Geometry and Analysis of Surfaces

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Notes on this lecture on geometric analysis.

The URL for these Beamer Slides from the lecture "Geometry & Analysis of Surfaces" given January 20, 2011:

http://www.math.utah.edu/~treiberg/GeomAnalSurfSlides.pdf

- Surfaces of Euclidean Space.
- Induced Metric and Riemannian Surfaces.
- Hyperbolic Space.
- Riemann Surfaces and Parabolicity.
- Complete Manifolds with Finite Total Curvature.

Some examples of what should be surfaces.

- Graphs of functions
$$\mathbb{G}^2 = \{(x, y, z) \in \mathbf{R}^3 : \ z = f(x, y) \text{ and } (x, y) \in U \}$$
where $U \subset \mathbf{R}^2$ is an open set.

- Level sets, *e.g.*,
$$\mathbb{S}^2 = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\}$$
This is the standard unit sphere.

- Parameterized Surfaces, *e.g.*,
$$\mathbb{T}^2 = \big\{ \big((A + a\cos\phi)\cos\theta, (A + a\cos\phi)\sin\theta, a\sin\phi \big) \big) : \theta, \ \phi \in \mathbf{R} \big\}$$
is the torus with radii $A > a > 0$ constructed as a surface of
revolution about the z-axis.

A surface can locally be given by a curvilinear coordinate chart, also called a parameterization. Let $U \subset \mathbf{R}^2$ be open. Let

$$X : U \to \mathbf{R}^3$$

be a $\mathcal{C}^1$ function. Then we want $M = X(U)$ to be a surface. At each point $P \in X(U)$ we can identify tangent vectors to the surface. If $P = X(a)$ some $a \in U$, then

$$X_i(a) = \frac{\partial X}{\partial u_i}(a)$$

for $i = 1, 2$ are vectors in $\mathbf{R}^3$ tangent to the coordinate curves. To avoid singularities at P, we shall assume that all $X_1(P)$ and $X_2(P)$ are linearly independent vectors. Then the tangent plane to the surface at P is

$$T_P M = \text{span}\{X_1(P), X_2(P)\}.$$

For the graph $\mathbb{G}^2 = \{(x, y, z) \in \mathbf{R}^3 : z = f(x, y) \text{ and } (x, y) \in U \}$ one coordinate chart covers the whole surface, $X : U \to \mathbb{G}^2 \cap V = \mathbb{G}^2$, where

$$X(u_1, u_2) = (u_1, u_2, f(u_1, u_2)).$$

where $V = \{(u_1, u_2, u_3) : (u_1, u_2) \in U \text{ and } u_3 \in \mathbf{R}\}$.
The tangent vectors are thus

$$
\begin{aligned}
X_1(u_1, u_2) &= \left(1, 0, \frac{\partial f}{\partial u_1}(u_1, u_2)\right), \\
X_2(u_1, u_2) &= \left(0, 1, \frac{\partial f}{\partial u_2}(u_1, u_2)\right)
\end{aligned}
\tag{1}
$$

which are linearly independent for every $(u_1, u_2) \in U$.

Definition

A connected subset $M \subset \mathbf{R}^3$ is a *regular surface* if to each $P \in M$, there is an open neighborhood $P \in V \subset \mathbf{R}^2$, and a map

$$X : U \to V \cap M$$

of an open set $U \subset \mathbf{R}^2$ onto $V \cap M$ such that

1. X is differentiable. (In fact, we shall assume X is smooth ($\mathcal{C}^\infty$)
2. X is a homeomorphism (X is continuous and has a continuous inverse)
3. The tangent vectors $X_1(a)$ and $X_2(a)$ are linearly independent for all $a \in U$.
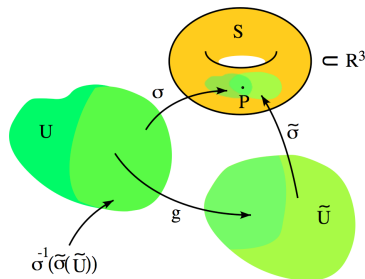
Figure: Extrinsic: Coordinate Charts for Surface in $\mathbb{E}^3$

Suppose $S \subset \mathbf{R}^3$ is a surface and at $P \in S$ there are two coordinate charts $\sigma : U \to S$ and $\tilde{\sigma} : \tilde{U} \to S$ such that U and $\tilde{U}$ are open subsets of $\mathbf{R}^2$ and $P \in \sigma(U) \cap \tilde{\sigma}(\tilde{U})$. Then on the overlap, for $u \in \sigma^{-1}(\tilde{\sigma}(\tilde{U}))$, we have the transition function

$$\tilde{u} = g(u) = \tilde{\sigma}^{-1}(\sigma(u))$$

which gives the change of coordinates map. These maps are smooth diffeomorphisms on overlaps, by the Inverse Function Theorem.

To abstract the idea of a regular surface, we drop the requirement that $M \subset \mathbf{R}^3$ and just require that there is a topological space M that has a collection of coordinate charts, an atlas, whose transition functions are smooth and consistently defined. Such an abstract surface is called a differentiable manifold and the atlas of charts with corresponding transition functions is called a differential structure.

Question: Are there differentiable manifolds that do not arise as submanifolds of Euclidean space with the induced differential structure?
Answer: No. (Provided we allow big codimension.)

Theorem (Whitney's Embedding Theorem.)

Let M^n be an abstract, smooth differentiable manifold of dimension n. Then M^n is diffeomorphic to some $W^n \subset \mathbf{R}^N$, an embedded regular submanifold provided $N \geq 2n + 1$.

The Euclidean structure of $\mathbf{R}^3$, the usual dot product, gives a way to measure lengths and angles of vectors. If $V = (v_1, v_2, v_3)$ then its length

$$|V| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{V \bullet V}$$

If $W = (w_1, w_2, w_3)$ then the angle $\alpha = \angle(V, W)$ is given by

$$\cos \alpha = \frac{V \bullet W}{|V| \, |W|}.$$

If $\gamma : [a, b] \to M \subset \mathbf{R}^3$ is a continuously differentiable curve, its length is

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)| \, dt.$$

11. Induced Riemannian Metric.

If the curve is confined to a coordinate patch $\gamma([a, b]) \subset X(U) \subset M$, then we may factor through the coordinate chart. There are continuously differentiable $u(t) = (u_1(t), u_2(t)) \in U$ so that

$$\gamma(t) = X(u_1(t), u_2(t)) \qquad \text{for all } t \in [a, b].$$

Then the tangent vector may be written

$$\dot{\gamma}(t) = X_1(u_1(t), u_2(t))\, \dot{u}_1(t) + X_2(u_1(t), u_2(t))\, \dot{u}_2(t)$$

so its length is

$$|\dot{\gamma}|^2 = X_1 \bullet X_1\, \dot{u}_1^2 + 2X_1 \bullet X_2\, \dot{u}_1 \dot{u}_2 + X_2 \bullet X_2\, \dot{u}_2^2$$

For $i, j = 1, 2$ the Riemannian Metric is is given by the matrix function

$$g_{ij}(u) = X_i(u) \bullet X_j(u)$$

Evidently, $g_{ij}(u)$ is a smoothly varying, symmetric and positive definite.

Thus $|\dot{\gamma}(t)|^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij}(u(t)) \, \dot{u}_i(t) \, \dot{u}_j(t)$.

The length of the curve on the surface is determined by its velocity in the coordinate patch $\dot{u}(t)$ and the metric $g_{ij}(u)$.

A vector field on the surface is also determined by functions in U using the basis. Thus if V and W are tangent vector fields, they may be written

$$V(u) = v^1(u)X_1(u) + v^2(u)X_2(u), \qquad W(u) = w^1(u)X_1(u) + w^2(u)X_2(u)$$

The $\mathbf{R}^3$ dot product can also be expressed by the metric. Thus

$$V \bullet W = \langle V, W \rangle = \sum_{i,j=1}^{2} g_{ij} \, v^i \, w^j.$$

where $\langle \cdot, \cdot \rangle$ is an inner product on T_pM that varies smoothly on M. This Riemannian metric is also called the First Fundamental Form.

If V and W are nonvanishing vector fields on M then their angle $\alpha = \angle(V, W)$ satisfies

$$\cos \alpha = \frac{\langle V, W \rangle}{|V| \, |V|}$$

which depends on coordinates of the vector fields and the metric. If $D \subset U$ is a piecewise smooth subdomain in the patch, the area if $X(D) \subset M$ is also determined by the metric

$$A(X(D)) = \int_D |X_1 \times X_2| \, du_1 \, du_2 = \int_D \sqrt{\det(g_{ij}(u))} \, du_1 \, du_2$$

since if $\beta = \angle(X_1, X_2)$ then

$$|X_1 \times X_2|^2 = \sin^2 \beta \, |X_1|^2 \, |X_2|^2 = (1 - \cos^2 \beta) \, |X_1|^2 \, |X_2|^2$$
$$= |X_1|^2 \, |X_2|^2 - (X_1 \bullet X_2)^2 = g_{11} g_{22} - g_{12}^2.$$

For the graph $\mathbb{G}^2 = \{(x, y, z) \in \mathbf{R}^3 : z = f(x, y) \text{ and } (x, y) \in U \}$ take the patch $X(u_1, u_2) = (u_1, u_2, f(u_1, u_2))$.

The metric components are $g_{ij} = X_i \bullet X_j$ so using (1),

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{pmatrix}$$

where $f_i = \dfrac{\partial f}{\partial u_i}$. Thus gives the usual formula for area

$$\det(g_{ij}) = 1 + f_1^2 + f_2^2$$

so

$$A(X(D)) = \int_D \sqrt{1 + f_1^2 + f_2^2}\, du_1\, du_2.$$

If we endow an abstract differentiable manifold M^n with a Riemannian Metric, a smoothly varying inner product on each tangent space that is consistently defined on overlapping coordinate patches, the resulting object is a Riemannian Manifold.

Question: Are there Riemannian manifolds that do not arise as submanifolds of Euclidean space with the induced differential structure and Riemannian metric?

Answer: No. (Provided we allow big codimension.)

Theorem (Nash's Isometric Immersion Theorem.)

Let M^n be an abstract, smooth Riemannian manifold of dimension n. Then M^n is isometric to a smooth immersed submanifold $W^n \subset \mathbf{R}^N$ with induced Riemannian metric provided that $N \geq n^2 + 10n + 3$.

John Nash had to invent some heavy duty PDE's (the Nash Implicit Function Theorem) to solve $X_i \bullet X_j = g_{ij}$ for X.

Extrinsic Geometry deals with how M sits in its ambient space.

How to measure the shape of a regular surface $M \subset \mathbf{R}^3$? Suppose that $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthonormal tangent vectors at $P \in M$ and $\mathbf{e}_3$ is a unit vector perpendicular to $T_P M$. Then near P, the surface may be parameterized as the graph over its tangent plane, where $f(u_1, u_2)$ is the "height" above the tangent plane

$$X(u_1, u_2) = P + u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + f(u_1, u_2)\mathbf{e}_3. \tag{2}$$

So $f(0) = 0$ and $Df(0) = 0$. The Hessian of f at 0 gives the shape operator at P. It is also called the Second Fundamental Form.

$$h_{ij}(P) = \frac{\partial^2 f}{\partial u_i \, \partial u_j}(0)$$

The Mean Curvature and Gaussian Curvature at P are

$$H(P) = \frac{1}{2} \operatorname{tr}(h_{ij}(P)), \qquad K(P) = \det(h_{ij}(P)).$$

The sphere about zero of radius $r > 0$ is an example

$$\mathbb{S}_r^2 = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = r^2\}.$$

Let $P = (0, 0, -r)$ be the south pole. By a rotation (an isometry of $\mathbb{R}^3$), any point of $\mathbb{S}_r^2$ can be moved to P with the surface coinciding. Thus the computation of $H(P)$ and $K(P)$ will be the same at all points of $\mathbb{S}_r^2$. If $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$, the height function of (2) near zero is given by

$$f(u_1, u_2) = r - \sqrt{r^2 - u_1^2 - u_2^2}.$$

The Hessian is

$$\frac{\partial^2 f}{\partial u_i\, u_j}(u) = \begin{pmatrix} \frac{r^2 - u_2^2}{\left(r^2 - u_1^2 - u_2^2\right)^{3/2}} & \frac{-u_1 u_2}{\left(r^2 - u_1^2 - u_2^2\right)^{3/2}} \\ \frac{-u_1 u_2^2}{\left(r^2 - u_1^2 - u_2^2\right)^{3/2}} & \frac{r^2 - u_1^2}{\left(r^2 - u_1^2 - u_2^2\right)^{3/2}} \end{pmatrix}$$

Thus the second fundamental form at P is

$$h_{ij}(P) = f_{ij}(0) = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \qquad \text{so } H(P) = \frac{1}{r} \text{ and } K(P) = \frac{1}{r^2}$$

If $X(u_1, u_2) = (u_1, u_2, f(u_1, u_2))$, by correcting for the slope at different points one finds for all $(u_1, u_2) \in U$,

$$H(u_1, u_2) = \frac{\left(1 + f_2^2\right) f_{11} - 2f_1 f_2 f_{12} + \left(1 + f_1^2\right) f_{22}}{2 \left(1 + f_1^2 + f_2^2\right)^{3/2}},$$

$$K(u_1, u_2) = \frac{f_{11} f_{22} - f_{12}^2}{\left(1 + f_1^2 + f_2^2\right)^2}.$$

So far, the formula for the Gauss Curvature has been given in terms of the second fundamental form and thus may depend on the extrinsic geometry of the surface. However, Gauss discovered a formula that he deemed excellent:

Theorem (Gauss's Theorema Egregium 1828)

Let $M^2 \subset \mathbf{R}^3$ be a smooth regular surface. Then the Gauss Curvature may be computed intrinsically from the metric and its first and second derivatives.

In other words, the Gauss Curvature coincides at corresponding points of isometric surfaces.

Thus the Gauss Curvature is an invariant that can be computed in abstract Riemannian manifolds.

The Latin word has the same root as "egregious" or "gregarious."

By a theorem of Korn and Lichtenstein, near every $P \in M$, a smooth regular surface, there is a coordinate chart in which the metric takes a nice form: $|X_1|^2 = g_{11} = \phi^2 = g_{22} = |X_2|^2$ and $X_1 \bullet X_2 = g_{12} = g_{21} = 0$.
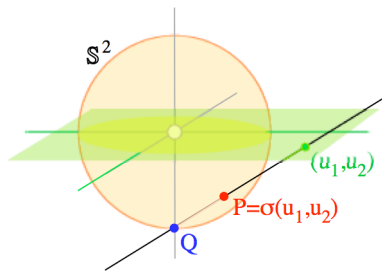
$$ds^2 = \phi(u)^2 \left(du_1^2 + du_2^2 \right)$$

ϕ^2 is called the conformal factor. The rectilinear coordinate grid in U is locally streched by the factor $\phi(u) > 0$. *X preserves angles.*

In these coordinates, the Gauss curvature takes the form

$$K(u) = -\frac{1}{\phi(u)^2} \Delta \log(\phi) \qquad \text{where } \Delta = \frac{\partial^2}{\partial u_1{}^2} + \frac{\partial^2}{\partial u_2{}^2}$$

is the U Laplacian.

Figure: Stereographic Projection. $P = \sigma(u_1, u_2)$ is the point on the sphere corresponding to $(u_1, u_2) \in \mathbf{R}^2$.

For the unit sphere $\mathbb{S}^2$ centered at the origin, imagine a line through the south pole Q and some other point $P \in \mathbb{S}^2$. This line crosses the $z = 0$ plane at some coordinate $x = u_1$ and $y = u_2$. Then we can express P in terms of (u_1, u_2). Thus $\sigma : U = \mathbf{R}^2 \to \mathbb{S}^2 - \{Q\}$ is a coordinate chart for the sphere called stereographic coordinates.

$$\sigma(u_1, u_2) = \left(\frac{2u_1}{1+u_1^2+u_2^2}, \frac{2u_2}{1+u_1^2+u_2^2}, \frac{1-u_1^2-u_2^2}{1+u_1^2+u_2^2} \right)$$

The tangent vectors for stereographic projection are

$$X_1 = \left(\frac{2 - 2u_1^2 + 2u_2^2}{\left(1 + u_1^2 + u_2^2\right)^2}, -\frac{4u_1 u_2}{\left(1 + u_1^2 + u_2^2\right)^2}, -\frac{4u_1}{\left(1 + u_1^2 + u_2^2\right)^2} \right),$$

$$X_2 = \left(-\frac{4u_1 u_2}{\left(1 + u_1^2 + u_2^2\right)^2}, \frac{2 + 2u_1^2 - 2u_2^2}{\left(1 + u_1^2 + u_2^2\right)^2}, -\frac{4u_2}{\left(1 + u_1^2 + u_2^2\right)^2} \right)$$

so that (u_1, u_2) are isothermal coordinates

$$X_1 \bullet X_2 = 0, \qquad \phi(u_1, u_2) = \sqrt{X_1 \bullet X_1} = \sqrt{X_2 \bullet X_2} = \frac{2}{1 + u_1^2 + u_2^2}.$$

Thus

$$K = -\frac{1}{\phi^2} \, \Delta \log(\phi) = 1.$$

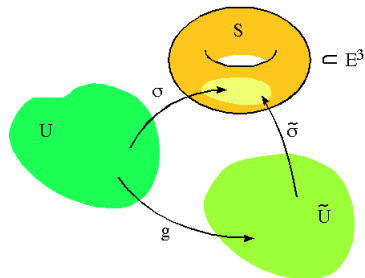In complex coordinates $z = u_1 + iu_2$, conformal factor $\phi(z) = \dfrac{2}{1 + |z|^2}$.

Figure: Transition between two isothermal coordinate charts.

Let two overlapping isothermal charts be given, $\sigma : U \to S$, $\tilde{\sigma} : \tilde{U} \to S$ with corresponding conformal factors

$$\phi(z)^2 \, |dz|^2 = \sigma^* \left(ds^2 \right), \qquad \tilde{\phi}(\tilde{z})^2 \, |d\tilde{z}|^2 = \tilde{\sigma}^* \left(ds^2 \right).$$

Written in complex notation, $z = x + iy$ so $|dz|^2 = dx^2 + dy^2$.

The induced metrics are consistently defined. The transition function $g : U \rightarrow \tilde{U}$ given by $g = \tilde{\sigma}^{-1} \circ \sigma$ turns out to be holomorphic (if orientation preserved) since angles are preserved. The transition identifies local metrics by a change of variables

$$\phi(z)^2 \, |dz|^2 = \tilde{\phi}(g(z))^2 \left| \frac{dg}{dz} \right|^2 |dz|^2 = g^* \left(\tilde{\phi}(\tilde{z})^2 \, |d\tilde{z}|^2 \right).$$

Thus, oriented surfaces with a Riemannian metric have the structure of a Riemann Surface.

We don't need to embed the surface in Euclidean Space as long as we have a cover S by charts and define the INTRINSIC METRIC of S chartwise in a consistent way.

The Riemannian metric gives length and angles of vectors and lengths of curves. If $\gamma : [\alpha, \beta] \to S$ then $\gamma(t) = \sigma(u(t))$ so in the conformal metric,

$$\mathsf{L}(\gamma) = \int_\alpha^\beta \phi\big(u(t)\big) \, |\dot{u}(t)| \, dt.$$

The Riemannian metric induces a distance function on S. If $P, Q \in S$,

$$\mathsf{d}(P, Q) = \inf \left\{ \mathsf{L}(\gamma) : \begin{array}{l} \gamma : [\alpha, \beta] \to S \text{ is piecewise } C^1, \\ \gamma(\alpha) = P, \, \gamma(\beta) = Q \end{array} \right\}$$
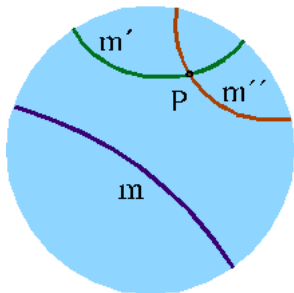
Theorem

(S, d) is a metric space.

Euclid's Postulates are the following:

1. any two points may be joined by a line segment;
2. any line segment may be extended to form a line;
3. a circle may be drawn with any given center and distance;
4. any two right angles are equal;
5. (Playfair's Version) Given any line m and a point p, there is a unique line through p and parallel to m.

In letters found after his death, Gauss had already realized in 1816 that there are geometries in which the Fifth Postulate fails. J. Bolyai and N. Lobachevski independently proved it in 1823 and 1826 by essentially constructing Poincaré's model. They assumed an axiom of Saccheri, who tried to reach a contradiction from it to prove the Fifth postulate.

Figure: m' and m'' are parallels to m through P. This is Poincaré's model of the Hyperbolic Plane $\mathbb{H}^2$. The space is the unit disk. Lines are diameters or arcs of circles that are perpendicular to the boundary circle.

5 Given any line m and a point p not in m, there are at least two lines through p and parallel to m.

This axiom is also known as the hyperbolic axiom. In 1854, Riemann showed a consistent geometry may also be constructed assuming instead that no lines are parallel, the elliptic axiom.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk. The Poincaré metric is

$$ds^2 = \phi(z)^2 \, |dz|^2 \qquad \text{where} \qquad \phi(z) = \frac{2}{1 - |z|^2}.$$

Thus

$$K = -\frac{1}{\phi^2} \, \Delta \log(\phi) = -1.$$

Theorem (Hilbert, 1901)

There is no C^2 isometric immersion $\sigma : \mathbb{H}^2 \to \mathbb{E}^3$.

The metric is invariant under rotation about the origin $z \mapsto e^{i\alpha}z$ ($\alpha \in \mathbf{R}$) and reflection $z \mapsto \bar{z}$. It is also invariant under the holomorphic self-maps of $\mathbb{D}$. Such maps $f : \mathbb{D} \to \mathbb{D}$ that fix the circle and map $p \in \mathbb{D}$ to 0 have the form

$$w = f(z) = \frac{e^{i\alpha}(z - p)}{1 - \bar{p}z}$$

They are isometries of the Poincaré plane because the pulled-back metric

$$
\begin{aligned}
f^*(ds^2) &= \phi(w)^2 |dw|^2 \\
&= \frac{4}{\left(1 - \frac{|z-p|^2}{|1-\bar{p}z|^2}\right)^2} \frac{(1-|p|^2)^2 |dz|^2}{|1-\bar{p}z|^4} \\
&= \frac{4(1-|p|^2)^2 |dz|^2}{(|1-\bar{p}z|^2 - |z-p|^2)^2} \\
&= \frac{4(1-|p|^2)^2 |dz|^2}{\left((1-\bar{p}z)(1-p\bar{z}) - (z-p)(\bar{z}-\bar{p})\right)^2} \\
&= \frac{4(1-|p|^2)^2 |dz|^2}{\left(1 - p\bar{z} - \bar{p}z + |p|^2|z|^2 - |z|^2 + \bar{p}z + p\bar{z} - |p|^2\right)^2} \\
&= \frac{4(1-|p|^2)^2 |dz|^2}{(1-|p|^2)^2(1-|z|^2)^2} = \frac{4|dz|^2}{(1-|z|^2)^2} = \phi(z)^2 |dz|^2.
\end{aligned}
$$

A geodesic is a curve that locally minimizes the length. A Calculus of Variations argument shows geodesics satisfy a 2nd order ODE.

If $\zeta : [a, b] \to U$ is minimizing in an isothermic patch, and $\eta : [a, b] \to \mathbb{C}$ is a variation such that $\eta(a) = \eta(b) = 0$, then the length $\mathsf{L}\left(\sigma(\zeta + \epsilon\eta)\right)$ is least when $\epsilon = 0$ so

$$
\begin{aligned}
0 &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathsf{L}(\zeta + \epsilon\eta) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_a^b \phi(\zeta + \epsilon\eta) \left|\dot{\zeta}(t) + \epsilon\dot{\eta}(t)\right| dt \\
&= \left. \left(\int_a^b \nabla\phi(\zeta + \epsilon\eta) \bullet \eta |\dot{\zeta} + \epsilon\dot{\eta}| + \phi(\zeta + \epsilon\eta) \frac{(\dot{\zeta} + \epsilon\dot{\eta}) \bullet \dot{\eta}}{|\dot{\zeta} + \epsilon\dot{\eta}|} dt \right) \right|_{\epsilon=0} \\
&= \int_a^b \left(\nabla\phi(\zeta)|\dot{\zeta}| - \frac{d}{dt}\left[\phi(\zeta)\frac{\dot{\zeta}}{|\dot{\zeta}|} \right] \right) \bullet \eta
\end{aligned}
$$

Since η is arbitrary, we deduce the Euler-Lagrange Equations. The geodesic satisfies the 2nd order ODE system

$$\frac{d}{dt}\left[\phi(\zeta)\frac{\dot{\zeta}}{|\dot{\zeta}|}\right] - \nabla\phi(\zeta)|\dot{\zeta}| = 0. \tag{3}$$

Combining ODE existence theorems with some geometry one gets

Theorem

For every $P \in S$ there is a neighborhood U such that if $Q_1, Q_2 \in U$ there there is a unique smooth distance realizing curve $\zeta : [\alpha, \beta] \to S$ from Q_1 to Q_2 such that $d(Q_1, Q_2) = L(\zeta)$, $\zeta([\alpha, \beta]) \subset U$ and ζ satisfies (3). Moreover, solutions of (3) are locally distance realizing.

For example, in $\mathbb{H}^2$, $\zeta(t) = (t, 0)$ is geodesic. $|\dot{\zeta}| = 1$,

$$\phi\big(\zeta(t)\big) = \frac{2}{1 - t^2}, \qquad \nabla\phi = \frac{4(u, v)}{(1 - u^2 - v^2)^2}.$$

Substituting

$$\frac{d}{dt}\left[\frac{2(1, 0)}{1 - t^2}\right] - \frac{4(t, 0)}{(1 - t^2)^2} = 0.$$

The length is independent of parametrization. Thus we may convert to arclength

$$s = \int_\alpha^t \phi\big(\zeta(t)\big)\,|\dot\zeta(t)|\,dt$$

so

$$\phi|\dot\zeta|\frac{d}{ds} = \frac{d}{dt}, \qquad \phi\zeta' = \frac{\dot\zeta}{|\dot\zeta|},$$

writing "$'$" for arclength derivatives.

$$\frac{d}{ds}\left[\phi(\zeta)\frac{\dot\zeta}{|\dot\zeta|}\right] - \nabla\ln\phi(\zeta)$$

or

$$\zeta'' + 2(\nabla\ln\phi \bullet \zeta')\zeta' - |\zeta'|^2\nabla\ln\phi = 0. \tag{4}$$

For example, in $\mathbb{E}^2$, $\phi \equiv 1$ so $\zeta'' = 0$ and $\zeta(s) = c_0 + c_1 s$: geodesics are straight lines. Note that any solution of (4) moves at a constant speed because $\phi(\zeta)|\zeta'|$ remains constant.

We shall assume our surfaces are complete. The geodesic equation is

$$\zeta'' + 2(\nabla \ln \phi \bullet \zeta')\zeta' - |\zeta'|^2 \nabla \ln \phi = 0. \qquad (5)$$

S is complete if solutions of the initial value problem for (5) can be infinitely extended.

Theorem (Hopf - Rinow)

S is complete if and only if (S, d) with the induced distance is a complete metric space. Completeness implies that for all $Q_1, Q_2 \in S$ there there is a distance realizing geodesic $\zeta : [\alpha, \beta] \to S$ from Q_1 to Q_2 such that $d(Q_1, Q_2) = L(\zeta)$.
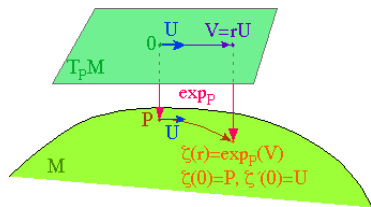
Figure: Exponential Map.

The solvability, uniqueness and smooth dependence of the initial value problem for (5) lets us define the exponential map

$$\exp_P : T_P M \to M.$$

Fix $P \in M$. The map takes a vector $V \in T_P M$ with length r and maps it to the endpoint of a geodesic of length r which starts at P and heads in the direction V. (So if $r = \phi(p)|V|$, let $\zeta(t)$ be the solution of the initial value problem for (5) with starting point $\zeta(0) = P$ and direction $\zeta'(0) = \frac{V}{r}$. Define $exp_P(V) = \zeta(r)$.)

The exponential map gives polar coordinates (also called normal coordinates) near P. If M is a surface then a unit vector $U(\theta) \in T_P M$ is determined by its angle θ. Thus any vector $V \in T_P M$ can be written $V = rU(\theta)$ for some unit vector $U(\theta)$ and scalar $r \geq 0$. The coordinate chart near P is

$$\sigma(r, \theta) = \exp_P\big(rU(\theta)\big).$$

The metric of M in polar coordinates turns out to be

$$ds^2 = dr^2 + J(r,\theta)^2 \, d\theta \qquad (6)$$

where $J \geq 0$ in $\mathcal{U}$.

(The fact that circles of radius r about P cross the geodesic rays emanating from P orthogonally, hence no cross term, is a lemma of Gauss.)

In these coordinates, the Gauss Curvature is

$$K = -\frac{J_{rr}}{J}.$$

It follows that the expansion of the area growth of the r-ball near $r = 0$ has the Euclidean value with a correction due to the Gauss curvature

$$A\big(B(0,r)\big) = \pi r^2 \left(1 - \frac{K(P)\, r^2}{12} + \cdots \right)$$

For example, if $P = 0$ in $\mathbb{H}^2$ then the length of the segment from $(0,0)$ to $(t,0)$ in $\mathbb{H}^2$ is

$$\rho = \int_0^t \frac{2\,dt}{1-t^2} = \ln\left(\frac{1+t}{1-t}\right) \quad \Longleftrightarrow \quad t = \tanh\left(\frac{\rho}{2}\right).$$

The exponential map takes (ρ, θ) in polar coordinates of $\mathbb{E}^2 = T_P\mathbb{H}^2$ to $(t, \theta) \in \mathbb{H}^2$. Pulling back the Poincaré metric

$$d\rho^2 + \sinh^2\!\rho\, d\theta^2 = \frac{\operatorname{sech}^4(\frac{\rho}{2})\,d\rho^2 + 4\tanh^2(\frac{\rho}{2})\,d\theta^2}{\left(1 - \tanh^2(\frac{\rho}{2})\right)^2} = \exp_P^*\!\left(\frac{4(dt^2 + t^2 d\theta^2)}{(1-t^2)^2}\right)$$

Thus

$$K = -\frac{J_{rr}}{J} = -\frac{1}{\sinh\rho}\,\frac{\partial^2}{\partial\rho^2}\,\sinh\rho = -1.$$

In polar coordinates, $\mathbb{H}^2 = (\mathbf{R}^2, d\rho^2 + \sinh^2\rho \, d\theta^2)$. Let $B(0, r)$ be the disk about the origin of radius r (measured in $\mathbb{H}^2$.). Then

$$L\big(\partial B(0, r)\big) = \int_0^{2\pi} \sinh r \, d\theta = 2\pi \sinh r,$$

$$A\big(B(0, r)\big) = \int_0^{2\pi} \int_0^r \sinh r \, dr \, d\theta = 2\pi(\cosh r - 1).$$

The Taylor expansion near $r = 0$ gives

$$A\big(B(0, r)\big) = \pi r^2 \left(1 + \frac{r^2}{12} + \cdots \right) = \pi r^2 \left(1 - \frac{K(0) \, r^2}{12} + \cdots \right)$$

thus $K(0) = -1$.

Let $ds^2 = \phi(z)^2 |dz|^2$ be the metric in an isothermal coordinate patch. The intrinsic area form, gradient and Laplacian are given by the formulas

$$dA = \phi^2 \, dx \, dy; \qquad |\nabla u|^2 = \frac{u_x^2 + u_y^2}{\phi^2}; \qquad \Delta u = \frac{u_{xx} + u_{yy}}{\phi^2} = \frac{4u_{z\bar{z}}}{\phi^2}.$$

Let $u \in C^1(\Omega)$ be a function on a domain $\Omega \subset S$. Then the energy or Dirichlet integral is invariant under conformal change of metric

$$\int_\Omega |\nabla u|^2 \, dA = \int_\Omega u_x^2 + u_y^2 \, dx \, dy.$$

A function is harmonic if $\Delta u = 0$ and subharmonic if $\Delta u \geq 0$ (at least weakly.) These notions agree regardless of conformal metric $\phi^2 |dz|^2$.

We seek generalizations of the Riemann mapping theorem to surfaces.

Theorem (Riemann Mapping Theorem)

Let $\Omega \subset \mathbf{R}^2$ be a simply connected open set that is not the whole plane. Then there is an analytic, one-to-one mapping onto the disk $f : \Omega \to \mathbb{D}$.

A noncompact, simply connected surface S is said to be parabolic if it is conformally equivalent to the plane. That is, there is a global isothermal coordinate chart $\sigma : \mathbb{C} \to S$. Otherwise the surface is called hyperbolic. The sphere $\mathbb{S}^2$ is compact, so it is neither parabolic nor hyperbolic.

Theorem (Koebe's Uniformization Theorem)

Let S be a simply connected Riemann Surface. Then S is conformally equivalent to the disk, the plane or to the sphere.

Conceivably, the topological disk could have many conformal structures, but the uniformization theorem tells us there are only two. The topological sphere has only one conformal structure.

For each $p \in S$, the positive Green's function $z \mapsto g(z, p)$ is harmonic for $z \in S - \{p\}$, $g(z, p) > 0$, $\inf_z g(z, p) = 0$ and in a isothermal patch around p, $g(z, p) + \ln |z - p|$ has a harmonic extension to a neighborhood of p (so $g(z, p) \to \infty$ as $z \to p$.)

Theorem

Let S be a simply connected noncompact Riemann surface. Then the following are equivalent.

- S is hyperbolic.
- S has a positive Green's function.
- S has a negative nonconstant subharmonic function.
- S has a bounded nonconstant harmonic function.

e. g., $u = ax + by$ is a bounded harmonic function on $\mathbb{D}$ hence on $\mathbb{H}^2$, but there are no bounded harmonic functions on $\mathbb{E}^2$ by Liouville's Theorem.

A surface S is said to have finite total curvature if

$$\int_S |K|\, dA < \infty.$$

Theorem (Blanc & Fiala, Huber)

Let S be a noncompact, complete surface with finite total curvature. Then S is conformally equivalent to a closed Riemann surface of genus g with finitely many punctures $\Sigma_g - \{p_1, \ldots, p_k\}$.

For example, $\mathbb{E}^2$ has zero total curvature and is conformal to $\mathbb{S}^2 - \{Q\}$ via stereographic projection.

To illustrate something of the ways of geometric analysis, we sketch the proof for the simply connected case.

If S is complete, then $\exp_p : T_PS \to S$ is onto. Let $\mathbf{e}_1, \mathbf{e}_2 \in T_pS$ be orthonormal vectors. Let $U(\theta) = \cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2$. Consider the unit speed geodesic $\gamma(t, \theta) = \exp_P(tU(\theta))$ from P in the $U(\theta)$ direction. For each $r > 0$ let $\Theta(r) \in \mathbb{S}^1$ be the set of directions θ such that $\gamma(\bullet, \theta)$ is minimizing over $[0, r]$.

Thus if $r_1 < r_2$ we have $\Theta(r_2) \subset \Theta(r_1)$. Let $\mathcal{U} = \cup_{r>0} rU(\Theta(r))$. It turns out that $\exp_P(\mathcal{U})$ covers all of S except for a set of measure zero.

The variation vector field measures the spread of geodesics as they are rotated about $P \in S$.

$$V = \frac{d}{d\theta}\gamma(t,\theta) \tag{7}$$

is perpendicular to $\dot{\gamma}(t,\theta)$ and has length $J(t,\theta)$ as in the metric in polar coordinates (6). By differentiating the geodesic equation (5) with respect to θ one finds the Jacobi Equation

$$J_{ss}(s,\theta) + K\big(\gamma(s,\theta)\big)\, J(s,\theta) = 0 \tag{8}$$

with initial conditions, $J(0,\theta) = 0$ and $J_s(0,\theta) = 1$.

For example, in $\mathbb{H}^2$, $J(s,\theta) = \sinh s$ and $K \equiv -1$.
For $\mathbb{E}^2$, $J(s,\theta) = s$ and $K \equiv 0$.

Lemma

Assume that S is a complete, noncompact, simply connected surface with finite total curvature

$$\int_S |K|\, dA = C < \infty.$$

Then

$$\mathrm{L}\big(\partial B(p, r)\big) \le (2\pi + C)r.$$

Proof Idea. By the Jacobi equation (7),

$$
\begin{aligned}
J_r(r, \theta) - 1 &= J_r(r, \theta) - J_r(0, \theta) \\
&= \int_0^r J_{rr}(r, \theta)\, dr \\
&= -\int_0^r K\big(\gamma(r, \theta)\big) J(r, \theta)\, dr.
\end{aligned}
$$

46. Growth of a geodesic circle proof.

The length $L(r) = \mathrm{L}\big(\partial B(p, r)\big)$ grows at a rate

$$\frac{dL}{dr} = \lim_{h \to 0+} \frac{L(r+h) - L(r)}{h}$$

$$= \lim_{h \to 0+} \frac{1}{h} \left(\int_{\Theta(r+h)} J(r+h, \theta) \, d\theta - \int_{\Theta(r)} J(r, \theta) \, d\theta \right)$$

$$= \lim_{h \to 0+} \left(\int_{\Theta(r+h)} \frac{J(r+h, \theta) - J(r, \theta)}{h} \, d\theta - \frac{1}{h} \int_{\Theta(r) - \Theta(r+h)} J(r, \theta) \, d\theta \right)$$

$$\leq \int_{\Theta(r)} J_r(r, \theta) \, d\theta$$

$$= \int_{\Theta(r)} \left(1 - \int_0^r K\big(\gamma(r, \theta)\big) J(r, \theta) \, dr \right) d\theta$$

$$\leq 2\pi + \int_0^r \int_{\Theta(r)} \big| K\big(\gamma(r, \theta)\big) \big| \, J(r, \theta) \, d\theta \, dr$$

$$= 2\pi + \int_{B(P, r)} |K| \, dA \quad \leq 2\pi + C.$$

Theorem (Blanc & Fiala)

Let S be a complete, noncompact, simply connected Riemann surface of finite total curvature. Then S is parabolic.

Proof idea. Suppose not. Then S has a global isothermal chart $\sigma : \mathbb{D} \to S$. Then $u = -\ln|z|$ is a positive Green's function on S. Let $\epsilon > 0$ and $K \subset \mathbb{D} - \overline{B(0,\epsilon)}$ be any compact domain. On the one hand, the energy is uniformly bounded

$$\mathcal{E}(K) = \int_K u_x^2 + u_y^2 \, dx \, dy \leq -2\pi \ln \epsilon.$$

This estimate is done in the background $\mathbb{D}$ metric. But energy is conformally invariant, so it will hold in the surface metric also.

Indeed, let $R = \sup\{|z| : z \in K\} < 1$ and $\mathcal{A} = \{z \in \mathbb{D} : \epsilon \leq |z| \leq R\}$ be an annulus containing K. Using the fact that $u > 0$ and $u_r < 0$ on K, by integrating by parts,

$$
\begin{aligned}
\mathcal{E}(K) \leq \mathcal{E}(\mathcal{A}) &= -\int_{\mathcal{A}} u \Delta_0 u \, dx \, dy + \oint_{|z|=\epsilon} u \frac{\partial u}{\partial n} + \oint_{|z|=R} u \frac{\partial u}{\partial n} \\
&\leq 0 - \oint_{|z|=\epsilon} \frac{\ln \epsilon}{\epsilon} + 0 \\
&\leq -2\pi \ln \epsilon
\end{aligned}
$$

since on $|z| = \epsilon$, $u = -\ln \epsilon$ and $u_r = -\frac{1}{\epsilon}$ and $\mathsf{L}(\{z : |z| = \epsilon\}) = 2\pi\epsilon$.

On the other hand, bounded total curvature will imply that the energy will grow to infinity. For the remainder of the argument, work in intrinsic polar coordinates. Let $B(0, r)$ denote the intrinsic ball for $r \geq 1$ and

$$\mathcal{E}(r) = \int_{B(0,r) - \overline{B(0,1)}} |Du|^2 \, dA = \int_1^r \int_{\partial B(0,r)} |Du|^2 \, ds \, dr$$

where s is length along $\partial B(0, r)$. By the Schwartz inequality,

$$L(r) \frac{d\mathcal{E}}{dr} = \int_{\partial B(0,r)} ds \int_{\partial B(0,r)} |Du|^2 \, ds \geq \left(\int_{\partial B(0,r)} |Du| \, ds \right)^2$$

$$\geq \left(\oint_{\partial B(0,r)} \frac{\partial u}{\partial n} \right)^2 = \left(\int_{B(0,r) - \overline{B(0,\delta)}} \Delta u \, dA - \oint_{\partial B(0,\delta)} \frac{\partial u}{\partial n} \right)^2 = c_0$$

for any fixed $\delta \in (0, 1]$ where $c_0 > 0$ is independent of r.
In fact, $c_0 \to 4\pi^2$ as $\delta \to 0$.

Now, using the length lemma,

$$\frac{d\mathcal{E}}{dr} \geq \frac{c_0}{(2\pi + C)r}.$$

Integrating, this says

$$\mathcal{E}(r) \geq \frac{c_0 \ln r}{(2\pi + C)} \to \infty \text{ as } r \to \infty.$$

This is a contradiction because the energy is invariant under conformal change and is uniformly bounded. $\square$

- To understand the argument, try using it to prove $\mathbb{E}^2$ is parabolic!

A harmonic map locally minimizes energy of a map between surfaces, generalizing harmonic functions and geodesics.

If $h : (S_1, \phi(z)^2 \, |dz|^2) \to (S_2, \psi(w)^2 \, |dw|^2)$ is harmonic, then it satisfies the PDE system

$$h_{z\bar{z}} + 2\frac{\psi_w(h)}{\psi(h)} \, h_z \, h_{\bar{z}} = 0.$$

Theorem

If $f : \tilde{S} \to S_1$ is a conformal diffeomorphism and $h : S_1 \to S_2$ is harmonic, then $h \circ f : \tilde{S} \to S_2$ is harmonic.

Theorem (Treibergs 1986)

Let $\mathcal{I} \subset \partial\mathbb{D}$ be any closed set with at least three distinct points and $\text{Conv}(\mathcal{I})$ its convex hull in $\mathbb{H}^2$, then there is a complete, spacelike entire constant mean curvature surface $\mathcal{S}$ in Minkowski Space such that the Gauss map $\mathcal{G} : S \rightarrow \text{Conv}(\mathcal{I})$ is a harmonic diffeomorphism.

Minkowski Space is $\mathbb{E}^{2,1} = (\mathbf{R}^3, dx^2 + dy^2 - dz^2)$.
An *entire, spacelike surface* $\mathcal{S} \subset \mathbb{E}^{2,1}$ is the graph of a function
$\mathcal{S} = \left\{ (x, y, u(x, y)) : (x, y) \in \mathbf{R}^2 \right\}$ such that $u_x^2 + u_y^2 < 1$.
The Gauss map is the map given by the future-pointing unit normal

$$\mathcal{G} = \frac{(u_x, u_y, 1)}{\sqrt{1 - u_x^2 - u_y^2}} : \mathcal{S} \rightarrow \mathcal{H}.$$

The hyperboloid $\mathcal{H} = \{(x, y, x) : x^2 + y^2 - z^2 = -1, \ z > 0\}$ consists of all future-pointing vectors of length -1.
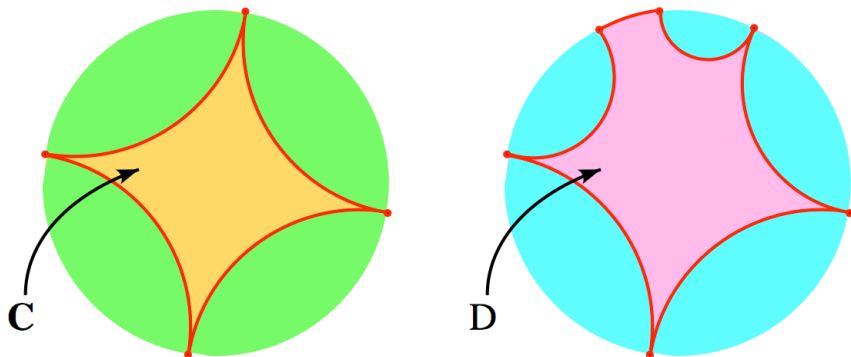The *hyperboloid model* $(\mathcal{H}, dx^2 + dy^2 - dz^2)$ is isometric to $\mathbb{H}^2$.

Figure: Harmonic diffeomorphisms

Corollary

Let $\mathcal{I} \subset \partial \mathbb{B}$ be a closed set with at least three points.

- If $\mathcal{I}$ is finite, there is a harmonic diffeomorphism

$$h : \mathbb{C} \to Conv(\mathcal{I}).$$

- If $\mathcal{I}$ has nonempty interior, then there is a harmonic diffeomorphism

$$h : \mathbb{B} \to Conv(\mathcal{I}).$$

Since $\mathcal{S}$ is convex, its total curvature is $-A\big(\mathcal{G}(\mathcal{S})\big) = \pi(2 - \sharp\mathcal{I})$ by the Gauss-Bonnet Theorem in $\mathbb{H}^2$. By Blanc-Fiala's Theorem, $\mathcal{S}$ is conformal to the plane if $\mathcal{I}$ is finite.

If $\mathcal{I}$ has nonempty interior then one can construct a nonconstant bounded harmonic function on $\mathcal{S}$ so it is conformal to the disk.

Thanks!