The Direct Method

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Notes for the first of three lectures on Calculus of Variations.

1. Andrejs Treibergs, “The Direct Method,” March 19, 2010,
2. Predrag Krtolica, “Falling Dominoes,” April 2, 2010,
3. Andrej Cherkaev, “ ‘Solving’ Problems that Have No Solutions,” April 9, 2010.

The URL for these Beamer Slides: *“The Direct Method”*

`http://www.math.utah.edu/~treiberg/DirectMethodSlides.pdf`

- Richard Courant, *Dirichlet's Principle, Conformal Mapping and Minimal Surfaces,* Dover (2005); origianlly pub. in series: Pure and Applied Mathematics 3, Interscience Publishers, Inc. (1950).
- Lawrence C. Evans, *Partial Differential Equations,* Graduate Studies in Math. 19, American Mathematical Society (1998).
- Mario Giaquinta, *Multiple Integrals in the Calculus of Variations,* Annals of Math. Studies 105, Princeton University Press, 1983.
- Robert McOwen, *Partial Differential Equations: Methods and Applications* 2nd ed., Prentice Hall (2003).
- Richard Schoen & S. T. Yau, *Lectures on Differential Geometry,* Conf. Proc. & Lecture Notes in Geometry and Topology 1, International Press (1994).

- Euler Lagrange Equations for Constrained Problems
- Solution of Minimization Problems
  - Dirichlet's principle and the Yamabe Problem.
  - Outline of the Direct Method.
  - Examples of failure of the Direct Method.
- Solve in a special case: Poisson's Minimization Problem.
  - Cast the minimization problem in Hilbert Space.
  - Coercivity of the functional.
  - Continuity of the functional.
  - Differentiability of the functional.
  - Convexity of the functional and uniqueness.
- Regularity

To find the shortest curve $\gamma(t) = (t, u(t))$ in the Euclidean plane from (a, y_1) to (b, y_2) we seek the function $u \in \mathcal{A}$, the admissible set

$$\mathcal{A} = \{w \in \mathcal{C}^1([a, b]) : w(a) = y_1,\ w(b) = y_2\}$$

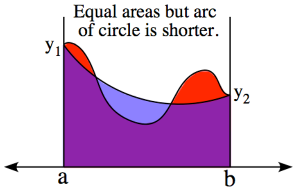
that minimizes the length integral

$$\mathrm{L}(u) = \int_\Omega \sqrt{1 + \dot{u}^2(t)}\, dt.$$

We saw that the minimizing curve would satisfy the Euler Equation

$$\frac{d}{dt}\left(\frac{\dot{u}}{\sqrt{1 + \dot{u}^2}}\right) = 0$$

whose solution is $u(t) = c_1 t + c_2$, a straight line.



If we assume also that the curve from (a, y_1) to (b, y_2) enclose a fixed area j_0 between $a \le x \le b$ and between γ and the x-axis, then the admissible set becomes curves with the given area and now seek $u \in \mathcal{A}'$ that minimizes the length integral

$$\mathrm{L}(u) = \int_\Omega \sqrt{1 + \dot{u}^2(t)}\, dt.$$

$$\mathcal{A}' = \left\{ w \in \mathcal{C}^1 : \begin{array}{c} w(a) = y_1, \\ w(b) = y_2, \\ \int_\Omega u(t)dt = j_0 \end{array} \right\}$$

This is the Isoperimetric Problem. We'll see that the corresponding differential equations, the Euler-Lagrange equations say that the curve has constant curvature, thus consists of an arc of a circle.

But this may be unsolvable! For a given choice of endpoints and j_0, there may not be a circular arc $\gamma(t) = (t, u(t))$ that connects the endpoints and encloses an area j_0.

More generally, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $f(x, u, p), g(x, u, p) \in \mathcal{C}^2(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n)$ and $\phi \in \mathcal{C}^1(\overline{\Omega})$. Suppose that we seek to find $u \in \mathcal{A} = \{x \in \mathcal{C}^1(\overline{\Omega}) : \; w = \phi \text{ on } \partial\Omega\}$ that achieves the extremum of the integral

$$I(u) = \int_\Omega f(x, u(x), Du(x))\, dx$$

subject to the constraint that

$$J(u) = \int_\Omega g(x, u(x), Du(x))\, dx = j_0$$

takes a prescribed value.

Choose two functions $\eta_i, \eta_2 \in \mathcal{C}^1(\overline{\Omega})$ such that

$$\eta_1(z) = \eta_2(z) = 0 \qquad \text{for all } z \in \partial\Omega.$$

Then, assuming that $u(x)$ is a solution of the constrained optimization problem, consider the two-parameter variation

$$U(x, \varepsilon_1, \varepsilon_2) = u(x) + \varepsilon_1 \eta_1(x) + \varepsilon_2 \eta_2(x)$$

which satisfies $U = \phi$ on $\partial\Omega$ so $U \in \mathcal{A}$. Substitute U

$$I(\varepsilon_1, \varepsilon_2) = \int_\Omega f(x, U, DU)\, dx$$
$$J(\varepsilon_1, \varepsilon_2) = \int_\Omega g(x, U, DU)\, dx$$

I is extremized in $\mathbf{R}^2$ when $\varepsilon_1 = \varepsilon_2 = 0$ subject to the constraint

$$J(\varepsilon_1, \varepsilon_2) = j_0.$$

Using Lagrange Multipliers, there is a constant λ so that solution is the critical point of the Lagrange function

$$\mathcal{L}(\varepsilon_1, \varepsilon_2) = I(\varepsilon_1, \varepsilon_2) + \lambda J(\varepsilon_1, \varepsilon_2) = \int_\Omega h(x, U, DU)\, dx$$

where

$$h(x, U, DU) = f(x, U, DU) + \lambda g(x, U, DU).$$

The extremum satisfies

$$\frac{\partial \mathcal{L}}{\partial \varepsilon_1} = \frac{\partial \mathcal{L}}{\partial \varepsilon_2} = 0 \quad \text{when } \varepsilon_1 = \varepsilon_2 = 0.$$

But for $i = 1, 2$,

$$\frac{\partial \mathcal{L}}{\partial \varepsilon_i} = \int_\Omega \left\{ \frac{\partial h}{\partial U}\frac{\partial U}{\partial \varepsilon_i} + \sum_{j=1}^n \frac{\partial h}{\partial p_j}\frac{\partial^2 U}{\partial \varepsilon_i\, \partial x_j} \right\} = \int_\Omega \left\{ \frac{\partial h}{\partial U}\eta_i + \sum_{j=1}^n \frac{\partial h}{\partial p_j}\frac{\partial \eta_i}{\partial x_j} \right\}$$

At the critical point when $\varepsilon_i = 0$ for $i = 1, 2$ so $U = u$,

$$\left.\frac{\partial \mathcal{L}}{\partial \varepsilon_i}\right|_{\eta_1=\varepsilon_2=0} = \int_\Omega \left\{ \frac{\partial h}{\partial u}\eta_i + D_p h \bullet D_x \eta_i \right\} dx = 0,$$

which is the weak form of the Euler Lagrange equations. Integrating the second term by parts (assuming it is OK to do so), using $\eta_i = 0$ on the boundary

$$\int_\Omega \eta_i \left\{ \frac{\partial h}{\partial u} - \operatorname{div}\left(D_p h\right) \right\} dt = 0.$$

As both η_i are arbitrary, we obtain the Euler-Lagrange equations

$$\frac{\partial h}{\partial u} - \operatorname{div}\left(D_p h\right) = 0.$$

In this case,

$$h = f + \lambda g = \sqrt{1+\dot{u}^2} + \lambda u.$$

Thus

$$\begin{aligned} 0 &= \frac{\partial h}{\partial u} - \frac{d}{dt}\left(\frac{\partial h}{\partial \dot{u}}\right) \\ &= \lambda - \frac{d}{dt}\left(\frac{\dot{u}}{\sqrt{1+\dot{u}^2}}\right) \\ &= \lambda - \frac{\ddot{u}}{\left(1+\dot{u}^2\right)^{3/2}} \\ &= \lambda - \kappa \end{aligned}$$

where κ is the curvature of the curve γ. Thus solutions of the E-L equations are curves of constant curvature which are arcs of circles.

The great strides of Riemann's Thesis (1851) and memoir on Abelian Functions (1857) relied on the statement that there exists a minimizer of an electrostatics problem which he did not prove! Riemann had learned it from Dirichlet's lectures. On a surface, which he thought of as a thin conducting sheet, he imagined the stationary electric current generated by connecting arbitrary points of the surface with the poles of an electric battery. The potential of such a current will be the solution of a boundary value problem. The corresponding variational problem is to find among all possible flows the one that produces the least quantity of heat.

(Dirichlet's Principle)

Let $G \subset \mathbf{R}^2$ (or in a smooth surface) be a compact domain and $\phi \in \mathcal{C}(\partial G)$. Then there is $u \in \mathcal{C}^1(G) \cap \mathcal{C}(\overline{G})$ that satisfies $u = \phi$ on ∂G and minimizes the Dirichlet Integral

$$D[u] = \int_G |Du|^2 \, dA.$$

Moreover, $\Delta u = 0$ on G.

Because $D[u] \geq 0$ it was assumed that minimizers exist. Weierstrass found the flaw in the argument and published his objections in 1869.

Dirichlet's Principle became the stimulus of much work. Finally, Hilbert proved the Dirichlet Principle in 1900 assuming appropriate smoothness.

Hilbert challenged mathematicians to solve what he considered to be the most important problems of the 20th century in his address to the International Congress of Mathematicians in Paris, 1900.

20th *"Has not every variational problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied, and provided also that if need be that the notion of solution be suitably extended?"*

19th *"Are the solutions of regular problems in the Calculus of Variations always necessarily analytic?"*

These questions have stimulated enormous effort and now things are much better understood. The existence and regularity theory for elliptic PDE's and variational problems is one of the greatest achievements of 20th century mathematics.

In 1960, Yamabe tried to prove the Poincaré Conjecture that every smooth closed orientable simply connected three manifold M is the three sphere using a variational method. He tried to find an Einstein metric $(\mathrm{Ric}(g) = cg)$ where Ric is the Ricci curvature and c is a constant, which would have to have constant sectional curvature in three dimensions. Consider the functional on the space $\mathcal{M}$ of all smooth Riemannian metrics on M. For $g \in \mathcal{M}$,

$$\mathcal{Q}(g) = \frac{\int_M R_g \, d\mu_g}{\left(\int_M d\mu_g\right)^{\frac{n-2}{n}}}$$

where $R_g =$ scalar curvature and $\mu_g =$ volume form for g. The Einstein Metrics are maxi-min critical points. g_0 is Einstein if it is critical so

$$\mathcal{Q}(g_0) = \sup_{g_1 \in \mathcal{M}} \inf_{g \in \mathcal{C}_{g_1}} \mathcal{Q}(g)$$

$\mathcal{C}_{g_1} = \{\rho g_1 : \rho \in \mathcal{C}^\infty(M) : \rho > 0\}$ are all metrics conformal to g_1.

The "mini" part for $n \geq 3$ is called the Yamabe Problem. For $n = 2$ this is equivalent to the Uniformization Theorem: every closed surface carries a constant curvature metric.

Theorem (Yamabe Problem.)

Let (M^n, g_0) be an $n \geq 3$ dimensional smooth compact Riemannian manifold without boundary. Then there is $\rho \in \mathcal{C}^\infty(M)$ with $\rho > 0$ so that

$$\mathcal{Q}(\rho g_0) = \inf_{g \in \mathcal{C}_{g_0}} \mathcal{Q}(g).$$

Moreover, $\tilde{g} = \rho g_0$ has constant scalar curvature $R_{\tilde{g}}$.

The Yamabe Problem was resolved in steps taken by Yamabe (1960), Trudinger (1974), Aubin (1976) and Schoen (1984).
The Poincaré Conjecture has since been practically resolved using Ricci Flow by Hamilton and Perelman (2004).

We illustrate with a simple minimization problem. Let $\Omega \subset \mathbf{R}^n$ be a connected bounded domain with smooth boundary. Let us assume that $f(x,u,p) \in \mathcal{C}^1(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n)$. For $u \in \mathcal{C}^1(\overline{\Omega})$ we define the functional

$$\mathcal{F}(u) = \int_\Omega f\big(x, u(x), Du(x)\big)\,dx$$

If $\phi \in \mathcal{C}^1(\overline{\Omega})$, then we define the set of admissible functions to be

$$\mathcal{A} = \left\{ u \in \mathcal{C}^1(\overline{\Omega}) : \; u = \phi \text{ on } \partial\Omega \;\right\}.$$

The minimization problem is to find $u_0 \in \mathcal{A}$ so that

$$\mathcal{F}(u_0) = \mathcal{I} := \inf_{u \in \mathcal{A}} \mathcal{F}(u).$$

18. Poisson Minimization Problem.

Let us consider a specific example of energy minimization. Let $\Omega \subset \mathbf{R}^n$ be a bounded, connected domain with smooth boundary. Let $\psi \in \mathcal{L}^2(\Omega)$ and $\phi \in \mathcal{C}^1(\overline{\Omega})$. Let us consider the energy functional defined for $u \in \mathcal{A}$

$$\mathcal{F}(u) = \int_\Omega \tfrac{1}{2}|Du|^2 + \psi u \, dx.$$

For $f(x, u, p) = \frac{1}{2}|p|^2 + \psi u$, the Euler equation is to find $u \in \mathcal{A}$ such that

$$0 = \frac{\partial f}{\partial u} - \operatorname{div}(D_p f) = \psi - \operatorname{div}(Du) \qquad \text{in } \Omega.$$

This is Poisson's Equation, usually written

$$\begin{aligned} \Delta u &= \psi, && \text{in } \Omega; \\ u &= \phi, && \text{on } \partial\Omega. \end{aligned}$$

A useful fact is that a solution of an elliptic PDE is gotten by minimizing energy. Taking $\psi = 0$ gives Dirichlet's Principle.
There are other methods to handle linear equations like this but we shall use it to illustrate the direct method.

We sketch the proof of

Theorem (Poisson's Minimization Problem.)

Let $\Omega \subset \mathbf{R}^n$ be a bounded, connected domain with smooth boundary. Let $\phi, \psi \in \mathcal{C}^\infty(\overline{\Omega})$. For $u \in C^1(\Omega)$, let

$$\mathcal{F}(u) = \int_\Omega \tfrac{1}{2}|Du|^2 + \psi u \, dx.$$

Then there is a unique $u_0 \in \mathcal{C}^\infty(\overline{\Omega})$ with $u_0 = \phi$ on $\partial\Omega$ such that

$$\mathcal{F}(u_0) = \inf_{u \in \mathcal{A}'} \mathcal{F}(u)$$

where $\mathcal{A}' = \left\{ u \in C(\overline{\Omega}) \cap \mathcal{C}^1(\Omega) : u = \phi \text{ on } \partial\Omega.) \right\}$. Also, $\Delta u_0 = \psi$ in Ω.

To find a u_0 that minimizes $\mathcal{F}(u)$ in $\mathcal{A}$.

1. Choose a minimizing sequence $\{u_n\} \subset \mathcal{A}$ so that as $n \to \infty$,

$$\mathcal{F}(u_n) \to \mathcal{I} = \inf_{u \in \mathcal{A}} \mathcal{F}(u).$$

2. Select a convergent subsequence

$$u_{n'} \to u_0 \in \mathcal{A} \text{ as } n' \to \infty.$$

3. Exchange limits

$$\mathcal{F}(u_0) = \mathcal{F}\left(\lim_{n' \to \infty} u_{n'}\right) = \lim_{n' \to \infty} \mathcal{F}(u_{n'}) = \mathcal{I}.$$

("$\leq$" lower semi-continuity will do.)

There may be problems with these steps!

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   $\mathcal{F}$ may not have a lower bound.

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   $\{u_n\}$ may not be compact.

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   (“$\leq$” lower semi-continuity will do.)

There may be problems with these steps!

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   $\mathcal{F}$ may not have a lower bound.

   $$\mathcal{F}(u_n) \to \mathcal{I} = \inf_{u \in \mathcal{A}} \mathcal{F}(u).$$

2. Select a convergent subsequence

   $\{u_n\}$ may not be compact.

   $$u_{n'} \to u_0 \in \mathcal{A} \text{ as } n' \to \infty.$$

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   $$\mathcal{F}(u_0) = \mathcal{F}\left(\lim_{n' \to \infty} u_{n'}\right) = \lim_{n' \to \infty} \mathcal{F}(u_{n'}) = \mathcal{I}.$$

   $\mathcal{F}$ may not be lower semi-continuous.

   (“$\leq$” lower semi-continuity will do.)

Let $[0,\pi] \subset \mathbf{R}^1$ and $\mathcal{A} = \{u \in \mathcal{C}^1([0,\pi]) : u(0) = u(\pi) = 0\}$.

- $\mathcal{F}$ is not bounded below: take
$$\mathcal{F}(u) = \int_0^\pi \dot{u}^2 - 2u^2\,dx \quad \text{and} \quad u_n(x) = n \sin x \in \mathcal{A}.$$
But $\mathcal{F}(u_n) = -\frac{\pi n^2}{2} \to -\infty$ as $n \to \infty$. (Also, u_n is unbounded.)
- No convergent subsequence in minimizing sequence: take
$$\mathcal{F}(u) = \int_0^\pi \left(\dot{u}^2 - 1\right)^2\,dx,\ u_n(x) = \sqrt{\frac{1}{n^2} + \frac{\pi^2}{4}} - \sqrt{\frac{1}{n^2} + \left(x - \frac{\pi}{2}\right)^2}$$
$u_n(x) \in \mathcal{A}$ and $\mathcal{F}(u_n) \to 0 = \inf_{v \in \mathcal{A}} \mathcal{F}(v)$ but every subsequence converges to $\frac{\pi}{2} - \left|x - \frac{\pi}{2}\right|$ which is not in $\mathcal{A}$.
- $\mathcal{F}$ is not lower semi-continuous: Let $g(p) = \begin{cases} p^2, & \text{if } p \neq 0; \\ 1, & \text{if } p = 0. \end{cases}$. Take
$$\mathcal{F}(u) = \int_0^\pi g(\dot{u})\,dx \text{ and } u_n(x) = \frac{1}{n} \sin x \to 0.$$
Then $u_n(x) \in \mathcal{A}$ and $\mathcal{F}(u_n) = \frac{\pi}{2n^2} \to 0 = \mathcal{I}$ as $n \to \infty$ but
$$\pi = \mathcal{F}(0) = \mathcal{F}\left(\lim_{n' \to \infty} u_n\right) > \lim_{n \to \infty} \mathcal{F}(u_n) = 0.$$

Let $\Omega \subset \mathbf{R}^n$ be a bounded, connected domain with smooth boundary. Let

$$\mathcal{H}^1(\Omega) := \left\{ u \in \mathcal{L}^2(\Omega) : \begin{array}{l} \text{all distributional derivatives} \\ \text{exist and } \frac{\partial u}{\partial x_i} \in \mathcal{L}^2(\Omega) \text{ for all } i. \end{array} \right\}$$

$\mathcal{H}^1$ is chosen because it suits the energy in our example. By Serrin's Theorem, $\mathcal{H}^1(\Omega)$ is also the completion of $\mathcal{C}^\infty(\overline{\Omega})$ under the norm

$$\|u\|^2_{\mathcal{H}^1} := \|u\|^2_{\mathcal{L}^2} + \|Du\|^2_{\mathcal{L}^2}.$$

Let $\mathcal{H}^1_0(\Omega)$ denote the completion under $\|\cdot\|_{\mathcal{H}^1}$ of

$$\left\{ u \in \mathcal{C}^\infty(\overline{\Omega}) : \operatorname{spt}(u) \subset \Omega \right\}.$$

where $\operatorname{spt} u = \overline{\{x \in \Omega : u(x) \neq 0\}}$. If $\phi \in \mathcal{C}^1(\overline{\Omega})$, the set of admissible functions is also extended to

$$\mathcal{A}^1 := \left\{ u \in \mathcal{H}^1(\Omega) : u - \phi \in \mathcal{H}^1_0(\Omega) \right\}.$$

Lemma ($\mathcal{F}$ is coercive)

For

$$\mathcal{F}(u) = \int_\Omega \frac{1}{2}|Du|^2 + \psi u\, dx$$

there are constants $c_1, c_2 > 0$ depending on ψ and Ω so that for all $u \in \mathcal{A}^1$,

$$\mathcal{F}(u) \geq c_1 \|u\|^2_{\mathcal{H}^1} - c_2.$$

It follows that $\mathcal{F}$ is bounded below by $-c_2$ and

$$\mathcal{I} = \inf_{v \in \mathcal{A}^1} \mathcal{F}(v)$$

exists and is finite.

Proof idea. The Lemma follows from the Poincaré Inequality: there is a constant $c_3(\Omega) > 0$ such that

$$\int_\Omega u^2 \le c_3^2 \int_\Omega |Du|^2 \qquad \text{for all } u \in \mathcal{H}_0^1(\Omega).$$

(Poincaré's Inequality follows from the Fundamental Theorem of Calculus.) Writing any $u \in \mathcal{A}^1$ as $u = v + \phi$ where $v \in \mathcal{H}_0^1(\Omega)$,

$$\begin{aligned}
\|v+\phi\|_{\mathcal{L}^2} &\le \|v\|_{\mathcal{L}^2} + \|\phi\|_{\mathcal{L}^2} \\
&\le c_3\|Dv\|_{\mathcal{L}^2} + \|\phi\|_{\mathcal{L}^2} \\
&\le c_3\|D(v+\phi)\|_{\mathcal{L}^2} + c_3\|D\phi\|_{\mathcal{L}^2} + \|\phi\|_{\mathcal{L}^2}
\end{aligned}$$

so

$$\begin{aligned}
\|v+\phi\|_{\mathcal{H}^1} &\le \|v+\phi\|_{\mathcal{L}^2} + \|D(v+\phi)\|_{\mathcal{L}^2} \\
&\le (1+c_3)\|D(v+\phi)\|_{\mathcal{L}^2} + c_3\|D\phi\|_{\mathcal{L}^2} + \|\phi\|_{\mathcal{L}^2}
\end{aligned}$$

Proof idea. The Lemma follows from the Poincaré Inequality: there is a constant $c_3(\Omega) > 0$ such that

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(Poincaré's Inequality follows from the Fundamental Theorem of Calculus.) Writing any $u \in \mathcal{A}^1$ as $u = v + \phi$ where $v \in \mathcal{H}_0^1(\Omega)$,

$$\begin{aligned} \|v+\phi\|_{\mathcal{L}^2} &\le \|v\|_{\mathcal{L}^2} + \|\phi\|_{\mathcal{L}^2} \\ &\le c_3\|Dv\|_{\mathcal{L}^2} + \|\phi\|_{\mathcal{L}^2} \\ &\le c_3\|D(v+\phi)\|_{\mathcal{L}^2} + c_3\|D\phi\|_{\mathcal{L}^2} + \|\phi\|_{\mathcal{L}^2} \end{aligned}$$

so

$$\begin{aligned} \|v+\phi\|_{\mathcal{H}^1} &\le \|v+\phi\|_{\mathcal{L}^2} + \|D(v+\phi)\|_{\mathcal{L}^2} \\ &\le (1+c_3)\|D(v+\phi)\|_{\mathcal{L}^2} + c_3\|D\phi\|_{\mathcal{L}^2} + \|\phi\|_{\mathcal{L}^2} \end{aligned}$$

It follows from this and the Schwartz Inequality

$$
\begin{aligned}
\mathcal{F}(v+\phi) &= \tfrac{1}{2}\|D(v+\phi)\|_{\mathcal{L}^2} + \int_\Omega \psi\cdot(v+\phi) \\
&\geq \frac{\|v+\phi\|_{\mathcal{H}^1} - c_3\|D\phi\|_{\mathcal{L}^2} - \|\phi\|_{\mathcal{L}^2}}{2(1+c_3)} - \sqrt{\|\psi\|_{\mathcal{L}^2}}\sqrt{\|\phi+v\|_{\mathcal{L}^2}} \\
&\geq \frac{\|v+\phi\|_{\mathcal{H}^1} - c_3\|D\phi\|_{\mathcal{L}^2} - \|\phi\|_{\mathcal{L}^2}}{2(1+c_3)} - \tfrac{1}{2\epsilon}\|\psi\|_{\mathcal{L}^2} - \tfrac{\epsilon}{2}\|\phi+v\|_{\mathcal{L}^2}
\end{aligned}
$$

where we have used the Peter-Paul inequality (Cauchy's Inequality): for any $\epsilon > 0$

$$2AB \leq \tfrac{1}{\epsilon}A^2 + \epsilon B^2.$$

Proof: $\left(\frac{1}{\sqrt{\epsilon}}A - \sqrt{\epsilon}B\right)^2 \geq 0.$

Split the first term in the sum

$$\mathcal{F}(v+\phi) \geq \frac{\|v+\phi\|_{\mathcal{H}^1}}{4(1+c_3)} - \frac{c_3\|D\phi\|_{\mathcal{L}^2} + \|\phi\|_{\mathcal{L}^2}}{2(1+c_3)} - \tfrac{1}{2\epsilon}\|\psi\|_{\mathcal{L}^2}$$
$$+ \left(\frac{\|v+\phi\|_{\mathcal{H}^1}}{4(1+c_3)} - \tfrac{\epsilon}{2}\|\phi + v\|_{\mathcal{L}^2} \right)$$

Choosing $\epsilon = \frac{1}{2(1+c_3)}$ the parenthesized term is nonnegative. Thus we get the desired inequality with

$$c_1 = \frac{1}{4(1+c_3)}$$

and

$$c_2 = \frac{c_3\|D\phi\|_{\mathcal{L}^2} + \|\phi\|_{\mathcal{L}^2}}{2(1+c_3)} + 2(1+c_3)\|\psi\|_{\mathcal{L}^2}. \qquad \square$$

Let us proceed with the direct method. Choose a minimizing sequence $u_n \in \mathcal{A}^1$ so that

$$\lim_{n\to\infty} \mathcal{F}(u_n) = \mathcal{I}.$$

Without loss of generality, $\mathcal{F}(u_n) < \mathcal{I} + 1$ for all n, therefore, by the Lemma, for all n,

$$\mathcal{I} + 1 \geq \mathcal{F}(u_n) \geq c_1 \|u_n\|_{\mathcal{H}^1} - c_2$$

which implies that the sequence is bounded in $\mathcal{H}^1$. For all n,

$$\|u_n\|_{\mathcal{H}^1} \leq \frac{\mathcal{I} + 1 + c_2}{c_1}.$$

FACT: In any Hilbert Space, *e.g.* in $\mathcal{H}^1$, any bounded sequence $\{u_n\}$ is weakly sequentially compact: there is a subsequence $\{u_{n'}\}$ that weakly converges in $\mathcal{H}^1$ to $u_0 \in \mathcal{H}^1$. That is, for any $v \in \mathcal{H}^1$,

$$\langle u_{n'}, v\rangle_{\mathcal{H}^1} \to \langle u_0, v\rangle_{\mathcal{H}^1} \qquad \text{as } n' \to \infty.$$

FACT: The embedding $\mathcal{H}^1(\Omega) \subset \mathcal{L}^2(\Omega)$ is compact. *i.e.*, by going to a sub-subsequence if necessary, we may assume $u_{n''} \to u_0$ in $\mathcal{L}^2(\Omega)$.

FACT: $\mathcal{A}^1$ is a closed subspace of $\mathcal{H}^1(\Omega)$. If all $u_{n'}$ belong to a closed subspace and $\{u_{n'}\}$ converges weakly to u_0 in $\mathcal{H}^1$, then u_0 also belongs to the closed subspace. *i.e.*, $u_0 \in \mathcal{A}^1$.

u_0 is the candidate to be the minimizer of the variational problem.

Lemma (SWLSC)

Let u_n be a minimizing sequence for

$$\mathcal{F}(u) = \int_\Omega \tfrac{1}{2}|Du|^2 + \psi u\,dx$$

such that $u_n \to u_0$ strongly in $\mathcal{L}^2(\Omega)$ and weakly in $\mathcal{H}^1(\Omega)$. Then

$$\mathcal{F}(u_0) \le \liminf_{n\to\infty} \mathcal{F}(u_n).$$

Proof. Since $u_n \to u_0$ in $\mathcal{L}^2(\Omega)$, $\int_\Omega \psi u_n \to \int_\Omega \psi u_0$ and $\|u_n\|_{\mathcal{L}^2} \to \|u_0\|_{\mathcal{L}^2}$. In any Hilbert Space the norm is SWLSC: $\|u_0\|_{\mathcal{H}^1} \le \liminf_{n\to\infty} \|u_n\|_{\mathcal{H}^1}$.

$$\begin{aligned}
\mathcal{F}(u_0) &= \tfrac{1}{2}\|Du_0\|^2_{\mathcal{L}^2} + \int \psi u_0 = \tfrac{1}{2}\|u_0\|^2_{\mathcal{H}^1} - \tfrac{1}{2}\|u_0\|^2_{\mathcal{L}^2} + \int \psi u_0 \\
&\le \liminf_{n\to\infty} \left\{ \tfrac{1}{2}\|u_n\|^2_{\mathcal{H}^1} - \tfrac{1}{2}\|u_n\|^2_{\mathcal{L}^2} + \int \psi u_n \right\} = \liminf_{n\to\infty} \mathcal{F}(u_n) = \mathcal{I}.
\end{aligned}$$

More generally, $\int f(x,u,Du)\,dx$ is SWLSC in $\mathcal{H}^1$ if

- $f \geq 0$,
- $f(x,u,p)$ is measurable in x for all (u,p),
- $f(x,u,p)$ is continuous in u for all p and almost every x.
- $f(x,u,p)$ is convex in p for all u and almost every x.

The directional derivative of the continuous functional $\mathcal{F} : \mathcal{H}^1 \to \mathbf{R}$ at u in the direction v is defined for $u, v \in \mathcal{H}^1$ to be

$$D\mathcal{F}(u)[v] = \lim_{\varepsilon \to 0} \frac{\mathcal{F}(u + \varepsilon v) - \mathcal{F}(u)}{\varepsilon}.$$

$\mathcal{F}$ is (Frechet) differentiable at u if

$$\mathcal{F}(u + v) - \mathcal{F}(u) = D\mathcal{F}(u)[v] + \mathbf{o}(\|v\|_{\mathcal{H}^1}) \text{ as } \|v\|_{\mathcal{H}^1} \to 0.$$

If $\mathcal{F}$ is differentiable at u then $D\mathcal{F}(u)[v]$ is linear in v.

$\mathcal{F}$ is continuously differentiable ($\mathcal{C}^1$) if the map $D\mathcal{F} : \mathcal{H}^1 \to (\mathcal{H}^1)^*$ is continuous.

$u_0 \in \mathcal{H}^1$ is a critical point of $\mathcal{F}$ if

$$D\mathcal{F}(u_0)[v] = 0 \text{ for all } v \in \mathcal{H}^1.$$

This is called the Euler Equation for $\mathcal{F}$ at u_0.

Lemma

$\mathcal{F}(u) = \int_\Omega \frac{1}{2}|Du|^2 + \psi u \, dx$ is continuously differentiable on $\mathcal{H}^1(\Omega)$.

Proof.

$$\begin{aligned} D\mathcal{F}(u)[v] &= \lim_{\varepsilon\to 0} \frac{\int_\Omega \frac{1}{2}|D(u+\varepsilon v)|^2 - \frac{1}{2}|Du|^2 + \psi(u+\varepsilon v) - \psi u \, dx}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \int_\Omega Du\cdot Dv + \tfrac{\epsilon}{2}|Dv|^2 + \psi v \, dx = \int_\Omega Du\cdot Dv + \psi v \, dx. \end{aligned}$$

$D\mathcal{F}(u)$ is evidently linear. It is also a bounded linear functional:

$$|D\mathcal{F}(u)[v]| \le \|Du\|_{\mathcal{L}^2}\|Dv\|_{\mathcal{L}^2} + \|\psi\|_{\mathcal{L}^2}\|v\|_{\mathcal{L}^2} \le \left(\|Du\|_{\mathcal{L}^2} + \|\psi\|_{\mathcal{L}^2}\right)\|v\|_{\mathcal{H}^1}.$$

$\mathcal{F}$ is differentiable:

$|\mathcal{F}(u+v) - \mathcal{F}(u) - D\mathcal{F}(u)[v]| = \frac{1}{2}\int_\Omega |Dv|^2 \, dx \le \frac{1}{2}\|v\|^2_{\mathcal{H}^1} = \mathbf{o}\left(\|v\|_{\mathcal{H}^1}\right)$ as $\|v\|_{\mathcal{H}^1} \to 0$.

$D\mathcal{F}$ is continuous: For $u, v, w \in \mathcal{H}^1$,

$$\begin{aligned} |D\mathcal{F}(u)[w] - D\mathcal{F}(v)[w]| &= \left| \int_\Omega (Du - Dv) \cdot Dw \, dx \right| \\ &\leq \|D(u-v)\|_{\mathcal{L}^2} \|Dw\|_{\mathcal{L}^2} \leq \|u-v\|_{\mathcal{H}^1} \|w\|_{\mathcal{H}^1}. \end{aligned}$$

Thus

$$\|D\mathcal{F}(u) - D\mathcal{F}(v)\|_{(\mathcal{H}^1)^*} = \sup_{w \neq 0} \frac{|D\mathcal{F}(u)[w] - D\mathcal{F}(v)[w]|}{\|w\|_{\mathcal{H}^1}} \to 0$$

as $\|u - v\|_{\mathcal{H}^1} \to 0$. $\square$

We have found $u_0 \in \mathcal{A}^1$ such that u_0 is the global minimizer. This means that for all $v \in \mathcal{H}_0^1(\Omega)$, $u_0 + tv \in \mathcal{A}^1$ and

$$f(t) = \mathcal{F}(u_0 + tv) \geq \mathcal{F}(u_0).$$

$f(t)$ achieves a minimum at $t = 0$, hence $0 = f'(0) = D\mathcal{F}(u_0)[v]$ or

$$\int_\Omega Du_0 \cdot Dv + \psi v \, dx = 0 \qquad \text{for all } v \in \mathcal{H}_0^1(\Omega).$$

In other words, u_0 is a weak solution of the Poisson Equation. u_0 satisfies the boundary condition in the sense that $u_0 - \phi \in \mathcal{H}_0^1(\Omega)$.

Definition

Suppose $\mathcal{F} : \mathcal{H}^1 \to \mathbf{R}$ ia a $\mathcal{C}^1$ functional. $\mathcal{F}$ is *convex* on $\mathcal{A}^1 \subset \mathcal{H}^1$ if

$$\mathcal{F}(u+w) - \mathcal{F}(u) \geq D\mathcal{F}(u)[w] \text{ whenever } u, u+w \in \mathcal{A}^1.$$

$\mathcal{F}$ is *strictly convex* if "=" holds iff $w = 0$.

Our Poisson functional is convex:

$$\begin{aligned} \mathcal{F}(u+w) - \mathcal{F}(u) &= \int_\Omega Du \cdot Dw + \tfrac{1}{2}|D(w)|^2 + \psi w \\ &\geq \int_\Omega Du \cdot Dw + \psi w = D\mathcal{F}(u)[w]. \end{aligned}$$

Lemma

Let u_0 be a minimizer of $\mathcal{F}(u) = \int_\Omega \frac{1}{2}|Du|^2 + \psi u \, dx$ in $\mathcal{A}^1$.
Then u_0 is unique.

Proof. Uniqueness follows from convexity. A special version for quadratic forms is the *Polarization Identity:*

$$|p|^2 + |q|^2 = 2\left|\tfrac{p+q}{2}\right|^2 + 2\left|\tfrac{p-q}{2}\right|^2.$$

Let u_0, u_1 both minimize $\mathcal{F}(u) = \int_\Omega |Du|^2 + \psi u$ on $\mathcal{A}^1$, namely $\mathcal{I} = \inf_{u\in\mathcal{A}^1} \mathcal{F}(u) = \mathcal{F}(u_1) = \mathcal{F}(u_2)$. Let $w = \frac{1}{2}(u_1 + u_2) \in \mathcal{A}^1$. By the Polarization Identity and the Poincaré Inequality

$$\begin{aligned}
4\mathcal{I} \le 4\mathcal{F}(w) &= \int_\Omega 2\left|\tfrac{Du_1+Du_2}{2}\right|^2 + 4\psi(x)\left(\tfrac{u_1+u_2}{2}\right) \\
&= \int_\Omega \left(|Du_1|^2 + 2\psi u_1\right) + \int_\Omega \left(|Du_2|^2 + 2\psi u_2\right) - \int_\Omega 2\left|\tfrac{Du_1-Du_2}{2}\right|^2 \\
&= 2\mathcal{I} + 2\mathcal{I} - \tfrac{1}{2}\int_\Omega |D(u_1-u_2)|^2\,dx \le 4\mathcal{I} - \frac{1}{2(1+c_3)}\|u_1-u_2\|_{\mathcal{H}^1}
\end{aligned}$$

since $u_1 - u_2 \in \mathcal{H}^1_0(\Omega)$. Hence $u_1 - u_2 = 0$: the minimizer is unique. $\square$

Using the Direct Method, we showed the existence of a unique weak solution $u_0 \in \mathcal{H}^1$ of the Dirichlet problem for Poisson's Equation. If the coefficients are known to be smoother, then the solution has more regularity. For example, this is a theorem proved in Math 6420.

Theorem

Suppose $\Omega \subset \mathbf{R}^n$ is a bounded connected domain with $\mathcal{C}^{2,\alpha}$ boundary. Suppose for some $0 < \alpha < 1$ that $\psi \in \mathcal{C}^{\alpha}(\overline{\Omega})$ and $\phi \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$. If $u \in \mathcal{A}^1$ is a weak solution to $\Delta u = \psi$, namely

$$\int_{\Omega} Du \cdot Dv + \psi v \, dx = 0 \qquad \text{for all } v \in \mathcal{H}_0^1(\Omega),$$

then $u \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$. If $\partial\Omega$, ψ and ϕ are in $\mathcal{C}^{\infty}(\overline{\Omega})$ then so is u.

$\mathcal{C}^{\alpha}(\overline{\Omega}) \subset \mathcal{C}(\overline{\Omega})$ is the subspace of α-Hölder continuous functions.
$\mathcal{C}^{2,\alpha}(\overline{\Omega}) \subset \mathcal{C}^{2}(\overline{\Omega})$ are functions whose second derivatives are in $\mathcal{C}^{\alpha}(\overline{\Omega})$.

Thanks!