} y^2 \sqrt{dx^2 + dy^2} := \mathcal{M}(\partial \tilde{G}).$$

Among all domains with given fixed surface moment $\int_{\tilde{G}} y^2 \, dx \, dy$, the semicircular arcs centered on the y-axis minimize $\mathcal{M}(\partial \tilde{G})$. If $\tilde{\mathcal{S}}(R) = \tilde{G}$ is a semicircle of radius R:

$$\mathcal{M}(\partial \tilde{\mathcal{S}}(R)) = \int_0^\pi R^3 \sin^2 t dt = \frac{\pi R^3}{2},$$

$$\mathcal{M}(\tilde{\mathcal{S}}(R)) = \int_0^\pi \int_0^R r^3 \sin^2 \theta dr \, d\theta = \frac{\pi R^4}{8}.$$

Solve for R and use semicircles are minimizers, for a general domain $\tilde{G}$,

$$\mathcal{M}(\partial \tilde{G}) \geq 2^{\frac{5}{4}} \pi^{\frac{1}{4}} \left\{ \int_{\tilde{G}} y^2 dx \, dy \right\}^{\frac{3}{4}}.$$

59. Pull back isoperimetric inequality.

Returning to the original variables, $dx\,dy = \alpha f \dot{f}\, d\rho\, d\theta$ so

$$\int_{\partial G} w^2 ds \geq \frac{1}{\alpha} 2^{\frac{5}{4}} \pi^{\frac{1}{4}} \left[\int_G f^2 \, \sin^2(\alpha\theta)\, \alpha f \dot{f}\, d\rho\, d\theta \right]^{\frac{3}{4}}$$

$$= \left(\frac{\pi}{2\alpha}\right)^{\frac{1}{4}} \left\{ \int_G \frac{4}{3} \left[\tan^{2\alpha}\left(\frac{\rho}{2}\right)\, \sin\rho \right]^{\frac{1}{3}} \, [2\alpha + \cos\rho]\, \tan^{2\alpha}\left(\frac{\rho}{2}\right)\, \sin^2 \alpha\theta\, d\rho\, d\theta \right\}^{\frac{3}{4}}$$

Choose β so that

$$\frac{2\alpha + 2}{2\alpha + 1} \leq \beta < \frac{4}{3}.$$

Regroup the integral inside the braces

$$I = \frac{4}{3\beta} \int_G \left[\tan^{2\alpha}\left(\frac{\rho}{2}\right)\, \sin\rho \right]^{\frac{4}{3} - \beta} \, [2\alpha + \cos\rho]$$

$$\beta \left[\tan^{2\alpha}\left(\frac{\rho}{2}\right)\, \sin\rho \right]^{\beta - 1} \, \tan^{2\alpha}\left(\frac{\rho}{2}\right)\, d\rho\, \sin^2 \alpha\theta\, d\theta.$$

Let $\Psi = \left[\tan^{2\alpha}\left(\frac{\rho}{2}\right)\,\sin\rho\right]^{\beta}$ so

$$\psi = \beta\left(\tan^{2\alpha}\left(\frac{\rho}{2}\right)\,\sin\rho\right)^{\beta-1}\left[2\alpha+\cos\rho\right]\,\tan^{2\alpha}\left(\frac{\rho}{2}\right)$$

and

$$\phi(z) = \frac{4}{3\beta}z^{\frac{4}{3\beta}-1} \Rightarrow \Phi(z) = z^{\frac{4}{3\beta}}.$$

So that ϕ is increasing, we require $\beta < \frac{4}{3}$. If $H_\theta = \{\rho \in [0,\pi) : (\rho,\theta) \in G\}$ is the slice of G in the ρ-direction then Szegő's inequality (6) implies

$$I \geq \int_0^{\pi/\alpha}\left(\beta\int_{H_\theta}\tan^{2\alpha\beta}\left(\frac{\rho}{2}\right)\,\sin^{\beta-1}\rho\,\left[2\alpha+\cos\rho\right]\,d\rho\right)^{\frac{4}{3\beta}}\sin^2\alpha\theta\,d\theta.$$

Equality holds if and only if $H_\theta = [0, r(\theta)]$ is an interval *a.e.*

61. Use Hölder's Inequality.

Next we let $p = \frac{4}{3\beta} > 1$, $q = \frac{4}{4-3\beta}$, and using measure $\sin^2 \alpha\theta \, d\theta$. Since $\int_0^{\pi/\alpha} d\nu = \int_0^{\pi/\alpha} \sin^2 \alpha\theta \, d\theta = \frac{\pi}{2\alpha}$, Hölder's Inequality implies $I \geq$

$$\left(\frac{2\alpha}{\pi}\right)^{\frac{4}{3\beta}-1} \left(\beta \int_0^{\pi/\alpha} \int_{H_\theta} \tan^{2\alpha\beta}\left(\frac{\rho}{2}\right) \sin^{\beta-1}\rho \, [2\alpha + \cos\rho] \, d\rho \, \sin^2\alpha\theta \, d\theta\right)^{\frac{4}{3\beta}}$$

We regroup the inside integral again:

$$J = \int_0^{\pi/\alpha} \int_{H_\theta} \tan^{2\alpha(\beta-1)}\left(\tfrac{\rho}{2}\right) \sin^{\beta-2}\rho \; [2\alpha + \cos\rho] \cdot \\ \cdot \tan^{2\alpha}\left(\tfrac{\rho}{2}\right) \sin\rho \, d\rho \, \sin^2\alpha\theta \, d\theta.$$

Let us denote

$$Z(r) = \int_0^r \tan^{2\alpha}\left(\tfrac{\rho}{2}\right) \sin\rho \, d\rho.$$

and define $\bar{r}(r,\theta)$ by

$$Z(\bar{r}) = \int_0^r \tan^{2\alpha}\left(\tfrac{\rho}{2}\right) \chi_{H_\theta}(\rho) \sin\rho \, d\rho$$

where χ_H denotes the characteristic function of H. The integrand $\tan^{2\alpha}(\rho/2)\sin\rho$ is positive and increasing for the range of ρ we are considering, and so $\bar{r}(r,\theta) \le r$ with equality if and only if $H_\theta \cap [0,r] = [0,r]$ *a.e.*

If we require $(2\alpha+1)\beta \geq 2\alpha+2$, then the factor

$$g_\beta(\rho) = \tan^{2\alpha(\beta-1)}\left(\frac{\rho}{2}\right)\sin^{\beta-2}\rho\,[2\alpha+\cos\rho]$$

is increasing in ρ. Thus we can define Φ_β by

$$\phi_\beta(y) = \beta g_\beta \circ Z^{-1}(y), \qquad \Phi_\beta(y) = \int_0^y \phi_\beta(s)\,ds. \tag{9}$$

Observe that Z and g_β are increasing, so ϕ_β is increasing and Φ_β is convex. Using $g_\beta(\bar{r}(\rho,\theta)) \leq g_\beta(\rho)$, we have

$$\begin{aligned} J \geq & \int_0^{\pi/\alpha}\int_{H_\theta} g_\beta(\bar{r}(\rho,\theta))\tan^{2\alpha}\left(\frac{\rho}{2}\right)\sin\rho\,d\rho\,\sin^2\alpha\theta\,d\theta \\ = & \frac{1}{\beta}\int_0^{\pi/\alpha}\int_{H_\theta}\phi_\beta\left(\int_0^\rho \tan^{2\alpha}\left(\frac{\rho'}{2}\right)\chi_{H_\theta}(\rho')\sin\rho'\,d\rho'\right)\cdot \\ & \cdot\tan^{2\alpha}\left(\frac{\rho}{2}\right)\sin\rho\,d\rho\,\sin^2\alpha\theta\,d\theta. \end{aligned}$$

64. Use Szegő's Inequality again and Jensen's Inequality.

Let $\psi(\rho) = \tan^{2\alpha}(\rho/2)\sin(\rho)\,\chi_{H_\theta}$.

$$J \geq \frac{1}{\beta}\int_0^{\pi/\alpha} \Phi_\beta\left(\int_{H_\theta} \tan^{2\alpha}\left(\frac{\rho}{2}\right)\,\sin\rho\, d\rho\right)\,\sin^2\alpha\theta\, d\theta$$

with equality if and only if $H_\theta = [0, r(\theta)]$ is an interval *a.e.* Next, by Jensen's inequality (with the measure $\sin^2\alpha\theta\, d\theta$),

$$J \geq \frac{\pi}{2\alpha\beta}\Phi_\beta\left(\frac{2\alpha}{\pi}\int_0^{\pi/\alpha}\int_{H_\theta} \tan^{2\alpha}\left(\frac{\rho}{2}\right)\,\sin^2\alpha\theta\,\sin\rho\, d\rho\, d\theta\right)$$

with equality if and only if $\bar{r}(\theta)$ is *a.e.* constant. Substituting back,

$$\begin{aligned} I &\geq \left(\frac{2\alpha}{\pi}\right)^{\frac{4}{3\beta}-1}(\beta J)^{\frac{4}{3\beta}} \\ &\geq \frac{\pi}{2\alpha}\left\{\Phi_\beta\left(\frac{2\alpha}{\pi}\int_0^{\pi/\alpha}\int_{H_\theta} \tan^{2\alpha}\left(\frac{\rho}{2}\right)\,\sin^2\alpha\theta\,\sin\rho\, d\rho\, d\theta\right)\right\}^{\frac{4}{3\beta}}. \end{aligned}$$

65. Combine.

$$\int_{\partial G} w^2\, ds \geq \left(\frac{\pi}{2\alpha}\right)^{\frac{1}{4}} I^{\frac{3}{4}}$$

$$\geq \frac{\pi}{2\alpha}\, \Phi_\beta^{\frac{1}{\beta}} \left(\frac{2\alpha}{\pi} \int_0^{\pi/\alpha} \int_{H_\theta} \tan^{2\alpha}\left(\frac{\rho}{2}\right) \sin^2 \alpha\theta \sin\rho\, d\rho\, d\theta\right)$$

where equality holds if and only if also $\rho(\theta)$ is constant *a.e.* Taking a limit as $\beta \to \frac{4}{3}$ from below implies the inequality holds for $\beta = \frac{4}{3}$.

66. Solve for $\Phi_\beta^{1/\beta}(Y)$.

Since it depends only on $\int_G w^2 da$, it would be the same for any function v^* whose level sets $G^*_\eta = \{x : v^*(x) \ge \eta\}$ give the same $\zeta(\eta) = \int_{G_\eta} w^2 da$ as the spherical rearrangement whose levels are sectors $G^*_\eta = \mathcal{S}(r(\eta))$. We express things in terms of $r(\eta)$. Now

$$\frac{2\alpha}{\pi} y = \frac{2\alpha}{\pi}\zeta(\eta) = \frac{2\alpha}{\pi}\int_{\mathcal{S}\left(r(\eta)\right)} w^2\, da = Z\big(r(\eta)\big)$$

so, changing variables $s = Z(r)$

$$\begin{aligned}\Phi_\beta(Y) &= \int_0^Y \phi_\beta(s)\, ds = \beta \int_0^{Z^{-1}(Y)} g_\beta(r)\, \tan^{2\alpha}\left(\tfrac{r}{2}\right) \sin r\, dr \\ &= \beta \int_0^{Z^{-1}(Y)} \left[\tan^{2\alpha}\left(\tfrac{r}{2}\right) \sin r\right]^{\beta-1} [2\alpha + \cos r]\ \tan^{2\alpha}\left(\tfrac{r}{2}\right)\ dr \\ &= \left[\tan^{2\alpha}\left(\frac{Z^{-1}(Y)}{2}\right) \sin(Z^{-1}(Y))\right]^{\beta}.\end{aligned}$$

We get the same equation for all β. Thus we set $\Upsilon_\alpha = \Phi_\beta^{\frac{1}{\beta}}$. □

Let $G \subset \mathcal{W}$. It suffices to estimate the Rayleigh quotient for admissible functions $u \in C_0^2(G)$ that are twice continuously differentiable and compactly supported in G. Any admissible function may be written $u = vw$ for $v \in C_0^2(G)$. The divergence theorem shows

$$\int_G |du|^2 \, da = \int_G w^2 \, |dv|^2 \, da.$$

Let G_t denote the points of G satisfying $v \geq t$. Putting

$$\zeta(t) = \int_{G_t} w^2 \, da, \tag{10}$$

we see that $\zeta(0) = \hat{\zeta} \geq \zeta(t) \geq 0 = \zeta(\hat{v})$, where $\hat{v} = \max_G v$,

$$\frac{\partial \zeta}{\partial t} = -\int_{\partial G_t} \frac{w^2}{|dv|} \, ds$$

and

$$\int_G w^2 \, v^2 \, da = \int_0^{\hat{v}} 2t \, \zeta(t) \, dt = \int_0^{\hat{\zeta}} t^2 d\zeta.$$

Then, using the coarea formula, Schwarz's inequality, isoperimetric inequality, and changing variables to $y = \zeta(t)$, the inequality implies

$$
\begin{aligned}
\int_G w^2 \, |dv|^2 \, da &\geq \int_0^{\hat{v}} \left\{ \int_{\partial G_t} w^2 \, |dv| \, ds \right\} dt \\
&\geq \int_0^{\hat{v}} \frac{\left\{ \int_{\partial G_t} w^2 \, ds \right\}^2}{\int_{\partial G_t} \dfrac{w^2}{|dv|} \, ds} \, dt \\
&\geq \frac{\pi^2}{4\alpha^2} \int_0^{\hat{v}} \frac{\Upsilon_\alpha^2 \left(\dfrac{2\alpha}{\pi} \zeta(t) \right)}{-\dfrac{\partial \zeta}{\partial t}} \, dt.
\end{aligned}
$$

Changing variables to $y = \zeta(t)$ we have

$$\int_0^{\hat{\zeta}} \Upsilon_\alpha^2\left(\frac{2\alpha}{\pi}y\right)\left(\frac{\partial t}{\partial y}\right)^2 dy \geq \mu \int_0^{\hat{\zeta}} t(y)^2\, dy$$

where μ is the least eigenvalue of the boundary value problem

$$\frac{\partial}{\partial y}\left(\Upsilon_\alpha^2\left(\frac{2\alpha}{\pi}y\right)\frac{\partial q}{\partial y}\right) + \mu\, q = 0,$$

$$q(\hat{\zeta}) = 0, \qquad \lim_{y\to 0+} \Upsilon_\alpha^2\left(\frac{2\alpha}{\pi}y\right)\frac{\partial q}{\partial y} = 0.$$

Now perform the change variables so that the domain is now $[0, r^*]$, $Z(r^*) = \frac{2\alpha}{\pi}\hat{\zeta}$, and μ is now the least eigenvalue of

$$\frac{\partial}{\partial r}\left(\tan^{2\alpha}\left(\frac{r}{2}\right)\sin(r)\,\frac{\partial q}{\partial r}\right) + \frac{\pi^2\mu}{4\alpha^2}\tan^{2\alpha}\left(\frac{r}{2}\right)\sin(r)q = 0, \tag{11}$$

$$q(r^*) = 0, \qquad \lim_{r\to 0+}\tan^{2\alpha}\left(\frac{r}{2}\right)\sin(r)\,\frac{\partial q}{\partial r} = 0. \tag{12}$$

Note that (11) is the eigenequation for the spherical sector $\mathcal{S}(r^*)$. Hence $\frac{\pi^2\mu}{4\alpha^2} = \lambda_1(\mathcal{S}(r^*))$.
Reassembling using equations

$$\int_G |du|^2\,da \geq \lambda_1\big(\mathcal{S}(r^*)\big)\int_G u^2\,da,$$

which implies the Theorem. □

The eigenvalue $\lambda^* = \lambda_1(\mathcal{S}(r^*))$ occurs as the eigenvalue of the problem (11), (12) on $[0, r^*]$, which may be rewritten

$$\sin(r)\, q'' + [2\alpha + \cos(r)]\, q' + \lambda^*\, q = 0;$$
$$\lim_{r\to 0-} \tan^{2\alpha}\left(\tfrac{r}{2}\right)\, \sin(r)\, \tfrac{dq}{dr}(r) = 0, \qquad\qquad q(r^*) = 0.$$

Making the change of variable $x = \frac{1-\cos r}{2}$ transforms the ODE to the hypergeometric equation on $[0, 1]$

$$x(1-x)\,\ddot{y} + [c - (a+b+1)x]\,\dot{y} - ab\,y = 0,$$

with

$$a, b = \tfrac{1 \pm \sqrt{1+4\lambda^*}}{2}, \qquad c = \alpha + 1.$$

The solution to the hypergeometric equation is Gauß's ordinary hypergeometric function, given by
$${}_2\mathrm{F}_1(a, b; c; x) = 1 + \tfrac{ab}{c}\,\tfrac{x}{1!} + \tfrac{a(a+1)b(b+1)}{c(c+1)}\,\tfrac{x^2}{2!} + \tfrac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\,\tfrac{x^3}{3!} + \cdots .$$

We find the eigenvalue by a shooting method. Given r^*, λ^* is the first positive root of the function

$$\lambda \mapsto {}_2\mathsf{F}_1\left(\frac{1-\sqrt{1+4\lambda}}{2}, \frac{1+\sqrt{1+4\lambda}}{2}; \alpha+1; \frac{1-\cos r^*}{2}\right). \tag{13}$$

73. Eigenvalues from the new Spherical Estimates.

G	$\mathcal{I}(G)$	r^*	$\lambda_1(G)$	$\lambda_1(\mathcal{S}(r^*))$
$\mathcal{W}$	∞	π	$(\alpha+1)\alpha$	$(\alpha+1)\alpha$
$\mathcal{S}(\frac{\pi}{2})$	$\frac{\pi}{2\alpha} Z\left(\frac{\pi}{2}\right)$	$\frac{\pi}{2}$	$(\alpha+1)(\alpha+2)$	$(\alpha+1)(\alpha+2)$
$\mathcal{S}(r)$	$\frac{\pi}{2\alpha} Z(r)$	r	λ^*	λ^*
$\mathcal{W}$ $\alpha=\frac{3}{2}$	∞	3.14159265	3.75	3.75
$\mathcal{S}(\delta)$ $\alpha=\frac{3}{2}$	2.07876577	2.18627604	5.00463538	5.00463538
$\mathcal{S}(\varepsilon)$ $\alpha=\frac{3}{2}$	0.90871989	1.91063324	6.19561775	6.19561775
$\mathcal{S}(\frac{\pi}{2})$ $\alpha=\frac{3}{2}$	0.30118555	1.57079633	8.75	8.75
$\mathcal{T}$	1.88896324	2.15399460	5.1590...	5.11641465
$\hat{\mathcal{T}}$	1.90831355	2.15742981	?	5.10421518

Table: Domains and eigenvalues. In this table $\delta = \cos^{-1}(-1/\sqrt{3})$ and $\epsilon = \cos^{-1}(-1/3)$.

Consider the example of the geodesic triangle $\mathcal{T} = \mathcal{T}_2 \subset \mathbb{S}^2$. Writing

$$\mathcal{T} = \left\{(\rho, \theta) : 0 \le \theta \le \tfrac{2\pi}{3}, \quad 0 \le \rho \le r(\theta)\right\}$$

we find

$$r(\theta) = \tfrac{\pi}{2} + \mathrm{Atn}\left(\frac{\cos(\theta - \frac{\pi}{3})}{\sqrt{2}}\right).$$

At the vertex we have $\alpha = \frac{3}{2}$ so that

$$Z(r) = \int_0^r \tan^3\left(\tfrac{\rho}{2}\right) \sin\rho \, d\rho = 4\tan\left(\tfrac{r}{2}\right) + \sin r - 3r.$$

$\lambda_1(\mathcal{T})$ was found numerically in [RT]. Using MAPLE©, we numerically integrate

$$\mathcal{I}(\mathcal{T}) = \int_0^{\pi/\alpha} Z(r(\theta)) \sin^2(\alpha\theta) \, d\theta$$

and solve $\frac{\pi}{2\alpha} Z(r^*) = \mathcal{I}(\mathcal{T})$ for r^* and λ^* to get the other values in the $\mathcal{T}$ line in Table 1.

To avoid the quadrature, we observe the estimate

$$Z(r(\theta)) \leq T(\theta) := A_1 + A_2 \cos\Big(\theta - \frac{\pi}{3}\Big) + A_3\Big(1 - \cos(6\theta)\Big),$$

where A_1 and A_2 are chosen so that the functions agree at $\theta = 0$ and $\theta = \frac{\pi}{3}$ and the A_3 is chosen to make the second derivatives agree at $\frac{\pi}{3}$. The inequality follows since the second derivative of the difference goes from negative to positive in $0 < \theta < \pi/3$.
This corresponds to the larger domain $\hat{\mathcal{T}}$ whose radius function is $\hat{r}(\theta) = Z^{-1}(T(\theta))$. Then

$$\frac{\pi}{2\alpha} Z(\hat{r}^*) = \int_{\hat{\mathcal{T}}} w^2 \, da = \int_0^{\frac{2\pi}{3}} T(\theta) \sin^2\left(\frac{3}{2}\theta\right) \, d\theta = \frac{\pi}{3} A_1 + \frac{9\sqrt{3}}{16} A_2 + \frac{\pi}{3} A_3.$$

Using these values we obtain the last row of Table 1. By eigenvalue monotonicity, if $\hat{\mathcal{T}} \supset \mathcal{T}$ then $\lambda_1(\mathcal{T}) \geq \lambda_1(\hat{\mathcal{T}})$.

Theorem (Ratzkin 2009)

Let Ω be a nice domain in the cone

$$\mathcal{W}_n = \{(r, \theta) : r \geq 0, \quad \theta \in \mathcal{D}_{n-1}\}$$

where $\mathcal{D}_{n-1} \subset \mathbb{S}^{n-1}$ is a convex domain. Choose r_0 so that

$$\int_\Omega w^2 \, dV = \int_{\mathrm{Sect}_n(r_0, \mathcal{D}_{n-1})} w^2 \, dV$$

where $w = r^\alpha \psi(\theta)$, ψ is the first eigenfunction of $\mathcal{D}_{n-1}$ and

$$\alpha = \tfrac{n-2}{2} + \sqrt{\left(\tfrac{n-2}{2}\right)^2 + \lambda_1(\mathcal{D}_{n-1})}.$$

Then $\lambda_1(\Omega) \geq \lambda_1(\mathrm{Sect}_n(r_0, \mathcal{D}_{n-1})$, with equality if and only if $\Omega = \mathrm{Sect}_n(r_0, \mathcal{D}_{n-1})$.

Theorem

There is a nice domain $\mathcal{Q}_2 \subset \mathbb{S}^2$ such that $\mathcal{T}_2 \subset \mathcal{Q}_2$ and such that

$$\lambda_1(\mathcal{Q}_2) = 5.102 > \lambda_{cr}.$$

Corollary

$$\lambda_1(\mathcal{T}_3) > \lambda_1(\mathrm{Sect}_3(\mathcal{Q}_2, \delta(3))) > 8$$

so

$$\mathbb{E}\tau_4 < \infty.$$

The idea was motivated by Rayleigh (1877) and Polya-Szego (1952) who studied the dependence of the eigenvalue on planar nearly circular domains of the form

$$r \leq c + \varepsilon f(\theta).$$

Idea of proof. Let $U = R(r)\Theta(\theta)$ solve

$$\Delta_2 U + \mu U = 0 \qquad \text{on } \mathrm{Sect}_2\left(\left[0, \tfrac{2}{3}\pi\right], \pi\right).$$

Fix $\mu = 5.102$. Fix $\theta \in \left[0, \frac{2}{3}\pi\right]$. For each angular eigenmode $\ell \in \mathbb{N}$,

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0 &\qquad& \text{on } \left[0, \tfrac{2}{3}\pi\right], \\ \Theta &= 0 && \text{at } \theta \in \left\{0, \tfrac{2}{3}\pi\right\} \end{aligned}$$

Thus the angular part $\Theta_\ell(\theta) = \sin\left(\frac{3}{2}\ell\theta\right)$ and $\lambda = \frac{9}{4}\ell^2$.

Solve for $R_\ell(r)$, the radial part of eigenfunction. Define $\mathcal{Q}_2$ as the nodal domain of

$$\Phi = \Theta_1(\theta)R_1(r) + \varepsilon\Theta_3(\theta)R_3(r)$$

for appropriate $\varepsilon \neq 0$. By construction, $\lambda_1(\mathcal{Q}_2) = \mu$.

If $r_1 > 0$ is the first zero of $R_1(r)$, then also by construction,

$$\mu = \lambda_1\left(\mathrm{Sect}_2\left(\left[0, \tfrac{2}{3}\pi\right], r_1\right)\right).$$

$\mathcal{Q}_2$ is a perurbation of the sector $\mathrm{Sect}_2\left(\left[0, \frac{2}{3}\pi\right], r_1\right)$, the nodal domain of $\Theta_1(\theta)R_1(r)$, which does not contain $\mathcal{T}_2$.

The radial part satisfies

$$\begin{aligned} \sin^2 r\, \ddot{R} + \sin r \cos r\, \dot{R} + \left(\mu \sin^2 r - \lambda\right) R &= 0 \quad \text{on } [0,\pi) \\ R &= 0 \quad \text{at } r = 0 \end{aligned}$$

Since $n = 2$, $m = \sqrt{\lambda}$. Putting $R(r) = \sin^m r\, u(r)$ as in the Truncated Cone Lemma, the equation becomes

$$\sin^2 r\, \ddot{u} + (1 + 2m) \sin r\, \cos r\, \dot{u} + (\mu - m - \lambda) \sin^2 r\, u = 0.$$

Using $\lambda_\ell = \frac{9}{4}\ell^2$, the solution is hypergeometric

$$u_\ell(r) = {}_2\mathrm{F}_1\left(\tfrac{3}{2}\ell + 0.5 \pm \sqrt{\tfrac{1}{4} + \mu};\, 1 + \tfrac{3}{2}\ell;\, \tfrac{1}{2}(1 - \cos r)\right).$$

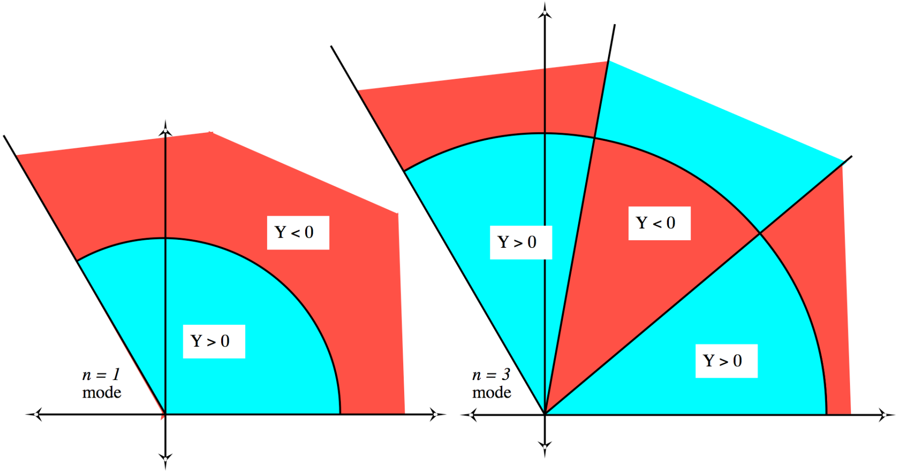
Finally, consider the $\mu = 5.102$ superposition of the $\ell = 1, 3$ modes

$$\Phi = \big(\sin r\big)^{3/2}\, u_1(r)\, \sin(\tfrac{3}{2}\theta) - .0003\big(\sin r\big)^{9/2}\, u_3(r)\, \sin(\tfrac{9}{2}\theta).$$

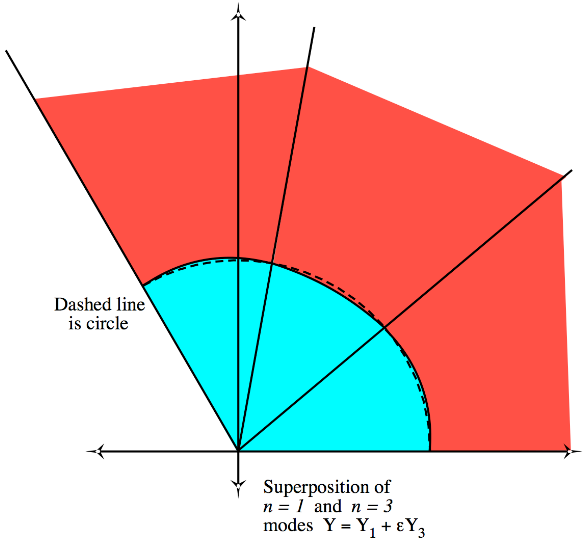
Let $\mathcal{Q}_2$ be its nodal domain. $\lambda_1(\mathcal{Q}_2) = 5.102$ by construction.

It remains to show for this ε we have $\mathcal{T}_2 \subset \mathcal{Q}_2$.

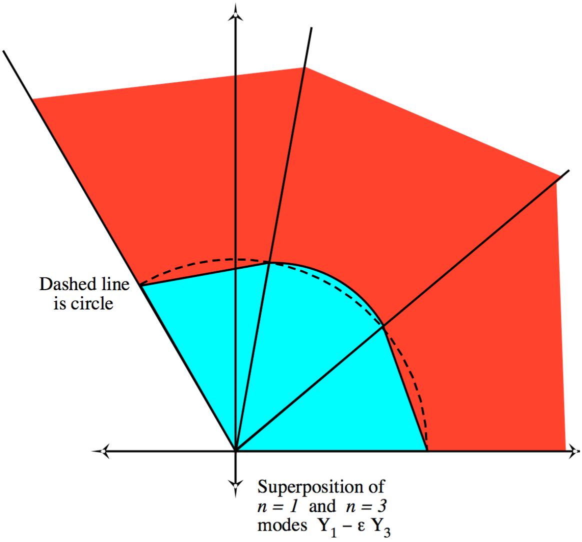
82. First and third modes.



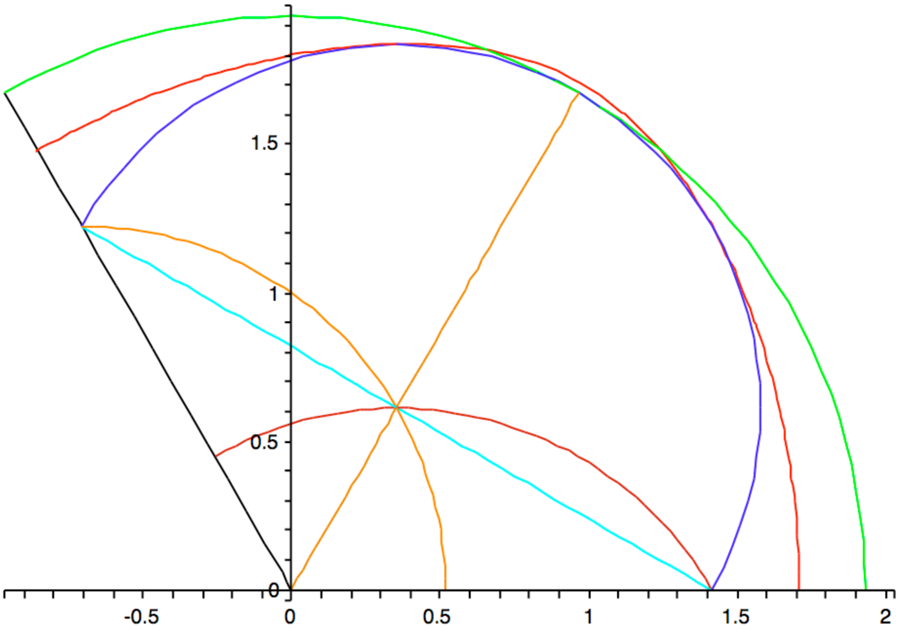
83. Superposition of first and third modes. Perturbed nodal domain.



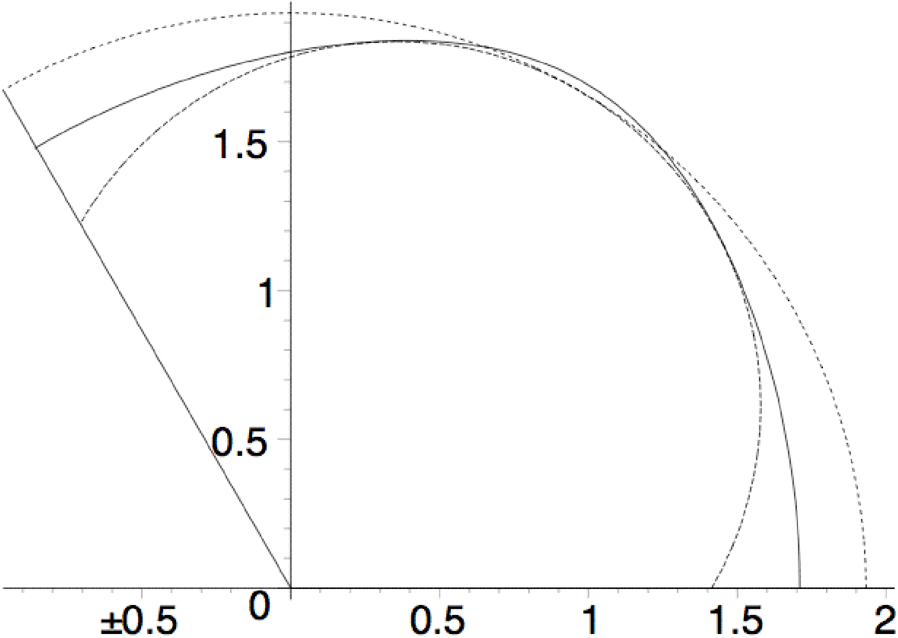
84. Superposition of first and third modes. Perturbed nodal domain.



85. Stereographic projection $\mathcal{T}_2 \subset \mathcal{Q}_2 \not\subset \hat{\mathcal{T}}_2$



86. Stereographic projection $\mathcal{T}_2 \subset \mathcal{Q}_2 \not\subset \hat{\mathcal{T}}_2$



It remains to check that $\mathcal{T}_2 \subset \mathcal{Q}_2$. As Υ is a perturbation of u_1, its nodal set is a perturbation of the sector $\mathrm{Sect}_2(\mathcal{T}_1, r_1)$ (with $r_1 < \delta(2)$.) Converting to the stereographic image $\rho = \tan(r/2)$, the radius of the circular outer edge $\rho(\theta)$ of $\mathcal{T}_2$ satisfies

$$\left(\rho(\theta)\cos\theta - \frac{\sqrt{2}}{4}\right)^2 + \left(\rho(\theta)\sin\theta - \frac{\sqrt{6}}{2}\right)^2 = \frac{3}{2}$$

so that

$$\rho(\theta) = \frac{\sqrt{2}\cos\left(\theta - \frac{\pi}{3}\right) + \sqrt{2\cos^2\left(\theta - \frac{\pi}{3}\right) + 4}}{2}.$$

Dropping the $\sin(r)^{3/2} \sin\left(\frac{3}{2}\theta\right)$ factor, it remains to prove that

$$\Psi(r) = u_1(r) - .0003\big(\sin r\big)^3\, u_3(r)\ \left(4\cos(\tfrac{3}{2}\theta)^2 - 1\right) \geq 0$$

$$\text{for all } 0 \leq r \leq 2\arctan\left(\rho(\theta)\right)\ \textit{and}\ 0 \leq \theta \leq \tfrac{2}{3}\pi$$

This is easily seen when plotted by a computer algebra system like MAPLE. Ψ and its derivatives are known. The result follows by finitely many function evaluations and esimates on the derivative of Υ to show $\Psi > 0$ on $\mathcal{T}_2$.

Numerical Computation.

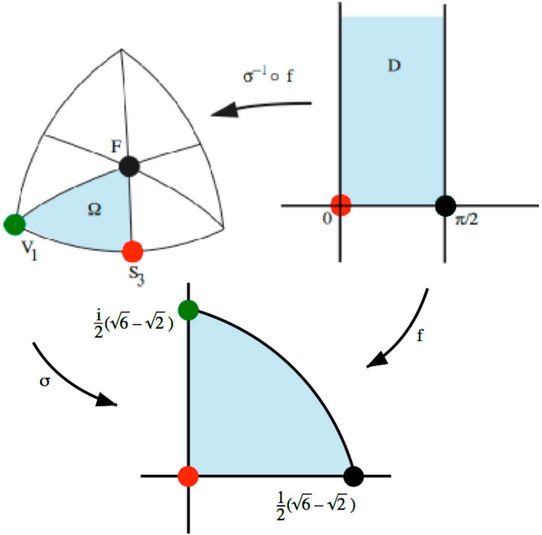
$$\lambda_1(\mathcal{T}_2) \approx 5.159...$$

We use a Sinc-Galerkin-Collocation scheme to approximate the eigenvalue of the triangle $\mathcal{T}_2$ using an idea of Stenger.

First, conformally map $\mathcal{T}_2$ to semiinfinite strip.
Let the vertices of $\mathcal{T}_2$ be V_1, V_2, V_3. The midpoint of the edge $\overline{V_1 V_2}$ is denoted S_3 and the center point of $\mathcal{T}_2$ is denoted F.

Ω is $\frac{1}{6}$ of $\mathcal{T}_2$ with vertices F, V_1, S_3. The eigenfunction u_1 of $\mathcal{T}_2$ when restricted to Ω is the first eigenfunction with Dirichlet condition on the segment $\overline{V_1 S_3}$ and Neumann condition on other two edges.

90. Conformally map infinite strip to a sixth of $\mathcal{T}_2$



Let $P = (X, Y, Z) \in \mathbb{S}^2 \subset \mathbb{R}^3$. Stereographic projection is

$$w = \sigma(P) = \frac{X + iY}{1 + Z}$$

so that the metric and Laplacian of the sphere is

$$ds^2 = \frac{4|dw|^2}{(1 + |w|^2)^2}, \qquad \Delta = \frac{(1 + |w|^2)}{4}\Delta_w$$

where $\Delta_w = 4\frac{\partial^2}{\partial w \partial \bar{w}}$. Placing S_3 at the north pole. $\sigma(S_3) = 0$. Rotate so

$$\sigma(F) = \frac{\sqrt{6} - \sqrt{2}}{2}, \qquad \sigma(V_1) = \frac{\sqrt{6} - \sqrt{2}}{2}i.$$

Let $D = \left\{z \in \mathbb{C} : 0 < \Re e z < \frac{\pi}{2}, 0 < \Im m z\right\}$ be the strip.
If $f : D \to \sigma(\Omega)$ is the conformal map so $f(0) = \sigma(S_3)$, $f(\frac{\pi}{2}) = \sigma(F)$ and $f(\infty) = \sigma(V_1)$. Pulling back to D,

$$\begin{aligned} \Delta^* u + \lambda u &= 0, && \text{if } z \in D, \\ u &= 0 && \text{if } \Re e z = 0 \text{ and } \Im m z > 0, \\ \frac{\partial u}{\partial n} &= 0 && \text{if } \Im m z = 0, \text{ or } \Re e z = \frac{\pi}{2} \end{aligned}$$

where $\Delta^* = f^* \Delta = \Delta_z$ is the pulled back Laplacian.

Schwarz triangle mapping $z \in D$ or from $\sin^2 z \in \mathcal{H}$ of the upper halfplane to $w \in \sigma(\Omega)$ is given by

$$\cos^2 z = \frac{(w^4 + 2\sqrt{3}\, w^2 - 1)^3}{(w^4 - 2\sqrt{3}\, w^2 - 1)^3} = \frac{\prod_{j=1}^4 \left(w - \sigma(F_j)\right)^3}{\prod_{j=1}^4 \left(w - \sigma(V_j)\right)^3}$$

where the coordinate for S_3 is $w = 0$, for V_1 is $w = \frac{i}{2}\left(\sqrt{6} - \sqrt{2}\right)$ and for F is $w = \frac{1}{2}\left(\sqrt{6} - \sqrt{2}\right)$. The corresponding points are $z = 0, \infty, \pi/2$. Thus we may compute f. Writing $g(z) = \cos^{2/3} z$,

$$f(z) = \sqrt{\frac{1 - g}{\sqrt{3}\,(1 + g) + 2\sqrt{1 + g + g^2}}}$$

Pulling back under $w = f(z)$, the conformal weight takes the form

$$\frac{4\left|\frac{df}{dz}\right|^2}{(1+|f|^2)^2} = \frac{\frac{4}{3}\left|\sqrt{3}(1+g)+2\sqrt{1+g+g^2}\right|}{|g|\left(\left|\sqrt{3}(1+g)+2\sqrt{1+g+g^2}\right|+|1-g|\right)^2}.$$

The branch cuts for the square and cube roots may be taken above the negative real axis. Thus $g(D)$ lies in the fourth quadrant so the denominator in f is nonvanishing.

Convert to eigenvalue problem of integral operator.

Let $G(z, z')$ denote the Green's function for the problem on D

$$\begin{aligned} \Delta^* u = f, &\qquad \text{if } z \in D, \\ u = 0 &\qquad \text{if } \Re e z = 0 \text{ and } \Im m z > 0, \\ \frac{\partial u}{\partial n} = 0 &\qquad \text{if } \Im m z = 0 \text{ or if } \Re e z = \frac{\pi}{2} \end{aligned}$$

The Green's function may be found by the method of images. Denote $w = \sin z = x + iy$, $w^* = -x + yi$ and $\omega = \sin \zeta = \xi + i\eta$, we get $\overline{w^*} = (\bar{w})^*$. Thus the Green's function is $G(z; \zeta) =$

$$\frac{1}{2\pi}\Big(\ln|w - \omega| - \ln|w^* - \omega| + \ln|\bar{w} - \omega| - \ln|\bar{w}^* - \omega|\Big) \quad =$$

$$G(x, y; \xi, \eta) = \frac{1}{4\pi}\ln\left(\frac{[(x-\xi)^2+(y-\eta)^2][(x-\xi)^2+(y+\eta)^2]}{[(x+\xi)^2+(y-\eta)^2][(x+\xi)^2+(y+\eta)^2]}\right)$$

Pulling back by f, restate as eigenvalue problem for the integral operator

$$\frac{1}{\lambda}u(z) = -4\int_D \frac{G(z;z')\,|df(z')|^2\,u(z')\,dz'}{(1+|f(z')|^2)^2} =: \mathcal{A}u(z)$$

The operator has logarithmic and algebraic singularities at the points $z' = 0, \frac{\pi}{2}$ and $z = z'$.

Approximate $f^* u(z)$ in an m-dimensional space X_m of SINC functions with the same symmetries. Take a basis $\{\phi_1, \dots, \phi_m\}$ of X_m. At the sinc points $z_i \in D$,

$$\phi_i(z_k) = \delta_{ik}.$$

$\mathcal{P}_\ell$ is the ℓ-th coefficient via point-evaluation

$$\mathcal{P}_\ell f = f(z_\ell),$$

so the "projection" to X_m is (a collocation)

$$(\mathcal{P} f)(\zeta) = \sum_k f(z_k)\, \phi_k(\zeta).$$

The integral operator shall be computed numerically via sinc quadrature.

The matrix of the transformation $A_{\ell k} = \mathcal{P}_\ell \mathcal{A} \phi_k$, whose largest eigenvalue approximates $\mu_m \to \dfrac{1}{\lambda_1}$ as $m \to \infty$. It is an upper bound $\lambda_1 \leq \dfrac{1}{\mu_m}$.

98. Basis given by products of sinc functions

For $z = x + iy$ let the basis $\phi_{jk}(z) = \alpha_j(x) \times \beta_k(y)$, where

$$\begin{aligned} \alpha_j(x) &= S(j,h) \circ \ln\left(\frac{x}{\frac{\pi}{2} - x}\right), \\ \alpha_{n+1}(x) &= \sin^2(x) - \sum_{\ell=-n}^{n} \sin^2(x_\ell)\, \alpha_\ell(x), \end{aligned}$$

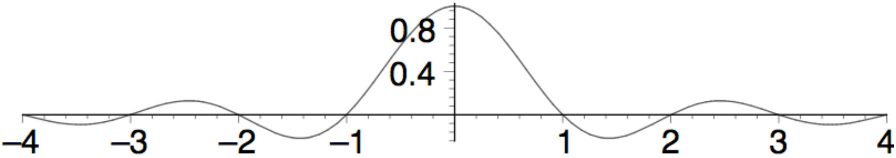
where $j = -n \ldots, n$ with sinc points $x_j = \dfrac{\pi e^{hj}}{2(1+e^{hj})}$ and

$$\begin{aligned} \beta_k(x) &= S(k,h) \circ \ln(\sinh y), \\ \beta_{n+1}(y) &= \operatorname{sech}(y) - \sum_{\ell=-n}^{n} \operatorname{sech}(y_\ell)\, \beta_\ell(x), \end{aligned}$$

for $k = -n, \ldots, n$ with sinc points $y_k = \sinh^{-1}(e^{hk})$. We let $h = \dfrac{\pi}{\sqrt{2n}}$, $x_{n+1} = \dfrac{\pi}{2}$ and $y_{n+1} = 0$.

$$S(j,h)(x) = \begin{cases} \dfrac{\sin\left(\frac{\pi(x-jh)}{h}\right)}{\frac{\pi(x-jh)}{h}}, & \text{if } x \neq jh, \\ 1, & \text{if } x = jh. \end{cases}$$

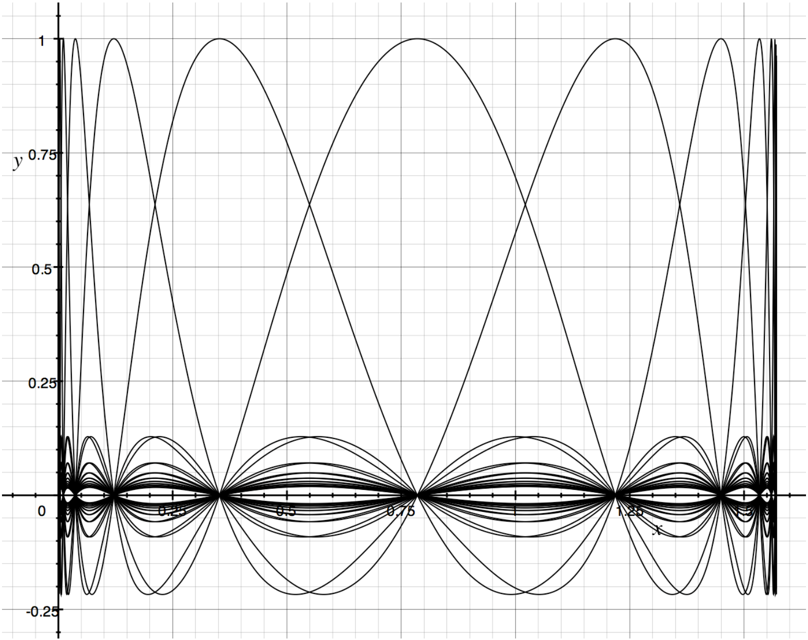
where $h = \dfrac{\pi}{\sqrt{2n}}$. Thus the dimension is $m = (2n+2)^2$.



Sinc function $S(0,1)(x)$.

Approximate $f^*u(z) \approx \mathcal{P}f^*u(z) = b^{ij}\phi_{ij}(z)$
(sum over $i, j = -n \ldots, n+1$.)
Let $b^{ij} = \mathcal{P}_{ij}f^*u = f^*u(x_i + y_j\sqrt{-1})$. Thus the approximation $\mathcal{P}f^*u$ is a collocation, as it equals f^*u at the sinc points.

100. Basis functions $\alpha_i(x)$ on $(0, \frac{\pi}{2})$ when $n = 17$.



Thus, the matrix is approximated by

$$A_{ij,pq} = \int_D G(x_i, y_j, \xi, \eta)\beta_{pq}(\xi, \eta)\Psi(\xi, \eta)\, d\xi\, d\eta$$
$$\approx \sum_{\iota,\kappa} v_\iota w_\kappa G(x_i, y_j, x_\iota, y_\kappa)\beta_{pq}(x_\iota, y_\kappa)\, \Psi(x_\iota, y_\kappa)$$

where

$$\Psi(\xi, \eta) = \frac{4|df(\xi + i\eta)|^2}{(1 + |f(\xi + i\eta)|^2)^2}$$

The approximating sum is carried over $4m$ terms, corresponding to sinc quadratures in the four regions bounded by singularities (*e.g.* in case $0 < x_i < \frac{\pi}{2}$ and $0 < y_j$):

$$\begin{aligned}
D_I &= \{\xi + i\eta : 0 < \xi < x_i,\ 0 < \eta < y_j\}; \\
D_{II} &= \{\xi + i\eta : x_i < \xi < \pi/2,\ 0 < \eta < y_j\}; \\
D_{III} &= \{\xi + i\eta : 0 < \xi < x_i,\ y_j < \eta\}; \\
D_{IV} &= \{\xi + i\eta : x_i < \xi,\ y_j < \eta\}.
\end{aligned}$$

and v_ι, w_κ are the corresponding weights.

Dimension	Eigenvalue Estimate
16	5.948293885960918
36	5.458635965290180
64	5.386598832939550
100	5.262319373675790
144	5.227827463701747
196	5.177342919223594
256	5.169086379011730
324	5.149086464180199
400	5.150079323070225
484	5.143150823134755
576	5.147209806571762
676	5.145614813257604

Dimension	Eigenvalue Estimate
784	5.149974059002415
900	5.150237866877259
1024	5.153693139833067
1156	5.154249270892947
1296	5.156376203334823
1444	5.156740724104297
1600	5.157841188308859
1764	5.158003066920548
1936	5.158526741103521
2116	5.158585939808193
2304	5.158832705984016
2500	5.158849530710926
2704	5.158968860560663

Thanks!