Eigenvalues of Spherical Triangles and a Brownian Pursuit Problem

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Joint work with Jesse Ratzkin, University of Cape Town, South Africa.

The URL for Beamer Slides for my October 14 talk, “Eigenvalues of spherical triangles and a Brownian pursuit problem”

http://www.math.utah.edu/~treiberg/EigenvalCapture.pdf

References.

- J. Ratzkin, Eigenvalues of Euclidean wedge domains in higher dimensions.
3. Outline.

- Eigenvalues
- Capture problem.
- Reduction to geometric eigenvalue problem.
- Eigenvalue basic properties
- Eigenvalue computation for simple domains.
- Recap proof in known cases.
- Domain perturbation and proof in remaining case.

Analytic arguments up to finding a few roots of exact expressions involving special functions via computer algebra system Maple.

- Numerical Computation.

Run on department’s mainframe.
A number $\lambda \in \mathbb{C}$ is an eigenvalue of a nice domain $\mathcal{D}_n \subset \mathbb{R}^n$ (or in $\mathcal{M}^n$) if there is a nonzero eigenfunction $U \in C(\overline{\mathcal{D}_n}) \cap C^2(\mathcal{D}_n)$ satisfying

\[
\begin{cases} 
\Delta U + \lambda U = 0 & \text{for } x \in \mathcal{D}_n \\
U = 0 & \text{if } x \in \partial \mathcal{D}_n.
\end{cases}
\] (1)

where, e.g., the Laplacian on $\mathbb{R}^n$ is

$$
\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

Compact domains have discrete spectrum

$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \cdots \to \infty$, where each eigenvalue has finite multiplicity.

Corresponding to $\lambda_j$ are eigenfunctions $U_j \in C(\overline{\mathcal{D}_n}) \cap C^2(\mathcal{D}_n)$ which may be chosen orthonormal with respect to $L^2$. 
5. Gradient, Divergence, Laplacian.

On a manifold, if $G = [g_{ij}(x)]$ is the Riemannian metric, then gradient, divergence and Laplacian are defined so that the usual Green’s formulas continue to hold on the manifold. If $V(x) = (v^1(x), \ldots, v^n(x))$ is a $C^1$ vector field in local coordinates $x = (x_1, \ldots, x_n)$ on a Riemannian manifold and $u \in C^2(M)$, then using the inverse matrix $g^{ij} = [g_{ij}]^{-1}$,

\[
\text{grad } u = \begin{pmatrix} \cdots, & \sum_{j=1}^{n} g^{ij} \frac{\partial}{\partial x_j} u, & \cdots \end{pmatrix}
\]

\[
\text{div } V = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left( \sqrt{g} v^i \right)
\]

\[
\Delta u = \text{div \ grad } u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x_i} u \right)
\]
5. Gradient, Divergence, Laplacian.

On a manifold, if $\mathcal{G} = [g_{ij}(x)]$ is the Riemannian metric, then gradient, divergence and Laplacian are defined so that the usual Green’s formulas continue to hold on the manifold. If $V(x) = (v^1(x), \ldots, v^n(x))$ is a $C^1$ vector field in local coordinates $x = (x_1, \ldots, x_n)$ on a Riemannian manifold and $u \in C^2(M)$, then using the inverse matrix $g^{ij} = [g_{ij}]^{-1}$,

When $M$ is Euclidean with $g_{ij} = \delta_{ij}$

$$\text{grad } u = \left( \cdots, \sum_{j=1}^n g^{ij} \frac{\partial}{\partial x_j} u, \cdots \right) = \left( \cdots, \frac{\partial u}{\partial x_i}, \cdots \right);$$

$$\text{div } V = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( \sqrt{g} v^i \right) = \sum_{j=1}^n \frac{\partial v^j}{\partial x_j};$$

$$\Delta u = \text{div grad } u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( \sqrt{g} g^{ij} \frac{\partial}{\partial x_i} u \right) = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}.$$
Suppose that a domain vibrates according to the wave equation. What frequencies are heard? Let $\rho$ be the density and $\tau$ be the tension. Then the amount of a small transverse vibration is given by $v(x, t)$ where $x \in M$ and $t \geq 0$,

$$\frac{\partial^2 v}{\partial t^2} = \frac{\tau}{\rho} \Delta v.$$ 

We seek solutions of the form $v(x, t) = T(t)u(x)$. Thus

$$T''(t)u(x) = \frac{\tau}{\rho} T(t) \Delta u(x).$$

We can separate variables. The only way a $t$-expression equals an $x$-expression is if both equal $\lambda = \text{const.}$

$$\frac{\rho T''(t)}{\tau T(t)} = -\lambda = \frac{\Delta u(x)}{u(x)}$$

which results in two equations

$$\Delta u + \lambda u = 0,$$
$$\rho \rho T'' + \lambda \tau T = 0.$$
7. Frequencies.

When \( \lambda > 0 \), the time equation

\[ \rho T'' + \lambda \tau T = 0 \]

has the solution

\[ T(t) = A \cos \left( \sqrt{\frac{\tau \lambda}{\rho}} t \right) + B \sin \left( \sqrt{\frac{\tau \lambda}{\rho}} t \right). \]

Thus the time dependence is sinusoidal. Its frequency is

\[ \frac{1}{2\pi} \sqrt{\frac{\tau \lambda}{\rho}} \]

cycles per unit time. The frequency increases with the eigenvalue \( \lambda \) and tension \( \tau \) and decreases with density \( \rho \).

The lowest frequency corresponds to smallest positive eigenvalue \( \lambda_1 > 0 \). Thus \( \lambda_1 \) is called the \textit{fundamental} eigenvalue.
8. Basic Properties.

Theorem

Let $\Omega$ be a piecewise $C^1$ domain in a smooth manifold.

1. Let $\lambda$ be an eigenvalue and $u$ its corresponding eigenfunction. Then $u \in C_0^\infty(\Omega)$.

2. For all $\lambda \in \text{spec}(\Omega)$, the eigenspace $\mathcal{E}_\lambda = \{ u : \Delta u + \lambda u = 0 \}$ is finite dimensional. Its dimension is called the multiplicity $m_\lambda$.

3. The $\lambda_1$ eigenspace is one dimensional $m_1 = 1$.

4. The set of Dirichlet eigenvalues is discrete and tends to infinity. The eigenvalues can be ordered

$$\text{spec}(\Omega) = \{ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \}$$

5. Let $u_i$ denote the $\lambda_i$ eigenfunction. If $\lambda_i \neq \lambda_j$ then $u_i$ and $u_j$ are orthogonal. By adjusting bases in the eigenspaces $\mathcal{E}_\lambda$ we may assume $\{ u_1, u_2, \ldots \}$ is a complete orthonormal basis in $L^2(\Omega)$. 

Proof Sketch. To see orthogonality (5), suppose \( \lambda_i \neq \lambda_j \) and \( u_i \) and \( u_j \) are corresponding eigenfunctions. Then

\[
(\lambda_i - \lambda_j) \int_M u_i u_j = \int_M - (\Delta u_i) u_j + u_i \Delta u_j = 0
\]

by Green’s formula.

Since eigenfunction \( u_j \) satisfy on \((M, g)\)

\[
\Delta u_j + \lambda_j u_j = 0,
\]

(2)
eigenvalues scale like \( \frac{1}{\text{distance}^2} \). So if we scale the lengths of curves by a factor \( s \) on the manifold by multiplying the metric, \( s^2 g \), then the eigenvalue becomes

\[
\lambda_j(M, s^2 g) = \frac{\lambda_j(M, g)}{s^2}.
\]

“Bigger tambourines have lower tones.”
For example in the rectangle $R = [0, a] \times [0, b] \subset \mathbb{R}^2$, the functions

$$u(x, y) = \sin \left( \frac{\pi k x}{a} \right) \sin \left( \frac{\pi \ell y}{b} \right)$$

with $k, \ell \in \mathbb{N}$ satisfy $\Delta u + \lambda u = 0$ with

$$\lambda = \pi^2 \left( \frac{k^2}{a^2} + \frac{\ell^2}{b^2} \right).$$

These turn out to be all the eigenfunctions. So $\lambda_1 = \pi^2 \left( \frac{1}{a^2} + \frac{1}{b^2} \right)$.

Note that if the area is fixed $ab = A$ then $\lambda_1$ is minimized when $R$ is a square and $a = b$. 

10. Eigenvalues of a rectangle.
A complete set of eigenfunctions of $S^1_a$, the circle of length $a$ are generated by

$$f(\theta) = A \cos\left(\frac{2\pi j \theta}{a}\right) + B \sin\left(\frac{2\pi j \theta}{a}\right)$$

so

$$\text{spec}(S^1_a) = \left\{ \frac{4\pi^2}{a^2} j^2 : j \in \mathbb{Z} \right\}$$
12. Example: Unit sphere $S^n$.

The sphere is the hypersurface $S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \}$ with the induced metric. Using spherical coordinates $\theta \in S^n$ and $r \geq 0$, the Laplacian $\Delta_{\mathbb{R}^{n+1}}$ in $\mathbb{R}^{n+1}$ may be expressed in terms of the spherical Laplacian $\Delta_\theta$

$$\Delta_{\mathbb{R}^{n+1}} = \frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\theta.$$

A homogeneous functions of degree $d$ satisfies $u(r\theta) = r^d u(\theta)$ for all $\theta$ and $r \geq 0$. It turns out that harmonic homogeneous polynomials restrict to a complete set of eigenfunctions of the sphere. Indeed if $\Delta_{\mathbb{R}^{n+1}} u = 0$ and $u$ is homogeneous of degree $d$, then

$$0 = \Delta_{\mathbb{R}^{n+1}} u = d(d - 1)r^{d-2}u + ndr^{d-2}u + r^{d-2} \Delta_\theta u.$$

Thus on the sphere, $r = 1$ so

$$0 = \Delta_\theta u + d(d + n - 1)u.$$
Thus on the sphere $S^n$, for $d = 0, 1, 2, \ldots$,

$$\lambda_d = d(d + n - 1).$$

The dimension of the harmonic polynomials of degree $d$ gives the multiplicity

$$m_d = \binom{n+d}{d} - \binom{n+d-2}{d-2}.$$ 

For example if $n = 1$ then $m_0 = 1$ and $m_d = 2$ for $d \geq 1$ corresponding to Fourier series. For example $\Re e(z^d)$ is a harmonic polynomial that restricts to $u(\theta) = \cos(d\theta)$ on $S^1$. 

If $n = 2$ then $m_d = 2d + 1$. For example, the coordinate function $u(x_1, x_2, x_3) = x_1$ is harmonic homogeneous of degree one that restricts to an eigenfunction with $\lambda_1 = 2$. Its multiplicity is three, corresponding to the three coordinates.

$$\text{spec}(S^2) = \{0, 2, 2, 2, 6, 6, 6, 6, 6, 6, 12, \ldots, 12, 20, \ldots\}$$
15. Eigenvalues minimize the Rayleigh Quotient.

Since eigenfunction \( U_j \) satisfy

\[
\begin{aligned}
\Delta_n U_j + \lambda_j U_j &= 0 & \text{for } x \in D_n \\
U_j &= 0 & \text{if } x \in \partial D_n.
\end{aligned}
\]  

(3)

they scale like \( \frac{1}{\text{distance}^2} \). So for \( D_n \subset \mathbb{R}^n \), \( \lambda_n(sD_n) = \frac{\lambda_n(D_n)}{s^2} \).

The first eigenvalue has a variational characterization. \( U_1 \) minimizes the Rayleigh Quotient

\[
\lambda_1 = \inf_{\substack{u \in H^1_0(D_n), \\ u \neq 0}} \frac{\int_{D_n} |du|^2}{\int_{D_n} u^2} := \inf_{u} \mathcal{R}(u)
\]
16. Properties of eigenvalues that follow from Rayleigh Quotient

- **UPPER BOUND PRINCIPLE:**
  If $0 \not\equiv f \in H_0^1(\Omega)$ then $\lambda_1(\Omega) \leq \mathcal{R}(f)$.

- **NODAL DOMAIN PRINCIPLE:**
  If $U$ satisfies $\Delta_n U + \mu U = 0$ in $\mathcal{D}_n$, $U = 0$ on $\partial\mathcal{D}_n$ and $\Omega \subset \mathcal{D}_n$ is a nodal domain (component of $U^{-1}((0, \infty))$) then $u_1 = U$ is a first eigenfunction of $\Omega$ and $\mu = \lambda_1(\Omega)$.

- **MONOTONICITY PRINCIPLE:**
  If $\Omega_1 \subset \Omega_2$ then $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$.
  (Hence $U_1 > 0$ in $\mathcal{D}_n$.)
The domain $\Omega$ is a nodal domain of the cubic spherical harmonic

$$h(x, y, z) = xyz.$$ 

Thus

$$\lambda_1(\Omega) = 12.$$ 

**Figure:** First Octant Triangle $\Omega \subset S^2$
18. Symmetrization lowers the first eigenvalue.

**Faber-Krahn/Spener Inequality:**
For nice $\Omega \subset S^n$ or $\Omega \subset \mathbb{R}^n$. If $|B_{R^*}| = |\Omega|$ then $\lambda_1(B_{R^*}) \leq \lambda_1(\Omega)$. 
“$=$” implies $\Omega$ is isometric to $B_{R^*}$. Rayleigh (1877) for analytic disk near round disk. Faber-Krahn (1923) for $\Omega \subset \mathbb{R}^n$. Sperner (1955) for $\Omega \subset S^n$.

Proved by symmetrization argument and Isoperimetric Inequality.
Let $u > 0$ be the first eigenfunction of $\Omega$. Let $u^*$ be the spherical rearrangement, *i.e.*, $u^*(x) = u^*(|x|)$ is defined on $\Omega^* = B_{R^*}$ such that

$$|\{x \in \Omega : u(x) > t\}| = |\{x \in \Omega^* : u^*(x) > t\}|$$

for all $t > 0$.

Then

$$\int_{\Omega} u^2 = \int_{\Omega^*} (u^*)^2 \quad \text{but} \quad \int_{\Omega} |du|^2 \geq \int_{\Omega^*} |du^*|^2$$
The domain $\Omega$ is not a nodal domain. Odd reflection does not produce an eigenfunction on $S^2$. It does lift to an eigenfunction of the two-fold cover with branching at the vertices.

Numerical computation

$$\lambda_1(\Omega) \approx 5.159\ldots$$

Let $\Omega^*$ be spherical cap with same area $|\Omega| = |\Omega^*|$. Then

$$\lambda_1(\Omega^*) = 4.93604187$$

Figure: Tetrahedral Triangle $\Omega \subset S^2$
Let $X_1(t), \ldots, X_n(t)$ be $n$ predators, $X_0(t)$ the prey, all doing independent standard Brownian motions on $\mathbb{R}$.

Suppose the predators start to the left of the prey:

$$X_j(0) < X_0(0) \quad \text{all } j = 1, \ldots, n.$$ 

The capture time is defined to be

$$\tau_n = \inf\{t > 0 : \exists j : X_j(t) \geq X_0(t)\}$$

**Conjecture (Bramson, Griffeath 1991)**

$$\mathbb{E}\tau_n = \infty \text{ for } n = 1, 2, 3 \quad \text{and} \quad \mathbb{E}\tau_n < \infty \text{ for } n \geq 4.$$  

Bramson & Griffeath gave a proof for $n \leq 3$ & did extensive simulation.
### Theorem (H. Kesten 1992)

\[ \mathbb{E} \tau_n < \infty \text{ for } n \gg 1. \]

### Theorem (W. Li & Q. M. Shao, 2001)

\[ \mathbb{E} \tau_n < \infty \text{ for } n \geq 5. \]

### Theorem (J. Ratzkin & T.)

\[ \mathbb{E} \tau_4 < \infty. \]
Then
\[ \mathbf{X}(t) = (X_0(t), \ldots, X_n(t)) \in \mathbb{R}^{n+1} \]
is an \((n + 1)\)-dimensional Brownian Motion in the cone
\[ C_{n+1} = \{(X_0, \ldots, X_n) \in \mathbb{R}^{n+1} : X_0 > X_i \text{ all } i = 1, \ldots, n\} \].

Its spherical angle is
\[ D_n = C_{n+1} \cap S^n. \]

Initial data \( \mathbf{X}(0) = \mathbf{b} \in (C_{n+1})^\circ \). Capture time becomes
\[ \tau_n(\mathbf{b}) = \inf\{t > 0 : \mathbf{X}(t) \notin C_{n+1}\} \]
23. Two predator cone $C_3$ and angle $D_2$
24. Capture probability depends on the eigenvalue.

**Theorem (De Blassie 1987)**

\[
P_b(\tau_n > t) \sim C(b)t^{-a} \quad \text{as } t \to \infty
\]

where \( a = a(n) \) depends on \( \lambda_1(D_n) \), the first Dirichlet eigenvalue, and \( D_n = C_{n+1} \cap S^n \) is the spherical angle of the cone.

\[
2a(n) = \left\{ \left( \frac{n-1}{2} \right)^2 + \lambda_1(D_n) \right\}^{\frac{1}{2}} - \frac{n-1}{2}
\]

Hence

\[
E\tau_n < \infty \quad \text{iff} \quad a(n) > 1 \quad \text{iff} \quad \lambda_1(D_n) > 2n + 2.
\]
Theorem (De Blassie 1987)

\[ \mathbb{P}_b(\tau_n > t) \sim C(b)t^{-a} \quad \text{as } t \to \infty \]

Bramson & Griffeath gave a proof for \( n \leq 3 \). They found by extensive simulation

\[ a(3) \approx 0.91 \quad \text{and} \quad a(4) \approx 1.032. \]

We prove

\[ 0.90671950 < a(3) < 0.995648748 \quad \text{and} \quad a(4) > 1.00007318. \]

Moreover, our numerical calculation gives

\[ a(3) \approx 0.9128... \quad \text{and} \quad a(4) \approx 1.0057... \]
Example. 1. $C_2 = \{(X_0, X_1) : X_0 > X_1\}$ is a halfplane so

$$D_1 = \left\{ (\cos \phi, \sin \phi) : -\frac{3}{4} \pi \leq \phi \leq \frac{1}{4} \pi \right\} \approx \left[ -\frac{3}{4} \pi, \frac{1}{4} \pi \right]$$

so $\lambda_1(D_1) = 1 \leq 4$ so $\mathbb{E}\tau_1 = \infty$. 
27. Capture probabilities satisfy the heat equation.

Spitzer (1958) estimated probability in cones of $\mathbb{R}^2$.

Use Burkholder’s (1977) PDE method. $u(x, t) = \mathbb{P}_x(\tau_n > t)$ satisfies the heat equation.

\[
\begin{align*}
 u_t &= \frac{1}{2} \Delta u \\
 u(x, 0) &= 1 \\
 u(x, t) &= 0
\end{align*}
\]  

$(x, t) \in \mathcal{C}_{n+1} \times [0, \infty)$  

$x \in \mathcal{C}_{n+1}$  

$(x, t) \in \partial \mathcal{C}_{n+1} \times (0, \infty)$
Write cone in polar coordinates \( r = |\mathbf{x}|, \theta = \frac{\mathbf{x}}{|\mathbf{x}|} \in \mathcal{D}_n \). Equation becomes

\[
2u_t = u_{rr} + \frac{n}{r} u_r + \frac{1}{r^2} \Delta_n u
\]

where \( \Delta_n \) is the Laplacian on \( \mathbb{S}^n \). Since there is self-similarity, look for solutions by separating variables \( p(r, \theta, t) = R(\xi)U(\theta) \) where \( \xi = \frac{r^2}{2t} \)

\[
\lambda_n(\mathcal{D}_n) = -\frac{\Delta_n U}{U} = \frac{4\xi^2 \ddot{R} + (4\xi^2 + 2(n + 1)\xi) \dot{R}}{R}
\]
Let $R(\xi) = \xi^a \rho(-\xi)$. Setting $\eta = -\xi$ the $\rho$ satisfies the Confluent Hypergeometric Equation:

$$
\eta \frac{\partial^2 \rho}{\partial \eta^2} + \left( 2a + \frac{n+1}{2} - \eta \right) \frac{\partial \rho}{\partial \eta} - a \rho = 0
$$

so

$$
\rho(\xi) = _1 \! F_1 \left( a; 2a + \frac{n+1}{2}; -\xi \right)
$$

where

$$
_1 \! F_1 (\alpha; \beta; z) = 1 + \frac{\alpha}{\beta} \frac{z}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} \frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} \frac{z^3}{3!} + \cdots
$$
Exit time from cone $C_{n+1} \subset \mathbb{R}^{n+1}$. Argue formal series

$$P_x(\tau_n > t) = \sum_{j=1}^{\infty} B_j \mathbf{1} \mathbf{F}_1\left( a_j, 2a_j + \frac{n+1}{2}, -\frac{|x|^2}{2t}\right) U_j\left( \frac{x}{|x|}\right) \left( \frac{|x|^2}{2t}\right)^{a_j}$$

converges uniformly on $K \times [T, \infty)$, where $K \subset \subset D_n$ and $T > 0$. Here

$$\Delta_n U_j + \lambda_j U_j = 0 \quad \text{for } x \in D_n$$

$$U_j = 0 \quad \text{if } x \in \partial D_n.$$ 

and

$$2a_j(n) = \left[ \left( \frac{n-1}{2}\right)^2 + \lambda_j(D_n) \right]^{\frac{1}{2}} - \frac{n-1}{2}.$$
31. Long time asymptotics.

Decay rate is given by

$$2a_j(n) = \left[ \left( \frac{n - 1}{2} \right)^2 + \lambda_j(D_n) \right]^{\frac{1}{2}} - \frac{n - 1}{2}.$$

**Corollary**

$$P_x(\tau_n > t) \sim B_1(U_1\left(\frac{x}{|x|}\right) \left(\frac{|x|^2}{2t}\right)^{a_1}.$$

Hence  \( \mathbb{E}\tau_n < \infty \)  iff  \( a = a_1 > 1 \)  iff  \( \lambda_1(D_n) > 2n + 2 \).
32. Geometry of the cone

- Cone splits line \( \mathcal{L} \)
  (all coordinates equal)

\[
C_{n+1} = \{ \mathbf{X} \in \mathbb{R}^{n+1} : X_i < X_0, \quad \forall i > 0 \} = \mathcal{L} \oplus \mathcal{V}_n
\]

where \( \mathcal{L} = \mathbb{R}(1, 1, \ldots, 1) \)

- Perpendicular part of the cone

\[
\mathcal{V}_n = C_{n+1} \cap (1, 1, \ldots, 1)^\perp.
\]

- Perp. part of cone angle

\[
\mathcal{T}_{n-1} = \mathcal{V}_n \cap \mathbb{S}^{n-1} = \mathcal{V}_n \cap \mathcal{D}_n.
\]

Dimension reduction:
suffices to estimate \( \mathcal{T}_{n-1} \).
\( \mathcal{T}_{n-1} \) is the face of the regular \((n + 1)\)-hedral tesselation in \( \mathbb{S}^{n-1} \).

At vertex \( v \in \mathcal{T}_{n-1} \), spherical angle of \( \mathcal{T}_{n-1} \) is \( \mathcal{T}_{n-2} \subset T_v \mathbb{S}^{n-1} \).

Let \( \mathbf{N} \in \mathbb{S}^n \cap \mathcal{L} \) and regard \( \mathbb{S}^{n-1} \subset T_{\mathbf{N}} \mathbb{S}^n \). In polar coordinates \( x = (r, \theta) \in \mathbb{S}^n \) where \( \theta \in \mathbb{S}^{n-1} \) and \( 0 \leq r \leq \pi \) and \( r = \text{dist}(x, \mathbf{N}) \).

The \( R_0 \)-truncated cone of any domain \( \mathcal{T}_{n-1} \subset \mathbb{S}^{n-1} \) is

\[
\text{Sect}_n(\mathcal{T}_{n-1}, R_0) = \{(r, \theta) \in \mathbb{S}^n : \theta \in \mathcal{T}_{n-1} \text{ and } 0 \leq r \leq R_0\}.
\]

\( \mathcal{D}_n = \text{Sect}_n(\mathcal{T}_{n-1}, \pi) \).
Let \( \mathbf{N} \in \mathbb{S}^n \), \( r = \text{dist}_{\mathbb{S}^n}(\cdot, \mathbf{N}) \) and \( \theta \in \mathbb{S}^{n-1} \subset T_{\mathbf{N}}\mathbb{S}^n \).

The Laplacian on \( u \in C^2(\mathbb{S}^n) \),

\[
\Delta_n u = \frac{\partial^2 u}{\partial r^2} + (n - 1) \cot r \frac{\partial u}{\partial r} + \csc^2 r \Delta_{n-1} u,
\]

**Lemma (Eigenvalues of domain in great sphere & of its suspension)**

If \( \mathcal{D}_n = \text{Sect}_n(T_{n-1}, \pi) \) then

\[
\lambda_1(\mathcal{D}_n) = \lambda_1(T_{n-1}) - \frac{n - 2}{2} + \sqrt{\frac{(n - 2)^2}{4}} + \lambda_1(T_{n-1}).
\]

In particular,

\[
\mathbb{E}\tau_n < \infty \quad \text{iff} \quad \lambda_1(\mathcal{D}_n) > 2n + 2 \quad \text{iff} \quad \lambda_1(T_{n-1}) > 2n.
\]
Example. Two policemen

If \( n = 2 \) then

\[
\mathcal{T}_1 \cong \left[ 0, \frac{2}{3} \pi \right] \cong \left\{ e^{i\phi} : 0 \leq \phi \leq \frac{2}{3} \pi \right\} \subset S^1
\]

then the eigenfunction on \( \mathcal{T}_1 \) is

\[
u_1 = \sin \left( \frac{3}{2} \theta \right) \quad \implies \quad \nu_1'' + \frac{9}{4} \nu_1 = 0 \quad \implies \quad \lambda_1 = \frac{9}{4} \leq 4 \quad \implies \quad \mathbb{E} \tau_2 = \infty.
\]
36. Two predator cone $C_3$ and angle $D_2$
Proof. Let $\mu = \lambda_1(D_n)$. Put $u(r, \theta) = R(r)u(\theta)$, where $R(0) = R(\pi) = 0$ and $u(\theta) = 0$ whenever $\theta \in \partial T_{n-1}$. Then

$$\frac{\sin^2 r \ddot{R} + (n - 1) \sin r \cos r \dot{R} + \mu \sin^2 r R}{R} = \lambda = -\frac{\Delta_{n-1} u}{u},$$

so $\lambda = \lambda_1(T_{n-1})$ and $u(\theta)$ is its first eigenfunction. Then

$$\sin^2 r \ddot{R} + (n - 1) \sin r \cos r \dot{R} + (\mu \sin^2 r - \lambda) R = 0,$$

Hence

$$R(r) = \sin^m r \quad \text{where} \quad m = \frac{2 - n}{2} + \sqrt{\frac{(2 - n)^2}{4} + \lambda}$$

$$\implies \mu = \lambda + m$$

Since $\mu$ is increasing in $\lambda$, we solve for $\lambda$ when $\mu = 2n + 2$. Answer: $\lambda = 2n$. \qed
In case $n = 3$ then $\mathcal{T}_2$ is a triangle. Let

$$\varphi(x) = \sin(\text{dist}(x, \partial \mathcal{T}_2)).$$

Then by the upper bound principle,

$$\lambda_1(\mathcal{T}_2) \leq \frac{\int_{\mathcal{T}_2} |d\varphi|^2}{\int_{\mathcal{T}_2} \varphi^2} = \frac{2\pi + \sqrt{3}}{\pi - \sqrt{3}} \approx 5.68641 \leq 6$$

$$\implies \mathbb{E} \tau_3 = \infty.$$ 

Hence $a(3) < .995649.$
39. Five or more predators.

**Theorem (W. Li & Q. M. Shao, 2001)**

\[ \mathbb{E} \tau_n < \infty \quad \text{for} \quad n \geq 5. \]

**Proof Idea.** Let \( B_{cr}^{n-1} \subset \mathbb{S}^{n-1} \) satisfy \( \lambda_1(B_{cr}^{n-1}) = 2n \). Let \( B_R^{n-1} \) satisfy \( |B_R^{n-1}| = |T_{n-1}| \). Suppose that \( B_R^{n-1} \subset B_{cr}^{n-1} \). By the Faber-Krahn/Sperner Inequality and the monotonicity principle,

\[ \lambda_1(T_{n-1}) > \lambda_1(B_{R*}^{n-1}) \geq \lambda_1(B_{cr}^{n-1}) = 2n \quad \implies \quad \mathbb{E} \tau_n < \infty. \]

Li & Shao show that \( B_{R*}^{n-1} \) is smaller than \( B_{cr}^{n-1} \) iff \( n \geq 5 \). Compare radii. \( R^* \) satisfies

\[ |T_{n-1}| = \frac{|\mathbb{S}^{n-1}|}{n + 1} = |B_{R*}^{n-1}| = |\mathbb{S}^{n-2}| \int_0^{R^*} \sin^{n-2} \rho \, d\rho. \]
40. $R_{cr}$ computation

Luckily, $R_{cr}$ is easy! A harmonic function on $\mathbb{R}^n$ restricts to an eigenfunctions on $S^{n-1}$: in polar coordinates $(r, \theta) \in \mathbb{R}^n$, the function

$$h(x_1, \ldots, x_n) = (n - 1)x_1^2 - x_2^2 - \cdots - x_n^2$$

is homogeneous $h(r\theta) = r^2 h(\theta)$ and is harmonic $\Delta h = 0$ so

$$0 = h_{rr} + \frac{n - 1}{r} h_r + \frac{1}{r^2} \Delta_{n-1} h$$

$$= 2h + 2(n - 1)h + \Delta_{n-1} h$$

$$= \Delta_{n-1} h + 2nh.$$ 

The nodal domain is a ball $B_{cr}^{n-1}$ with $\lambda_1(B_{cr}^{n-1}) = 2n$. Its radius is $R_{cr} = \text{Atn} \sqrt{n - 1}$. 
For all $n$, $\lambda_1(\mathcal{T}_n) > \lambda_1(B^n_{R^*})$.

| $n$ | $|\mathcal{T}_n| = |B^n_{R^*}|$ | $R^*$ | $\lambda_1(B^n_{R^*})$ | $R_{cr} = \text{Atn} \left( \sqrt{n - 1} \right)$ |
|-----|---------------------|-------|-------------------|-----------------------------|
| 2   | 3.141592654         | 1.047197551 | 4.93604187       | 0.78539816                 |
| 3   | 3.947841762         | 1.056569480 | 7.84104544       | 0.95531662                 |
| 4   | 4.386490846         | 1.068200504 | 10.8876959       | 1.04719755                 |
| 5   | 4.429468100         | 1.080033938 | 14.0396033       | 1.10714872                 |
The $I_{n-1}$ bulge in the middle. The diameter is the distance from a vertex of $I_{n-1}$ to the center of the opposite face.

$$\delta(n-1) = \text{diam}(I_{n-1}) = \arccos \left( -\sqrt{\frac{n-1}{2n}} \right).$$

Since the spherical angle at a vertex of $I_n$ is $I_{n-1}$, we can construct outer comparison domains inductively

$$\hat{I}_1 = I_1 = [0, \frac{2}{3}\pi]$$

$$\hat{I}_n = \text{Sect}_n \left( \hat{I}_{n-1}, \delta(n) \right) \quad \text{for } n \geq 2$$

By induction, and the monotonicity principle, for all $n$,

$$I_n \subset \hat{I}_n \implies \lambda_1(I_n) \geq \lambda_1(\hat{I}_n).$$

Similarly, we construct inner comparison domains $\bar{I}_n$.
43. Table of eigenvalues of $\tilde{T}$ and $\hat{T}$

\[ \mathbb{E} \tau_n < \infty \iff \lambda_1(\mathcal{T}_{n-1}) > 2n \iff \lambda_1(\hat{T}_{n-1}) > 2n. \]

Computed using the Truncated Cone Lemma. ($\tilde{T}_n \subset T_n \subset \hat{T}_n$.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{vol}(\tilde{T}_n)$</th>
<th>$\lambda_1(\tilde{T}_n)$</th>
<th>$\text{vol}(\hat{T}_n)$</th>
<th>$\lambda_1(\hat{T}_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.094395103</td>
<td>2.2500000000</td>
<td>2.094395103</td>
<td>2.2500000000</td>
</tr>
<tr>
<td>2</td>
<td>2.792526804</td>
<td>6.195617753</td>
<td>3.303594680</td>
<td>5.004635381</td>
</tr>
<tr>
<td>3</td>
<td>2.884035172</td>
<td>12.04009682</td>
<td>4.482940454</td>
<td>7.884040724</td>
</tr>
<tr>
<td>4</td>
<td>2.491806389</td>
<td>19.93880798</td>
<td>5.445852727</td>
<td>10.77018488</td>
</tr>
<tr>
<td>5</td>
<td>1.877352230</td>
<td>30.01419568</td>
<td>6.039182278</td>
<td>13.62031916</td>
</tr>
</tbody>
</table>
Lemma (Truncated Cone Eigenvalues.)

$T_{n-1} \subset \mathbb{S}^{n-1}$ is nice, proper so $\lambda = \lambda_1(T_{n-1})$ and $0 < r < \pi$. Then

$$\lambda_1(\text{Sect}_n(T_{n-1}, r)) = \mu_1(n, \lambda_1(T_{n-1}), r)$$

where $\mu_1$ is the least $\mu > 0$ so the solution $R(\rho)$ of the equation

$$\sin^2 \rho \ddot{R} + (n - 1) \sin \rho \cos \rho \dot{R} + (\mu \sin^2 \rho - \lambda) R = 0.$$  

is positive on $(0, r)$ and $R(r) = 0$. If $r \geq \frac{\pi}{2}$, it is the unique $\mu \in (m + \lambda, 3m + \lambda + n)$ such that $2F_1(\alpha_1, \beta_1; \gamma_1; \frac{1}{2}(1 - \cos r)) = 0$, where $2F_1(\alpha, \beta, \gamma, z)$ is the hypergeometric function, and

$$\alpha_1, \beta_1 = \frac{1 + \sqrt{(n-2)^2 + 4\lambda} \pm \sqrt{(n-1)^2 + 4\mu}}{2},$$

$$\gamma_1 = \frac{2 + \sqrt{(n-2)^2 + 4\lambda}}{2}.$$
Lemma (Spherical Cap Eigenvalues.)

For the ball $B^n_r \subset S^n$ the first eigenvalue is given by $\lambda_1(B^n_r) = \mu_2(n, r)$ where $\mu_2$ is the least $\mu > 0$ so a solution $R(\rho)$ of the equation

$$\sin^2 \rho \ddot{R} + (n - 1) \sin \rho \cos \rho \dot{R} + \mu \sin^2 \rho R = 0.$$ 

is positive on $(0, r)$ and $R(r) = 0$. If $r \leq \frac{\pi}{2}$, it can be computed as the unique value $\mu \in (0, n)$ such that

$$2F_1 \left( \alpha_2, \beta_2; \gamma_2; \frac{1}{2}(1 - \cos r) \right) = 0,$$

$$\alpha_2, \beta_2 = \frac{n-1 \pm \sqrt{(n-1)^2 + 4\mu}}{2}, \quad \gamma_2 = \frac{n}{2}.$$ 

For the spherical cap $B^n_r$, the radial eigenfunction $u(\theta) \equiv 1$ and $\lambda = 0$. 
46. Proof of the lemmata.

**Proof.** Let $R(r) = \sin^m(r) u(r)$ on $[0, R_0]$ where

$$m = -\frac{n-2}{2} + \sqrt{\frac{(n-2)^2}{4} + \lambda_1(T_{n-1})}.$$ 

$u(t) \neq 0$ for $t \in [0, R_0]$ but $u(R_0) = 0$. Substituting $u(r) = y(x)$ where $x = \frac{1}{2}(1 - \cos r)$ and writing "′" for $\frac{\partial}{\partial x}$ yields

$$x(1 - x) y'' + (m + \frac{1}{2}n - (2m + n)x) y' - (\lambda + m - \mu) y = 0.$$ 

Solution is the hypergeometric function $y(x) = {}_2F_1(\alpha, \beta; \gamma; x)$, taking

$$\alpha, \beta = \frac{2m+n-1 \pm \sqrt{(2m+n-1)^2 - 4\lambda - 4m + 4\mu}}{2},$$

$$\gamma = \frac{2m+n}{2},$$

Thus $R(r) = \sin^m r \ {}_2F_1(\alpha, \beta; \gamma; \frac{1}{2}(1 - \cos r))$ and $\mu$ is chosen so that $R(R_0) = 0$. The eigenvalue of the ball is gotten by a similar analysis.
Gauß's ordinary hypergeometric function is given by

\[ _2F_1(\alpha, \beta; \gamma; z) = \]

\[ 1 + \frac{\alpha \beta}{\gamma} \frac{z}{1!} + \frac{\alpha (\alpha + 1) \beta (\beta + 1)}{\gamma (\gamma + 1)} \frac{z^2}{2!} + \frac{\alpha (\alpha + 1) (\alpha + 2) \beta (\beta + 1) (\beta + 2)}{\gamma (\gamma + 1) (\gamma + 2)} \frac{z^3}{3!} + \ldots \]

Eigenfunctions of truncated cones on spheres can also be represented by other special functions.

These functions are regarded as known since they are canned in Maple. Finding a parameter that zeros an expression involving these functions is accomplished by simple root search.
The critical 2d estimate that implies the desired 3d estimate.

\[ \mathbb{E} \tau_4 < \infty \iff \lambda_1(T_3) > 8 = 2 \cdot 4. \]

For any domain \( Q_2 \subset S^2 \),

\[ \lambda_1(\text{Sect}_3(Q_2, \delta(3))) > 8 \iff \lambda_1(Q_2) > \lambda_{cr} = 5.101267527. \]

Using our PW eigenvalue estimate, we show that \( \lambda_1(T_2) \geq 5.11641465 \).

so that \( \lambda_1(T_3) > \lambda_1(\hat{T}_3) > 8 \).

In [RT], we found a domain \( Q_2 \subset S^2 \) such that \( T_2 \subset Q_2 \) and \( \lambda(Q_2) = 5.102 \). Thus

\[ T_3 \subseteq \hat{T}_3 \subseteq \text{Sect}_3(Q_2, \delta(3)). \]

and

\[ \lambda_1(T_3) \geq \lambda_1(\hat{T}_3) \geq \lambda_1(\text{Sect}_3(Q_2, \delta(3))) = 8.000878153. \]
49. Numerical calculation of $\lambda_1(T_2)$. (Test if idea could work.)

First pulling $T_2$ back to a rectangle in $\mathbb{R}^2$ by a conformal map and then using a sinc-collocation method, we find by

**Numerical result:** $\lambda_1(T_2) \approx 5.159 \ldots > \lambda_{cr}$  YES!

Thus the numerical values of the critical numbers are

$$\lambda_1(T_3) > \lambda_1(\text{Sect}_3(T_2, \delta(3))) \approx 8.000878153$$

so

$$a(3) \approx .9128\ldots \quad \text{and} \quad a(4) \rightarrow >1.0057\ldots$$

This provides a **numerical verification of the conjecture** $E_{\tau_4} < \infty$. 
Faber-Krahn type argument: apply isoperimetric inequality to level sets.

**Theorem (Payne-Weinberger 1960)**

Suppose that $\Omega \subset \mathbb{R}^2$ is a subdomain in the wedge $\mathcal{W} = \{(\rho, \theta): 0 \leq \rho, \ 0 \leq \theta \leq \pi/\alpha\}$, where $\alpha > 1$. Then

\[
\lambda_1(\Omega) \geq \lambda_1(\text{Sect}_2([0, \pi/\alpha], r))
\]

where $r$ is chosen so that for $w = \rho^\alpha \sin \alpha \theta$,

\[
\int_{\Omega} w^2 \, da = \int_{\text{Sect}_2([0, \pi/\alpha], r)} w^2 \, da
\]

In fact

\[
\lambda_1(\text{Sect}_2([0, \pi/\alpha], r)) = \left\{ \frac{4\alpha(\alpha+1)}{\pi} \int_G w^2 \, da \right\} - \frac{1}{\alpha+1} j_\alpha^2
\]

where $j_\alpha$ is the smallest zero of the Bessel function $J_\alpha$. 
(\rho, \theta) \text{ are polar coordinates of } S^2 \text{ with metric}

\[ ds^2 = d\rho^2 + \sin^2 \rho \, d\theta^2. \]

Sector in \( S^2 \) of angle \( \pi/\alpha \), for \( \alpha > 1 \)

\[ \mathcal{W} = \{(\rho, \theta) : 0 \leq \theta \leq \pi/\alpha, \ 0 \leq \rho < \pi\} \]

Let \( G \) be a domain such that \( \overline{G} \subset \mathcal{W} \) is compact.

Truncated sector

\[ S(r) := \{(\rho, \theta) : 0 \leq \theta \leq \pi/\alpha, \ 0 \leq \rho \leq r\} \]

A positive harmonic function in \( \mathcal{W} \), with zero boundary values

\[ w = \tan^{\alpha}(\frac{\rho}{2}) \sin \alpha \theta \]
Theorem for spherical wedge domains

For every subdomain $G$ with compact $\overline{G} \subset \mathcal{W}$, we have the estimate

$$\lambda_1(G) \geq \lambda_1(S(r^*)),$$

where $r^*$ is chosen such that

$$\int_G w^2 \, da = \int_{S(r^*)} w^2 \, da.$$

Equality holds if and only if $G$ is the sector $S(r^*)$. 

(5)
Szegő’s Lemma

Lemma

Let \( \psi, \phi : [0, \omega) \to [0, \infty) \) be locally integrable functions with \( \psi \) nonnegative and \( \phi \) nondecreasing. Let \( \Phi(y) = \int_0^y \phi(t) \, dt \) and \( \Psi(x) = \int_0^x \psi(s) \, ds \) be their primitives. Let \( E \subset [0, \omega) \) be a bounded measurable set. Then

\[
\Phi \left( \int_E \psi(x) \, dx \right) \leq \int_E \phi(\Psi(x)) \psi(x) \, dx. \tag{6}
\]

For \( \phi \) increasing, equality holds if and only if the measure of \( E \cap [0, R] \) is \( R \).

For example, if \( 0 \leq r_1 \leq r_2 \leq r_3, \leq \ldots, \leq r_{2n} \), by choosing \( \phi = py^{p-1} \) some \( p > 1 \) and \( \psi(x) = 1 \),

\[
\left( \sum_{i=1}^{2n} (-1)^i r_i \right)^p \leq \sum_{i=1}^{2n} (-1)^i (r_i)^p.
\]
Change variables $y = \Psi(x)$. $dy = \psi(x)dx$. Let $E'$ be the image of $E$ under the map $\Psi$. Because $\phi$ is nondecreasing, for $y \geq 0$,

$$
\phi \left( \int_{0}^{y} \chi_{E'} dy \right) \leq \phi(y).
$$

For $\phi$ increasing, equality holds iff $\mu(E' \cap [0, y]) = y$. Multiply by $\chi_{E'}$ and integrate:

$$
\Phi \left( \int_{E} \psi(x) dx \right) = \Phi \left( \int_{E'} dy \right)
= \int_{0}^{\omega} \phi \left( \int_{0}^{y} \chi_{E'} dt \right) \chi_{E'} dy
\leq \int_{0}^{\omega} \phi(y) \chi_{E'} dy
= \int_{E'} \phi(y) dy = \int_{E} \phi(\Psi(x)) \psi(x) dx.
$$
Let $G \subset \mathcal{W}$ be a domain with compact closure. Then there is a function $\Upsilon_\alpha = \mathcal{F} \circ Z^{-1}$ so that

$$\int_{\partial G} w^2 \, ds \geq \frac{\pi}{2\alpha} \Upsilon_\alpha \left( \frac{2\alpha}{\pi} \int_G w^2 \, da \right).$$

Here $\mathcal{F}(\rho) = \tan^{2\alpha}(\rho/2) \sin \rho$ and $Z$ is given by

$$Z(r) = \int_0^r \tan^{2\alpha} \left( \frac{\rho}{2} \right) \sin \rho \, d\rho.$$

Equality holds if and only if $G$ is a sector $S(r)$. 

**Lemma**
Map the domain $G$ into a domain $\tilde{G}$ in the upper halfplane using
\[ x = f(\rho) \cos \alpha \theta, \quad y = f(\rho) \sin \alpha \theta, \]
The Euclidean line element is $dx^2 + dy^2 = \dot{f}^2 d\rho^2 + \alpha^2 f^2 d\theta^2$. We claim for some $f$ the map satisfies
\[ \alpha^2 \tan^{4\alpha} \left( \frac{\rho}{2} \right) \sin^4 \alpha \theta (d\rho^2 + \sin^2 \rho d\theta^2) \geq y^4 (dx^2 + dy^2). \]
For this to be true pointwise, we need the inequalities to hold
\[ \alpha \tan^{2\alpha} \left( \frac{\rho}{2} \right) \geq f^2 \dot{f} = \left( \frac{f^3}{3} \right)', \tag{7} \]
\[ \sin \rho \tan^{2\alpha} \left( \frac{\rho}{2} \right) \geq f^3. \tag{8} \]
Use equality in inequality (8) to define $f = \tan^{\frac{2\alpha}{3}} \left( \frac{\rho}{2} \right) \sin^{\frac{1}{3}} \rho$.
Differentiating,
\[ f^2 \dot{f} = \frac{1}{3} \tan^{2\alpha} \left( \frac{\rho}{2} \right) [2\alpha + \cos \rho], \]
which implies that the inequality (7) holds as well.
57. Pull back the variational problem for moments of inertia to $W$.

Among $\tilde{G}$ in the upper halfspace $y > 0$, the calculus of variations problem

$$\text{minimize } \int_{\partial \tilde{G}} y^2 \, ds \quad \text{subject to } \int_{\tilde{G}} y^2 \, dxdy = \text{fixed}.$$ 

is solved by semicircles centered on the $x$-axis.

$$M(\tilde{G}) = M(\tilde{S}(r*))$$

$$M(\partial \tilde{G}) \geq M(\partial \tilde{S}(r*))$$
58. State variational problem as inequality.

Inequality (7) implies

\[ \alpha \int_{\partial \tilde{G}} w^2 \, ds \geq \int_{\partial \tilde{G}} y^2 \sqrt{dx^2 + dy^2} := \mathcal{M}(\partial \tilde{G}). \]

Among all domains with given fixed surface moment \( \int_{\tilde{G}} y^2 \, dx \, dy \), the semicircular arcs centered on the \( y \)-axis minimize \( \mathcal{M}(\partial \tilde{G}) \). If \( \tilde{S}(R) = \tilde{G} \) is a semicircle of radius \( R \):

\[ \mathcal{M}(\partial \tilde{S}(R)) = \int_{0}^{\pi} R^3 \sin^2 t \, dt = \frac{\pi R^3}{2}, \]

\[ \mathcal{M}(\tilde{S}(R)) = \int_{0}^{\pi} \int_{0}^{R} r^3 \sin^2 \theta \, dr \, d\theta = \frac{\pi R^4}{8}. \]

Solve for \( R \) and use semicircles are minimizers, for a general domain \( \tilde{G} \),

\[ \mathcal{M}(\partial \tilde{G}) \geq 2^\frac{5}{4} \pi^\frac{1}{4} \left\{ \int_{\tilde{G}} y^2 \, dx \, dy \right\}^{\frac{3}{4}}. \]
Returning to the original variables, \( dx \, dy = \alpha f \dot{f} \, d\rho \, d\theta \) so

\[
\int_{\partial G} w^2 \, ds \geq \frac{1}{\alpha} 2^{\frac{5}{4}} \pi^{\frac{1}{4}} \left[ \int_G f^2 \sin^2(\alpha\theta) \alpha f \dot{f} \, d\rho \, d\theta \right]^{\frac{3}{4}}
\]

\[
= \left( \frac{\pi}{2\alpha} \right)^{\frac{1}{4}} \left\{ \int_G \frac{4}{3} \left[ \tan^{2\alpha} \left( \frac{\rho}{2} \right) \sin \rho \right]^{\frac{1}{3}} \left[ 2\alpha + \cos \rho \right] \tan^{2\alpha} \left( \frac{\rho}{2} \right) \sin^2 \alpha \theta \, d\rho \, d\theta \right\}^{\frac{3}{4}}
\]

Choose \( \beta \) so that

\[
\frac{2\alpha + 2}{2\alpha + 1} \leq \beta < \frac{4}{3}.
\]

Regroup the integral inside the braces

\[
I = \frac{4}{3\beta} \int_G \left[ \tan^{2\alpha} \left( \frac{\rho}{2} \right) \sin \rho \right]^{\frac{4}{3} - \beta} \left[ 2\alpha + \cos \rho \right] \beta \left[ \tan^{2\alpha} \left( \frac{\rho}{2} \right) \sin \rho \right]^{\beta - 1} \tan^{2\alpha} \left( \frac{\rho}{2} \right) \, d\rho \, \sin^2 \alpha \theta \, d\theta.
\]
Let $\Psi = [\tan^{2\alpha} \left( \frac{\rho}{2} \right) \sin \rho]^{\beta}$ so

$$\psi = \beta \left( \tan^{2\alpha} \left( \frac{\rho}{2} \right) \sin \rho \right)^{\beta-1} [2\alpha + \cos \rho] \tan^{2\alpha} \left( \frac{\rho}{2} \right)$$

and

$$\phi(z) = \frac{4}{3\beta} z^{4\beta-1} \Rightarrow \Phi(z) = z^{\frac{4}{3\beta}}.$$

So that $\phi$ is increasing, we require $\beta < \frac{4}{3}$. If $H_\theta = \{\rho \in [0, \pi) : (\rho, \theta) \in G\}$ is the slice of $G$ in the $\rho$-direction then Szegő’s inequality (6) implies

$$I \geq \int_0^{\pi/\alpha} \left( \beta \int_{H_\theta} \tan^{2\alpha\beta} \left( \frac{\rho}{2} \right) \sin^{\beta-1} \rho \ [2\alpha + \cos \rho] \ d\rho \right)^{\frac{4}{3\beta}} \sin^{2\alpha} \alpha \theta \ d\theta.$$

Equality holds if and only if $H_\theta = [0, r(\theta)]$ is an interval a.e.
Next we let \( p = \frac{4}{3\beta} > 1 \), \( q = \frac{4}{4-3\beta} \), and using measure \( \sin^2 \alpha \theta \, d\theta \). Since \( \int_0^{\pi/\alpha} d\nu = \int_0^{\pi/\alpha} \sin^2 \alpha \theta \, d\theta = \frac{\pi}{2\alpha} \), Hölder’s Inequality implies \( I \geq \)

\[
\left( \frac{2\alpha}{\pi} \right)^{\frac{4}{3\beta} - 1} \left( \beta \int_0^{\pi/\alpha} \int_{H\theta} \tan^{2\alpha\beta} \left( \frac{\rho}{2} \right) \sin^{\beta-1} \rho \left[ 2\alpha + \cos \rho \right] \, d\rho \, \sin^2 \alpha \theta \, d\theta \right)^{\frac{4}{3\beta}}
\]
We regroup the inside integral again:

\[ J = \int_0^{\frac{\pi}{\alpha}} \int_{H_\theta} \tan^{2\alpha}(\beta-1)\left(\frac{\rho}{2}\right) \sin^{\beta-2}\rho \left[2\alpha + \cos \rho\right] \cdot \tan^{2\alpha}\left(\frac{\rho}{2}\right) \sin \rho \, d\rho \sin^2\alpha\theta \, d\theta. \]

Let us denote

\[ Z(r) = \int_0^r \tan^{2\alpha}\left(\frac{\rho}{2}\right) \sin \rho \, d\rho. \]

and define \( \bar{r}(r, \theta) \) by

\[ Z(\bar{r}) = \int_0^\bar{r} \tan^{2\alpha}\left(\frac{\rho}{2}\right) \chi_{H_\theta}(\rho) \sin \rho \, d\rho \]

where \( \chi_H \) denotes the characteristic function of \( H \). The integrand \( \tan^{2\alpha}(\rho/2)\sin \rho \) is positive and increasing for the range of \( \rho \) we are considering, and so \( \bar{r}(r, \theta) \leq r \) with equality if and only if \( H_\theta \cap [0, r] = [0, r] \) a.e.
63. Decompose increasing function as function of what you want.

If we require \((2\alpha + 1)\beta \geq 2\alpha + 2\), then the factor

\[
g_\beta(\rho) = \tan^{2\alpha(\beta-1)} \left( \frac{\rho}{2} \right) \sin^{\beta-2} \rho [2\alpha + \cos \rho]
\]

is increasing in \(\rho\). Thus we can define \(\Phi_\beta\) by

\[
\phi_\beta(y) = \beta g_\beta \circ Z^{-1}(y), \quad \Phi_\beta(y) = \int_0^y \phi_\beta(s) \, ds. \quad (9)
\]

Observe that \(Z\) and \(g_\beta\) are increasing, so \(\phi_\beta\) is increasing and \(\Phi_\beta\) is convex. Using \(g_\beta(\bar{r}(\rho, \theta)) \leq g_\beta(\rho)\), we have

\[
J \geq \int_0^{\pi/\alpha} \int_{H_\theta} g_\beta(\bar{r}(\rho, \theta)) \tan^{2\alpha} \left( \frac{\rho}{2} \right) \sin \rho \, d\rho \, \sin^2 \alpha \theta \, d\theta
\]

\[
= \frac{1}{\beta} \int_0^{\pi/\alpha} \int_{H_\theta} \phi_\beta \left( \int_0^\rho \tan^{2\alpha} \left( \frac{\rho'}{2} \right) \chi_{H_\theta}(\rho') \sin \rho' \, d\rho' \right) \cdot \tan^{2\alpha} \left( \frac{\rho}{2} \right) \sin \rho \, d\rho \, \sin^2 \alpha \theta \, d\theta.
\]
64. Use Szegő’s Inequality again and Jensen’s Inequality.

Let \( \psi(\rho) = \tan^{2\alpha}(\rho/2) \sin(\rho) \chi_{H_\theta} \).

\[
J \geq \frac{1}{\beta} \int_{0}^{\pi/\alpha} \Phi_\beta \left( \int_{H_\theta} \tan^{2\alpha}(\rho/2) \sin \rho \, d\rho \right) \sin^{2}\alpha \theta \, d\theta
\]

with equality if and only if \( H_\theta = [0, r(\theta)] \) is an interval a.e. Next, by Jensen’s inequality (with the measure \( \sin^{2}\alpha \theta \, d\theta \)),

\[
J \geq \frac{\pi}{2\alpha \beta} \Phi_\beta \left( \frac{2\alpha}{\pi} \int_{0}^{\pi/\alpha} \int_{H_\theta} \tan^{2\alpha}(\rho/2) \sin^{2}\alpha \theta \sin \rho \, d\rho \, d\theta \right)
\]

with equality if and only if \( \bar{r}(\theta) \) is a.e. constant. Substituting back,

\[
I \geq \left( \frac{2\alpha}{\pi} \right)^{\frac{4}{3\beta} - 1} \left( \beta J \right)^{\frac{4}{3\beta}}
\]

\[
\geq \frac{\pi}{2\alpha} \left\{ \Phi_\beta \left( \frac{2\alpha}{\pi} \int_{0}^{\pi/\alpha} \int_{H_\theta} \tan^{2\alpha}(\rho/2) \sin^{2}\alpha \theta \sin \rho \, d\rho \, d\theta \right) \right\}^{\frac{4}{3\beta}}.
\]
\[
\int_{\partial G} w^2 \, ds \geq \left( \frac{\pi}{2\alpha} \right)^{\frac{1}{4}} l^{\frac{3}{4}} \\
\geq \frac{\pi}{2\alpha} \Phi_{\beta}^\frac{1}{\beta} \left( \frac{2\alpha}{\pi} \int_0^{\pi/\alpha} \int_{H_\theta} \tan^{2\alpha} \left( \frac{\rho}{2} \right) \sin^2 \alpha \theta \sin \rho \, d\rho \, d\theta \right)
\]

where equality holds if and only if also \( \rho(\theta) \) is constant a.e. Taking a limit as \( \beta \to \frac{4}{3} \) from below implies the inequality holds for \( \beta = \frac{4}{3} \).
66. Solve for $\Phi^{1/\beta}(Y)$.

Since it depends only on $\int_G w^2 da$, it would be the same for any function $\nu^*$ whose level sets $G^*_{\eta} = \{x : \nu^*(x) \geq \eta\}$ give the same $\zeta(\eta) = \int_{G^*_{\eta}} w^2 da$ as the spherical rearrangement whose levels are sectors $G^*_{\eta} = S(r(\eta))$.

We express things in terms of $r(\eta)$. Now

$$\frac{2\alpha}{\pi} y = \frac{2\alpha}{\pi} \zeta(\eta) = \frac{2\alpha}{\pi} \int_{S(r(\eta))} w^2 \, da = Z(r(\eta))$$

so, changing variables $s = Z(r)$

$$\Phi_\beta(Y) = \int_0^Y \phi_\beta(s) \, ds = \beta \int_0^{Z^{-1}(Y)} g_\beta(r) \tan^{2\alpha}\left(\frac{r}{2}\right) \sin r \, dr$$

$$= \beta \int_0^{Z^{-1}(Y)} \left[\tan^{2\alpha}\left(\frac{r}{2}\right) \sin r\right]^{\beta-1} [2\alpha + \cos r] \tan^{2\alpha}\left(\frac{r}{2}\right) \, dr$$

$$= \left[\tan^{2\alpha}\left(\frac{Z^{-1}(Y)}{2}\right) \sin(Z^{-1}(Y))\right]^{\beta}. $$

We get the same equation for all $\beta$. Thus we set $\Upsilon_{\alpha} = \Phi^{1/\beta}_\beta$.  \qed
67. Rayleigh Quotient.

Let \( G \subset \mathcal{W} \). It suffices to estimate the Rayleigh quotient for admissible functions \( u \in C^2_0(G) \) that are twice continuously differentiable and compactly supported in \( G \). Any admissible function may be written \( u = \nu w \) for \( \nu \in C^2_0(G) \). The divergence theorem shows

\[
\int_G |du|^2 \, da = \int_G w^2 |d\nu|^2 \, da.
\]

Let \( G_t \) denote the points of \( G \) satisfying \( \nu \geq t \). Putting

\[
\zeta(t) = \int_{G_t} w^2 \, da,
\]

we see that \( \zeta(0) = \hat{\zeta} \geq \zeta(t) \geq 0 = \zeta(\hat{\nu}) \), where \( \hat{\nu} = \max_G \nu \),

\[
\frac{\partial \zeta}{\partial t} = -\int_{\partial G_t} \frac{w^2}{|d\nu|} \, ds
\]

and

\[
\int_G w^2 \nu^2 \, da = \int_0^{\hat{\nu}} 2t \zeta(t) \, dt = \int_0^{\hat{\zeta}} t^2 \, d\zeta.
\]
Then, using the coarea formula, Schwarz’s inequality, isoperimetric inequality, and changing variables to $y = \zeta(t)$, the inequality implies

$$\int_G w^2 |dv|^2 \, da \quad \geq \quad \int_0^\hat{v} \left\{ \int_{\partial G_t} w^2 |dv| \, ds \right\} \, dt \quad \geq \quad \int_0^\hat{v} \left( \int_{\partial G_t} w^2 \, ds \right)^2 \, dt \quad \geq \quad \int_0^\hat{v} \frac{\left( \int_{\partial G_t} w^2 \, ds \right)^2}{\int_{\partial G_t} |dv| \, ds} \, dt \quad \geq \quad \frac{\pi^2}{4\alpha^2} \int_0^\hat{v} \gamma_\alpha^2 \left( \frac{2\alpha}{\pi} \zeta(t) \right) \left( \frac{\partial \zeta}{\partial t} \right) \, dt.$$
Changing variables to \( y = \zeta(t) \) we have

\[
\int_0^{\xi} \gamma_\alpha^2 \left( \frac{2\alpha}{\pi} y \right) \left( \frac{\partial t}{\partial y} \right)^2 \, dy \geq \mu \int_0^{\xi} t(y)^2 \, dy
\]

where \( \mu \) is the least eigenvalue of the boundary value problem

\[
\frac{\partial}{\partial y} \left( \gamma_\alpha^2 \left( \frac{2\alpha}{\pi} y \right) \frac{\partial q}{\partial y} \right) + \mu q = 0,
\]

\[
q(\zeta) = 0,
\]

\[
\lim_{y \to 0^+} \gamma_\alpha^2 \left( \frac{2\alpha}{\pi} y \right) \frac{\partial q}{\partial y} = 0.
\]
Now perform the change variables so that the domain is now $[0, r^*]$, 

$Z(r^*) = \frac{2\alpha}{\pi} \hat{\zeta}$, and $\mu$ is now the least eigenvalue of

$$\frac{\partial}{\partial r} \left( \tan^{2\alpha} \left( \frac{r}{2} \right) \sin(r) \frac{\partial q}{\partial r} \right) + \frac{\pi^2 \mu}{4\alpha^2} \tan^{2\alpha} \left( \frac{r}{2} \right) \sin(r) q = 0,$$  \hspace{1cm} (11)

$q(r^*) = 0, \quad \lim_{r \to 0^+} \tan^{2\alpha} \left( \frac{r}{2} \right) \sin(r) \frac{\partial q}{\partial r} = 0. \hspace{1cm} (12)$

Note that (11) is the eigenequation for the spherical sector $S(r^*)$. Hence

$$\frac{\pi^2 \mu}{4\alpha^2} = \lambda_1(S(r^*)).$$

Reassembling using equations

$$\int_G |du|^2 \, da \geq \lambda_1(S(r^*)) \int_G u^2 \, da,$$

which implies the Theorem. \hfill \Box
The eigenvalue \( \lambda^* = \lambda_1(S(r^*)) \) occurs as the eigenvalue of the problem (11), (12) on \([0, r^*]\), which may be rewritten

\[
\sin(r) q'' + [2\alpha + \cos(r)] q' + \lambda^* q = 0; \\
\lim_{r \to 0} \tan^{2\alpha}(\frac{r}{2}) \sin(r) \frac{dq}{dr}(r) = 0, \quad q(r^*) = 0.
\]

Making the change of variable \( x = \frac{1 - \cos r}{2} \) transforms the ODE to the hypergeometric equation on \([0, 1]\)

\[
x(1 - x) \ddot{y} + [c - (a + b + 1)x] \dot{y} - ab y = 0,
\]

with

\[
a, b = \frac{1 \pm \sqrt{1 + 4\lambda^*}}{2}, \quad c = \alpha + 1.
\]

The solution to the hypergeometric equation is Gauß’s ordinary hypergeometric function, given by

\[
\_{2}F_{1}(a, b; c; x) = 1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)} \frac{x^3}{3!} + \cdots.
\]
We find the eigenvalue by a shooting method. Given \( r^* \), \( \lambda^* \) is the first positive root of the function

\[
\lambda \mapsto 2F_1 \left( \frac{1 - \sqrt{1 + 4\lambda}}{2}, \frac{1 + \sqrt{1 + 4\lambda}}{2}; \alpha + 1; \frac{1 - \cos r^*}{2} \right). \tag{13}
\]
73. Eigenvalues from the new Spherical Estimates.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\mathcal{I}(G)$</th>
<th>$r^*$</th>
<th>$\lambda_1(G)$</th>
<th>$\lambda_1(S(r^*))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{W}$</td>
<td>$\infty</td>
<td>\pi</td>
<td>$(\alpha + 1)\alpha$</td>
<td>$(\alpha + 1)\alpha$</td>
</tr>
<tr>
<td>$S(\frac{\pi}{2})$</td>
<td>$\frac{\pi}{2\alpha}Z(\frac{\pi}{2})$</td>
<td>$\frac{\pi}{2}$</td>
<td>$(\alpha + 1)(\alpha + 2)$</td>
<td>$(\alpha + 1)(\alpha + 2)$</td>
</tr>
<tr>
<td>$S(r)$</td>
<td>$\frac{\pi}{2\alpha}Z(r)$</td>
<td>$r$</td>
<td>$\lambda^*$</td>
<td>$\lambda^*$</td>
</tr>
<tr>
<td>$\mathcal{W}$</td>
<td>$\infty$</td>
<td>$3.14159265$</td>
<td>$3.75$</td>
<td>$3.75$</td>
</tr>
<tr>
<td>$\alpha = \frac{3}{2}$</td>
<td>$2.07876577$</td>
<td>$2.18627604$</td>
<td>$5.00463538$</td>
<td>$5.00463538$</td>
</tr>
<tr>
<td>$S(\delta)$</td>
<td>$0.90871989$</td>
<td>$1.91063324$</td>
<td>$6.19561775$</td>
<td>$6.19561775$</td>
</tr>
<tr>
<td>$\alpha = \frac{3}{2}$</td>
<td>$0.30118555$</td>
<td>$1.57079633$</td>
<td>$8.75$</td>
<td>$8.75$</td>
</tr>
<tr>
<td>$S(\varepsilon)$</td>
<td>$1.88896324$</td>
<td>$2.15399460$</td>
<td>$5.1590\ldots$</td>
<td>$5.11641465$</td>
</tr>
<tr>
<td>$\widehat{T}$</td>
<td>$1.90831355$</td>
<td>$2.15742981$</td>
<td>$?$</td>
<td>$5.10421518$</td>
</tr>
</tbody>
</table>

**Table:** Domains and eigenvalues. In this table $\delta = \cos^{-1}(-1/\sqrt{3})$ and $\epsilon = \cos^{-1}(-1/3)$. 
Consider the example of the geodesic triangle $\mathcal{T} = \mathcal{T}_2 \subset S^2$. Writing

$$\mathcal{T} = \{(\rho, \theta) : 0 \leq \theta \leq \frac{2\pi}{3}, \quad 0 \leq \rho \leq r(\theta)\}$$

we find

$$r(\theta) = \frac{\pi}{2} + \text{Atn} \left( \frac{\cos(\theta - \frac{\pi}{3})}{\sqrt{2}} \right).$$

At the vertex we have $\alpha = \frac{3}{2}$ so that

$$Z(r) = \int_0^r \tan^3 \left( \frac{\rho}{2} \right) \sin \rho \, d\rho = 4 \tan \left( \frac{r}{2} \right) + \sin r - 3r.$$ 

$\lambda_1(\mathcal{T})$ was found numerically in [RT]. Using MAPLE®, we numerically integrate

$$\mathcal{I}(\mathcal{T}) = \int_0^{\pi/\alpha} Z(r(\theta)) \sin^2(\alpha \theta) \, d\theta$$

and solve $\frac{\pi}{2\alpha} Z(r^*) = \mathcal{I}(\mathcal{T})$ for $r^*$ and $\lambda^*$ to get the other values in the $\mathcal{T}$ line in Table 1.
To avoid the quadrature, we observe the estimate

\[ Z(r(\theta)) \leq T(\theta) := A_1 + A_2 \cos\left(\theta - \frac{\pi}{3}\right) + A_3 \left(1 - \cos(6\theta)\right), \]

where \( A_1 \) and \( A_2 \) are chosen so that the functions agree at \( \theta = 0 \) and \( \theta = \frac{\pi}{3} \) and the \( A_3 \) is chosen to make the second derivatives agree at \( \frac{\pi}{3} \). The inequality follows since the second derivative of the difference goes from negative to positive in \( 0 < \theta < \pi/3 \).

This corresponds to the larger domain \( \hat{T} \) whose radius function is \( \hat{r}(\theta) = Z^{-1}(T(\theta)) \). Then

\[
\frac{\pi}{2\alpha} Z(\hat{r}^*) = \int_{\hat{T}} w^2 \, da = \int_{0}^{\frac{\pi}{3}} T(\theta) \sin^2\left(\frac{3}{2}\theta\right) \, d\theta = \frac{\pi}{3} A_1 + \frac{9\sqrt{3}}{16} A_2 + \frac{\pi}{3} A_3.
\]

Using these values we obtain the last row of Table 1. By eigenvalue monotonicity, if \( \hat{T} \supset T \) then \( \lambda_1(T) \geq \lambda_1(\hat{T}) \).
Theorem (Ratzkin 2009)

Let \( \Omega \) be a nice domain in the cone

\[
\mathcal{W}_n = \{(r, \theta) : r \geq 0, \quad \theta \in \mathcal{D}_{n-1}\}
\]

where \( \mathcal{D}_{n-1} \subset \mathbb{S}^{n-1} \) is a convex domain. Choose \( r_0 \) so that

\[
\int_{\Omega} w^2 \, dV = \int_{\text{Sect}_n(r_0, \mathcal{D}_{n-1})} w^2 \, dV
\]

where \( w = r^\alpha \psi(\theta) \), \( \psi \) is the first eigenfunction of \( \mathcal{D}_{n-1} \) and

\[
\alpha = \frac{n-2}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1(\mathcal{D}_{n-1})}.\]

Then \( \lambda_1(\Omega) \geq \lambda_1(\text{Sect}_n(r_0, \mathcal{D}_{n-1})) \), with equality if and only if \( \Omega = \text{Sect}_n(r_0, \mathcal{D}_{n-1}) \).
Theorem

There is a nice domain $Q_2 \subset S^2$ such that $T_2 \subset Q_2$ and such that

$$\lambda_1(Q_2) = 5.102 > \lambda_{cr}.$$ 

Corollary

$$\lambda_1(T_3) > \lambda_1(\text{Sect}_3(Q_2, \delta(3))) > 8$$

so

$$\mathbb{E}T_4 < \infty.$$ 

The idea was motivated by Rayleigh (1877) and Polya-Szego (1952) who studied the dependence of the eigenvalue on planar nearly circular domains of the form

$$r \leq c + \varepsilon f(\theta).$$
Construct $Q_2$ as nodal domain.

*Idea of proof.* Let $U = R(r)\Theta(\theta)$ solve

$$\Delta_2 U + \mu U = 0 \quad \text{on } \text{Sect}_2 \left([0, \frac{2}{3}\pi], \pi\right).$$

Fix $\mu = 5.102$. Fix $\theta \in [0, \frac{2}{3}\pi]$. For each angular eigenmode $\ell \in \mathbb{N}$,

$$\Theta'' + \lambda \Theta = 0 \quad \text{on } [0, \frac{2}{3}\pi],$$

$$\Theta = 0 \quad \text{at } \theta \in \{0, \frac{2}{3}\pi\}.$$

Thus the angular part $\Theta_\ell(\theta) = \sin \left(\frac{3}{2}\ell\theta\right)$ and $\lambda = \frac{9}{4}\ell^2$. 
Solve for $R_\ell(r)$, the radial part of eigenfunction. Define $Q_2$ as the nodal domain of
\[ \Phi = \Theta_1(\theta)R_1(r) + \varepsilon\Theta_3(\theta)R_3(r) \]
for appropriate $\varepsilon \neq 0$. By construction, $\lambda_1(Q_2) = \mu$.

If $r_1 > 0$ is the first zero of $R_1(r)$, then also by construction,
\[ \mu = \lambda_1 \left( \text{Sect}_2([0, \frac{2}{3}\pi], r_1) \right). \]

$Q_2$ is a perturbation of the sector $\text{Sect}_2([0, \frac{2}{3}\pi], r_1)$, the nodal domain of $\Theta_1(\theta)R_1(r)$, which does not contain $\mathcal{T}_2$. 
The radial part satisfies

\[
\sin^2 r \ddot{R} + \sin r \cos r \dot{R} + (\mu \sin^2 r - \lambda) R = 0 \quad \text{on } [0, \pi)
\]

\[
R = 0 \quad \text{at } r = 0
\]

Since \( n = 2 \), \( m = \sqrt{\lambda} \). Putting \( R(r) = \sin^m r \ u(r) \) as in the Truncated Cone Lemma, the equation becomes

\[
\sin^2 r \ddot{u} + (1 + 2m) \sin r \cos r \dot{u} + (\mu - m - \lambda) \sin^2 r \ u = 0.
\]

Using \( \lambda_\ell = \frac{9}{4} \ell^2 \), the solution is hypergeometric

\[
u_\ell(r) = _2F_1\left(\frac{3}{2} \ell + 0.5 \pm \sqrt{\frac{1}{4} + \mu}; 1 + \frac{3}{2} \ell; \frac{1}{2}(1 - \cos r)\right).
\]
Finally, consider the $\mu = 5.102$ superposition of the $\ell = 1, 3$ modes

$$
\Phi = (\sin r)^{3/2} u_1(r) \sin\left(\frac{3}{2} \theta\right) - .0003 (\sin r)^{9/2} u_3(r) \sin\left(\frac{9}{2} \theta\right).
$$

Let $Q_2$ be its nodal domain. $\lambda_1(Q_2) = 5.102$ by construction.

It remains to show for this $\varepsilon$ we have $T_2 \subset Q_2$. 
82. First and third modes.
Superposition of first and third modes. Perturbed nodal domain.

Dashed line is circle

Superposition of $n = 1$ and $n = 3$ modes $Y = Y_1 + \varepsilon Y_3$
84. Superposition of first and third modes. Perturbed nodal domain.

- Dashed line is circle
- Superposition of $n = 1$ and $n = 3$ modes $Y_1 - \varepsilon Y_3$
85. Stereographic projection $\mathcal{T}_2 \subset Q_2 \not\subset \mathring{\mathcal{T}}_2$
86. Stereographic projection $T_2 \subset Q_2 \not\subset \hat{T}_2$
It remains to check that $\mathcal{T}_2 \subset Q_2$. As $\Upsilon$ is a perturbation of $u_1$, its nodal set is a perturbation of the sector $\text{Sect}_2(\mathcal{T}_1, r_1)$ (with $r_1 < \delta(2)$.) Converting to the stereographic image $\rho = \tan \left( \frac{r}{2} \right)$, the radius of the circular outer edge $\rho(\theta)$ of $\mathcal{T}_2$ satisfies

$$
\left( \rho(\theta) \cos \theta - \frac{\sqrt{2}}{4} \right)^2 + \left( \rho(\theta) \sin \theta - \frac{\sqrt{6}}{2} \right)^2 = \frac{3}{2}
$$

so that

$$
\rho(\theta) = \frac{\sqrt{2} \cos (\theta - \frac{\pi}{3}) + \sqrt{2} \cos^2 (\theta - \frac{\pi}{3}) + 4}{2}.
$$
Dropping the \(\sin(r)^{3/2} \sin\left(\frac{3}{2} \theta\right)\) factor, it remains to prove that

\[
\Psi(r) = u_1(r) - 0.0003 (\sin r)^3 u_3(r) \left(4 \cos\left(\frac{3}{2} \theta\right)^2 - 1\right) \geq 0
\]

for all \(0 \leq r \leq 2 \arctan (\rho(\theta))\) and \(0 \leq \theta \leq \frac{2}{3} \pi\).

This is easily seen when plotted by a computer algebra system like Maple. \(\Psi\) and its derivatives are known. The result follows by finitely many function evaluations and estimates on the derivative of \(\Upsilon\) to show \(\Psi > 0\) on \(T_2\).
89. Numerical evaluation of $\lambda_1(T_2)$ details

### Numerical Computation.

\[
\lambda_1(T_2) \approx 5.159...
\]

We use a Sinc-Galerkin-Collocation scheme to approximate the eigenvalue of the triangle $T_2$ using an idea of Stenger.

**First, conformally map $T_2$ to semiinfinite strip.**

Let the vertices of $T_2$ be $V_1, V_2, V_3$. The midpoint of the edge $V_1V_2$ is denoted $S_3$ and the center point of $T_2$ is denoted $F$.

$\Omega$ is $\frac{1}{6}$ of $T_2$ with vertices $F, V_1, S_3$. The eigenfunction $u_1$ of $T_2$ when restricted to $\Omega$ is the first eigenfunction with Dirichlet condition on the segment $V_1S_3$ and Neumann condition on other two edges.
90. Conformally map infinite strip to a sixth of $\mathcal{T}_2$
Let \( P = (X, Y, Z) \in S^2 \subset \mathbb{R}^3 \). Stereographic projection is

\[
    w = \sigma(P) = \frac{X + iY}{1 + Z}
\]

so that the metric and Laplacian of the sphere is

\[
    ds^2 = \frac{4|dw|^2}{(1 + |w|^2)^2}, \quad \Delta = \frac{(1 + |w|^2)}{4} \Delta_w
\]

where \( \Delta_w = 4\frac{\partial^2}{\partial w \partial \bar{w}} \). Placing \( S_3 \) at the north pole. \( \sigma(S_3) = 0 \). Rotate so

\[
    \sigma(F) = \frac{\sqrt{6} - \sqrt{2}}{2}, \quad \sigma(V_1) = \frac{\sqrt{6} - \sqrt{2}}{2} i.
\]
Let $D = \{ z \in \mathbb{C} : 0 < \Re z < \frac{\pi}{2}, 0 < \Im mz \}$ be the strip.
If $f : D \to \sigma(\Omega)$ is the conformal map so $f(0) = \sigma(S_3)$, $f(\frac{\pi}{2}) = \sigma(F)$ and $f(\infty) = \sigma(V_1)$. Pulling back to $D$,

$$\Delta^* u + \lambda u = 0, \quad \text{if } z \in D,$$

$$u = 0 \quad \text{if } \Re z = 0 \text{ and } \Im mz > 0,$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{if } \Im mz = 0, \text{ or } \Re z = \frac{\pi}{2}$$

where $\Delta^* = f^* \Delta = \Delta_z$ is the pulled back Laplacian.
Schwarz triangle mapping $z \in D$ or from $\sin^2 z \in \mathcal{H}$ of the upper halfplane to $w \in \sigma(\Omega)$ is given by

$$\cos^2 z = \frac{(w^4 + 2\sqrt{3} w^2 - 1)^3}{(w^4 - 2\sqrt{3} w^2 - 1)^3} = \frac{\prod_{j=1}^{4} (w - \sigma(F_j))^3}{\prod_{j=1}^{4} (w - \sigma(V_j))^3}$$

where the coordinate for $S_3$ is $w = 0$, for $V_1$ is $w = \frac{i}{2} (\sqrt{6} - \sqrt{2})$ and for $F$ is $w = \frac{1}{2} (\sqrt{6} - \sqrt{2})$. The corresponding points are $z = 0, \infty, \pi/2$. Thus we may compute $f$. Writing $g(z) = \cos^{2/3} z$,

$$f(z) = \sqrt{\frac{1 - g}{\sqrt{3} (1 + g) + 2\sqrt{1 + g + g^2}}}$$
Pulling back under $w = f(z)$, the conformal weight takes the form

$$4 \left| \frac{df}{dz} \right|^2 = \frac{4}{3} \frac{\left| \sqrt{3}(1 + g) + 2\sqrt{1 + g + g^2} \right|}{|g| \left( \left| \sqrt{3}(1 + g) + 2\sqrt{1 + g + g^2} \right| + |1 - g| \right)^2}.$$

The branch cuts for the square and cube roots may be taken above the negative real axis. Thus $g(D)$ lies in the fourth quadrant so the denominator in $f$ is nonvanishing.
Convert to eigenvalue problem of integral operator.

Let \( G(z, z') \) denote the Green's function for the problem on \( D \)

\[
\Delta^* u = f, \quad \text{if } z \in D, \\
u = 0 \quad \text{if } \Re z = 0 \text{ and } \Im mz > 0, \\
\frac{\partial u}{\partial n} = 0 \quad \text{if } \Im mz = 0 \text{ or if } \Re z = \frac{\pi}{2}
\]

The Green's function may be found by the method of images. Denote \( w = \sin z = x + iy \), \( w^* = -x + yi \) and \( \omega = \sin \zeta = \xi + i\eta \), we get \( \overline{w^*} = (\overline{w})^* \). Thus the Green's function is

\[
G(z; \zeta) = \frac{1}{2\pi} \left( \ln |w - \omega| - \ln |w^* - \omega| + \ln |\overline{w} - \omega| - \ln |\overline{w^*} - \omega| \right)
\]

\[
G(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \left( \frac{[(x-\xi)^2+(y-\eta)^2][((x-\xi)^2+(y+\eta)^2]^{\frac{1}{2}}}{[(x+\xi)^2+(y-\eta)^2][((x+\xi)^2+(y+\eta)^2]^{\frac{1}{2}}} \right)
\]
Pulling back by $f$, restate as eigenvalue problem for the integral operator

$$\frac{1}{\lambda} u(z) = -4 \int_D \frac{G(z; z') |df(z')|^2 u(z') dz'}{(1 + |f(z')|^2)^2} =: Au(z)$$

The operator has logarithmic and algebraic singularities at the points $z' = 0$, $\frac{\pi}{2}$ and $z = z'$. 
Approximate $f^* u(z)$ in an $m$-dimensial space $X_m$ of SINC functions with the same symmetries. Take a basis $\{\phi_1, \ldots, \phi_m\}$ of $X_m$. At the sinc points $z_i \in D$,

$$\phi_i(z_k) = \delta_{ik}. $$

$P_\ell$ is the $\ell$-th coefficient via point-evaluation

$$P_\ell f = f(z_\ell),$$

so the “projection” to $X_m$ is (a collocation)

$$(Pf)(\zeta) = \sum_k f(z_k) \phi_k(\zeta).$$

The integral operator shall be computed numerically via sinc quadrature.

The matrix of the transformation $A_{\ell k} = P_\ell A \phi_k$, whose largest eigenvalue approximates $\mu_m \rightarrow \frac{1}{\lambda_1}$ as $m \rightarrow \infty$. It is an upper bound $\lambda_1 \leq \frac{1}{\mu_m}$. 
For $z = x + iy$ let the basis $\phi_{jk}(z) = \alpha_j(x) \times \beta_k(y)$, where

$$\alpha_j(x) = S(j, h) \circ \ln \left( \frac{x}{\frac{\pi}{2} - x} \right),$$

$$\alpha_{n+1}(x) = \sin^2(x) - \sum_{\ell=-n}^{n} \sin^2(x_\ell) \alpha_\ell(x),$$

where $j = -n \ldots, n$ with sinc points $x_j = \frac{\pi e^{hj}}{2(1 + e^{hj})}$ and

$$\beta_k(x) = S(k, h) \circ \ln(\sinh y),$$

$$\beta_{n+1}(y) = \text{sech}(y) - \sum_{\ell=-n}^{n} \text{sech}(y_\ell) \beta_\ell(x),$$

for $k = -n, \ldots, n$ with sinc points $y_k = \sinh^{-1}(e^{hk})$. We let $h = \frac{\pi}{\sqrt{2n}}$, $x_{n+1} = \frac{\pi}{2}$ and $y_{n+1} = 0$. 
The sinc cardinal function

\[ S(j, h)(x) = \begin{cases} 
\frac{\sin \left( \frac{\pi(x-jh)}{h} \right)}{\frac{\pi(x-jh)}{h}}, & \text{if } x \neq jh, \\
1, & \text{if } x = jh.
\end{cases} \]

where \( h = \frac{\pi}{\sqrt{2n}} \). Thus the dimension is \( m = (2n + 2)^2 \).

Approximate \( f^*u(z) \approx \mathcal{P}f^*u(z) = b^{ij} \phi_{ij}(z) \) (sum over \( i, j = -n \ldots, n + 1 \)).

Let \( b^{ij} = \mathcal{P}_{ij}f^*u = f^*u(x_i + y_j\sqrt{-1}) \). Thus the approximation \( \mathcal{P}f^*u \) is a collocation, as it equals \( f^*u \) at the sinc points.
100. Basis functions \( \alpha_i(x) \) on \((0, \frac{\pi}{2})\) when \( n = 17 \).
Thus, the matrix is approximated by

\[ A_{ij,pq} = \int_D G(x_i, y_j, \xi, \eta) \beta_{pq}(\xi, \eta) \psi(\xi, \eta) \, d\xi \, d\eta \approx \sum_{\iota, \kappa} v_\iota w_\kappa G(x_i, y_j, x_\iota, y_\kappa) \beta_{pq}(x_\iota, y_\kappa) \psi(x_\iota, y_\kappa) \]

where

\[ \psi(\xi, \eta) = \frac{4|df(\xi + i\eta)|^2}{(1 + |f(\xi + i\eta)|^2)^2} \]

The approximating sum is carried over \(4m\) terms, corresponding to sinc quadratures in the four regions bounded by singularities (e.g. in case \(0 < x_i < \frac{\pi}{2}\) and \(0 < y_j\):

\[
D_I = \{ \xi + i\eta : 0 < \xi < x_i, \, 0 < \eta < y_j \};
\]

\[
D_{II} = \{ \xi + i\eta : x_i < \xi < \pi/2, \, 0 < \eta < y_j \};
\]

\[
D_{III} = \{ \xi + i\eta : 0 < \xi < x_i, \, y_j < \eta \};
\]

\[
D_{IV} = \{ \xi + i\eta : x_i < \xi, \, y_j < \eta \}.
\]

and \(v_\iota, \, w_\kappa\) are the corresponding weights.
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Thanks!