Deforming Surfaces

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This is one of those topics that is usually discussed at the end of a text on curves and surfaces, and not often reached in a modern abbreviated treatment. Let me try to skip over many technicalities and state some results.

We imagine a two dimensional surface in three space as the image of a smooth vector function $M^2_0 = X_0(W)$ where

$$X_0 : W \to \mathbb{R}^3$$

is a smooth map on $W \subset \mathbb{R}^2$, an open subset. We imagine deforming the surface $M^2_\tau = X(W, \tau)$ by a smooth one-parameter family of maps

$$X : W \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$$

such that $X(u, v; 0) = X_0(u, v)$ for all points $(u, v) \in W$. 
6. Deformation Vector Field of Surfaces.

The deformation vector field is the velocity of the deformation

\[ Z(\bullet) = \frac{dX}{d\tau}(\bullet; 0). \]

Figure 1: Deformation of Surface

An infinitesimal isometric deformation is a deformation that preserves lengths on the surface up to first order.

If \( \gamma(\sigma) = (u(\sigma), v(\sigma)) \in \mathcal{W} \) is a curve for \( \sigma \in [a, b] \), then its length in \( M^2_\tau \) of \( \gamma \) is

\[
L(\gamma, \tau) = \int_a^b ds_\tau = \int_a^b \left| \frac{d}{d\sigma} X(\gamma(\sigma), \tau) \right| d\sigma
\]

where \( d\gamma[1] = \gamma' = (u', v') \), an element of arclength, is by chain rule

\[
ds_\tau = |dX \circ d\gamma[1]| d\sigma = \left| dX(\gamma(\sigma); \tau)[u'(\sigma), v'(\sigma)] \right| d\sigma
\]

\[
= \left( X_u(\gamma(\sigma); \tau) \cdot X_u(\gamma(\sigma); \tau) u'(\sigma)^2 + 2X_u(\gamma(\sigma); \tau) \cdot X_v(\gamma(\sigma); \tau) u'(\sigma) v'(\sigma) + X_v(\gamma(\sigma); \tau) \cdot X_v(\gamma(\sigma); \tau) v'(\sigma)^2 \right)^{\frac{1}{2}} d\sigma
\]

\[
= \left( (dX \cdot dX)[\gamma'(\sigma), \gamma'(\sigma)] \right)^{\frac{1}{2}} d\sigma
\]

which, for short is written \( ds^2_\tau = dX \cdot dX \).
Note that for a regular surface, $dX$ is full rank so that $dX \cdot dX$ is a symmetric, positive definite quadratic form called the \textit{metric form}.

To preserve the length of curves up to first order we must have for every curve $\gamma : [a, b] \to \mathcal{W}$,

$$0 = \frac{d}{d\tau} \bigg|_{\tau=0} L(\gamma) = \int_{a}^{b} \frac{(dX \cdot dZ)[\gamma'(\sigma), \gamma'(\sigma)]}{\left((dX \cdot dX)[\gamma'(\sigma), \gamma'(\sigma)]\right)^{\frac{1}{2}}} d\sigma$$

which implies that the quadratic form

$$dX \cdot dZ = 0.$$  \hspace{1cm} (1)

This is the equation for \textit{infinitesimal isometric deformation} for unknown vector field $Z$. 
9. Equation in Local Coordinates.

The infinitesimal deformation equation (1) is shorthand for the more cumbersome, but perhaps more familiar form

\[ 0 = X_u \cdot Z_u \]
\[ 0 = X_u \cdot Z_v + X_v \cdot Z_u \]
\[ 0 = X_v \cdot Z_v \]

where the vector functions

\[ X, Z : W \rightarrow \mathbb{R}^3 \]

give the position and deformation field in a local coordinates.

This is a system of three linear first order partial differential equations for the three unknown components of the vector function \( Z \).

If a surface moves as a rigid body, then the lengths of all curves on $M_2^2$ are preserved. Rigid motions are the composition of rotations and translations. The deformation due to a rigid motion may be written in terms of smooth rotation matrix $R(\tau)$ and translation vector $T(\tau)$:

$$X(\bullet; \tau) = R(\tau)X_0(\bullet) + T(\tau)$$

such that $R(0) = I$ and $T(0) = 0$. The deformation vector becomes

$$Z(\bullet) = \frac{d}{d\tau} \bigg|_{\tau=0} X(\bullet, \tau) = R'(0)X_0(\bullet) + T'(0) = A \times X_0(\bullet) + B.$$

Multiplying by the skew symmetric matrix $R'(0)$ (why?) is the same as taking a cross product with the rotation vector $A = (R'_{32}(0), R'_{13}(0), R'_{21}(0))$.

The translation vector $B = T'(0)$. Such a deformation is called an infinitesimal congruence. Observe that an infinitesimal congruence is an infinitesimal isometric deformation:

$$dX \bullet dZ = dX \bullet [A \times dX] = 0.$$
11. Why Derivative of Rotation $R'(0)$ is Skew.

$R(t)$ is a smooth family of rotations matrices with $R(0) = I$. Since $R$ preserves Euclidean inner product,

$$RV \cdot RW = V \cdot W$$

for all vectors $V, W$. Hence

$$R^T RV \cdot W = V \cdot W$$

which implies

$$R^T R = I.$$

Differentiating with respect to $t$,

$$(R')^T R + R^T R' = 0$$

Since $R(0) = I$, at $t = 0$,

$$R' + R = 0$$

so $R'(0)$ is skew.
Thus there are numbers $a, b, c$ so that

$$R'(0) V = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} av_2 + bv_3 \\ -av_1 + cv_3 \\ -bv_1 - cv_3 \end{pmatrix}$$

$$= \begin{vmatrix} i & j & k \\ -c & b & -a \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{pmatrix} -c \\ b \\ -a \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = A \times V$$

The vector $V$ is the rotation vector. It is parallel to the axis of the rotation. Its magnitude gives the angular velocity.
A surface $M^2$ is **infinitesimally rigid** if every vector field $Z$ on $M^2$ satisfying the infinitesimal rigidity equation (1)

$$dX \bullet dZ = 0$$

is an infinitesimal congruence. If there is a solution $Z$ which is not an infinitesimal congruence, then we call it an **infinitesimal flex**. An infinitesimal deformation gives a corresponding deformation, the **simple flex**, 

$$Y = X_0 + \tau Z$$

which preserves the metric to first order at $\tau = 0$ because $dY \bullet dY =$

$$dX \bullet dX + 2\tau dX \bullet dY + \tau^2 dY \bullet dY = dX \bullet dX + \tau^2 dY \bullet dY.$$ 

Note that both $Y = X \pm \tau Z$ have the same metric forms, i.e., are **isometric**.
Suppose that the surface $X_0$ has a flat planar region $F$, whose normal vector is $(0, 0, 1)$. Let $\psi : F \to \mathbb{R}$ be a smooth a “bump function” such that $\psi \geq 0$ and $\psi = 0$ near $\partial F$ and off $F$. Then

$$Z = (0, 0, \psi)$$

which is zero off $F$ is an infinitesimal deformation. ($dX$ and $dZ$ are perpendicular.)
At each point replace by smoothly varying orthonormal tangent vectors \( \{e_1, e_2\} \).
Unit normal is
\[
e_3 = e_1 \times e_2.
\]

Define dual basis of one-forms \( \{\omega^A\} \) by
\[
\omega^A(e_B) = \delta^A_B. \quad \text{So}
\]
\[
dX = \sum_{i=1}^{3} \omega^i e_i
\]
and metric is
\[
ds^2 = (\omega^1)^2 + (\omega^2)^2.
\]

Locally, a surface is \( X : \mathcal{W} \to \mathbb{R}^3 \).
Tangent plane is spanned by basis \( \{X_u, X_v\} \).
Extrinsic Geometry deals with how $M$ sits in its ambient space.

Near $P \in M$, the surface may be parameterized as the graph over its tangent plane, where $f(u_1, u_2)$ is the “height” above the tangent plane

$$X(u_1, u_2) = P + u_1 e_1(P) + u_2 e_2(P) + f(u_1, u_2) e_3(P). \quad (2)$$

So $f(0) = 0$ and $Df(0) = 0$. The Hessian of $f$ at 0 gives the shape operator at $P$. It is also called the Second Fundamental Form.

$$h_{ij}(P) = \frac{\partial^2 f}{\partial u_i \partial u_j}(0)$$

The Mean Curvature and Gaussian Curvature at $P$ are

$$H(P) = \frac{1}{2} \text{tr}(h_{ij}(P)), \quad K(P) = \det(h_{ij}(P)).$$

A one-form on $\mathbb{R}^3$ is an expression of the form

$$\alpha = f \, dx + g \, dy + h \, dz$$

where $f, g, h$ are smooth functions on $\mathbb{R}^3$. One-forms may be integrated along a curve $\Gamma$ given by

$$\gamma : [a, b] \rightarrow \mathbb{R}^3 \quad \text{where} \quad \gamma(t) = \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix}$$

by the formula

$$\int_{\Gamma} \alpha = \int_{a}^{b} \left( f(\gamma(t))u'(t) + g(\gamma(t))v'(t) + h(\gamma(t))w'(t) \right) \, dt.$$ 

The differential $d$ (or gradient) of a function $\phi$ on $\mathbb{R}^3$ is a one form

$$d\phi = \frac{\partial \phi}{\partial x} \, dx + \frac{\partial \phi}{\partial y} \, dy + \frac{\partial \phi}{\partial z} \, dz.$$
A two-form on $\mathbb{R}^3$ is an expression of the form

$$\beta = f \, dx \wedge dy + g \, dx \wedge dz + h \, dy \wedge dz$$

where $f, g, h$ are smooth functions on $\mathbb{R}^3$. Two-forms may be integrated along a surface $M = X(U)$ given by

$$X : U \to \mathbb{R}^3 \quad \text{where} \quad X(s, t) = \begin{pmatrix} u(s, t) \\ v(s, t) \\ w(s, t) \end{pmatrix}$$

and $U \subset \mathbb{R}^2$ is smooth bounded domain by

$$\int_M \beta = \int \int_U \left( f(X(s, t)) \frac{\partial (u, v)}{\partial (s, t)} + g(X(s, t)) \frac{\partial (u, w)}{\partial (s, t)} + h(X(s, t)) \frac{\partial (v, w)}{\partial (s, t)} \right) \, ds \, dt$$

where the Jacobean is

$$\frac{\partial (u, v)}{\partial (s, t)} = \frac{\partial u}{\partial s}(s, t) \frac{\partial v}{\partial t}(s, t) - \frac{\partial u}{\partial t}(s, t) \frac{\partial v}{\partial s}(s, t)$$
An example of a two-form is the area form for the surface $M$. In terms of the orthonormal coframe it is given by

$$\beta = dA = \omega^1 \wedge \omega^2$$

which is everywhere positive on positively oriented surfaces $M$. Its integral over any subset $N \subset M$ gives the area of the subset

$$A(N) = \int_N \omega^1 \wedge \omega^2.$$
The **exterior derivative** $d$ (or curl) of a one-form

$$\alpha = f \, dx + g \, dy + h \, dz$$

on $\mathbb{R}^3$ is a two-form

$$d\alpha = \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dx \wedge dy + \left( \frac{\partial h}{\partial x} - \frac{\partial f}{\partial z} \right) \, dx \wedge dz + \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \, dy \wedge dz.$$

**Stokes Theorem** in this notation is

$$\int_\Sigma d\alpha = \int_{\partial \Sigma} \alpha$$

where $\partial \Sigma$ is the collection of oriented boundary curves of the surface $\Sigma$. The right side is zero if the surface is closed (has no boundary such as a sphere or torus.)
Upper case Roman indices run over $A, B, C, \ldots = 1, 2, 3$ and lower case run over $i, j, k, \ldots = 1, 2$. Einstein Convention: repeated lower and upper indices are assumed to be summed.

Because $e_A \cdot e_A = 1$, taking the directional derivative $d(e_A \cdot e_A) = 0$, so that $d e_A \perp e_A$ and we may express the rate of rotation of the frame

$$d e_A = \omega_A^B e_B.$$ 

$\omega_A^B$ are called connection forms. Differentiation of $e_A \cdot e_B = \delta_{AB}$ implies $\omega_A^B$ is skew and satisfies

$$d \omega^A = \omega^B \wedge \omega_B^A.$$ (3)

Also, differentiating the normal defines $d e_3 = \omega_3^i e_i$. Moreover

$$\omega_3^i = -h_{ij} \omega^j$$

recovers the second fundamental form.
An ovaloid is a $C^3$ closed surface $M^2$ which is the boundary of a bounded convex domain of three space. It is strictly convex if the second fundamental form $h_{ij}$ with respect to the inner normal is positive definite.

**Theorem (Liebmann, 1899)**

*Strictly convex ovaloids are infinitesimally rigid.*

This question was asked by Jellet in 1854, who was unable to prove it.

In 1835 Minding posed a related problem: if two ovaloids are isometric, must they be congruent? For round spheres, this was proved by Liebmann and Hilbert in 1903. The first proof for ovaloids was given by Weyl in 1915.

Let $X$ be local coordinates in the neighborhood of a point of a surface $M^2 \subset \mathbb{R}^3$. Let $Z$ be an infinitesimal deformation. Then

$$dX \cdot dZ = 0.$$ 

This implies that there is a globally defined vector field $Y$, the bending vector, such that

$$dZ = Y \times dX.$$ 

Intuitively, this is because $dZ$ is perpendicular to $dX$.

Thus, the differential of an infinitesimal deformation $dZ$ gives a local rotation whose axis $Y$ varies from point to point.
24. Proof.

Because if we choose a right-handed orthonormal frame on $M$ such that $e_1$ and $e_2$ are tangent so $e_3$ is normal to $M$. Let $\{\omega^A\}$ be the dual coframe so $dX = \omega^i e_i$ and $Z = \zeta^A e_A$ (Einstein convention!) Then the covariant derivative is in coordinates $dZ = \zeta^B e^i \omega^j e_B$. Substituting,

\[
0 = (\omega^i e_i) \cdot (\zeta^B_j \omega^j e_B) = (\delta^B_{iB} \zeta^B_j) \omega^i \otimes \omega^j = (\zeta^1_1) \omega^1 \otimes \omega^1 + (\zeta^1_2 + \zeta^2_1) \omega^1 \otimes \omega^2 + (\zeta^2_2) \omega^2 \otimes \omega^2
\]

so $\zeta^1_1 = \zeta^2_2 = \zeta^2_1 + \zeta^1_2 = 0$. Hence $e_1 \times e_2 = e_3$ and so on implies

\[
dZ = \zeta^1_2 \omega^2 e_1 + \zeta^2_1 \omega^1 e_2 + (\zeta^3_1 \omega^1 + \zeta^3_2 \omega^2) e_3 = (\zeta^3_2 e_1 - \zeta^3_1 e_2 + \zeta^2_1 e_3) \times (\omega^j e_j) = Y \times dX.
\]

The desired $Y = y^A e_A = \zeta^3_2 e_1 - \zeta^3_1 e_2 + \zeta^2_1 e_3$. 
Theorem (Liebmann, 1899)

*Strictly convex ovaloids are infinitesimally rigid.*

**Figure 4:** Wilhelm Blaschke 1885–1962

Proof (Blaschke, 1921).

Here is the basic idea of Blaschke’s amazing proof using an integral formula.
The idea is to show that the local bending vector is a constant. In other words, its derivative \( dY \) should vanish.

Blaschke finds a one-form \( \theta \) such that

\[
d\theta = \Phi(X, dX, dY) \omega^1 \wedge \omega^2
\]

where \( \Phi \leq 0 \) is a nonpositive function if the surface is convex and \( \Phi = 0 \) implies \( dY = 0 \). Then by Stokes Theorem on the ovaloid \( M \),

\[
0 = \int_M d\theta = \int_M \Phi(X, dX, dY) \omega^1 \wedge \omega^2
\]

which implies \( \Phi = 0 \) at all points so \( dY = 0 \).
Differentiating \( dZ = Y \times dX \) we find

\[
0 = dY \times dX
\]

which means tangent planes are parallel to each other. Hence the covariant derivative may be written

\[
dY = y^i_j \omega^j_e_i
\]

such that \( y^1_1 + y^2_2 = 0 \).
Because, if we write the covariant derivative $dY = y^A_i \omega^i e_A$ we get

\[
0 = (y^A_i \omega^i e_A) \times (\omega^j e_j)
= \left( y^A_i \omega^i \wedge \omega^j \right) e_A \times e_j
= \omega^1 \wedge \omega^2 \left\{ (y^1_1 + y^2_2) e_1 \times e_2 - y^3_2 e_3 \times e_1 + y^3_1 e_3 \times e_2 \right\}
\]
so that $y^1_1 + y^2_2 = y^3_1 = y^3_2 = 0$. 

Differentiating $dY = y^i_j \omega^i e_i$, 

$$0 = dy^i_j \wedge \omega^j e_i + y^i_j \omega^k \wedge \omega^j_k e_i - y^i_j \omega^j \wedge \omega^j_i e_B$$

$$= \left(dy^i_j - y^i_k \omega_j^k + y^j_k \omega_k^i\right) \omega^j e_i - y^i_j h_{ik} \omega^j \wedge \omega^k e_3$$

This gives a first order system of PDE's for $y^i_j$. The $e_3$ coefficient

$$0 = (y^i_1 h_{i2} - y^i_2 h_{i1}) \omega^1 \wedge \omega^2$$

Or using $y^1_1 + y^2_2 = 0$.

$$0 = -y^1_2 h_{11} + 2y^1_1 h_{12} + y^2_1 h_{22}$$ \hspace{1cm} (4)
Using the determinant of a matrix whose three column vectors are given, we consider the one-form defined globally on $M^2$,

$$\theta = \det (X, Y, dY)$$

Differentiating using $y^3_j = y^1_1 + y^2_2 = 0$ and $\xi^A = X \cdot e_A$,

$$d\theta = \det (dX, Y, dY) + \det (X, dY, dY)$$

$$= \det \left( \omega^i e_i, \ y^B e_B, \ y^k j \omega^j e_k \right) + \det \left( \xi^A e_A, \ y^i j \omega^j e_i, \ y^k \ell \omega^\ell e_k \right)$$

$$= -y^3 \left( y^2_2 + y^1_1 \right) \omega^1 \wedge \omega^2 + \xi^3 \left( y^1_j y^2_\ell - y^2_j y^1_\ell \right) \omega^j \wedge \omega^\ell$$

$$= 2\xi^3 \left( y^1_1 y^2_2 - y^2_1 y^1_2 \right) \omega^1 \wedge \omega^2$$
Assume $h_{ij}$ is symmetric and positive definite and $y^{ij}$ satisfies

$$0 = -y^{12}h_{11} + 2y^{11}h_{12} + y^{21}h_{22}; \quad y^{11} + y^{22} = 0. \quad (5)$$

Then

$$\det(y^{ij}) = y^{11}y^{22} - y^{21}y^{12} \leq 0$$

and is equal to zero only if all $y^{ij} = 0$. 
Positive definite means $h_{11} > 0$, $h_{22} > 0$ and $\det(h_{ij}) > 0$.

There are many proofs. We give an elementary argument.

Proof.

In case $h_{12} = 0$ we have

$$0 = -y_{12}^1 h_{11} + y_{21}^2 h_{22} \quad (6)$$

so $y_{12}^1$ and $y_{21}^2$ have the same sign so

$$\det(y_{ij}^i) = -(y_{11}^1)^2 - y_{21}^2 y_{12}^1 \leq 0.$$ 

If equal to zero then $y_{11}^1 = -y_{22}^2 = 0$ and, say, $y_{12}^1 = 0$ which imples $y_{21}^2 = 0$ by (6). The case $y_{21}^2 = 0$ is similar.
In case $h_{12} \neq 0$ we have by (5), $y^{1}_{1} = \frac{y^{1}_{2}h_{11}-y^{2}_{1}h_{22}}{2h_{12}}$. Hence

$$
\text{det}(y_{ij}) = -\left(\frac{y^{1}_{2}h_{11}-y^{2}_{1}h_{22}}{2h_{12}}\right)^{2} - y^{1}_{2}y^{2}_{1}
$$

$$
= -\left[\frac{y^{1}_{2}h_{11}-y^{2}_{1}h_{22}}{2h_{12}}\right]^{2} + 4h^{2}_{12}y^{2}_{1}y^{1}_{2} 
$$

$$
= -\frac{\left[\frac{y^{1}_{2}h_{11}+y^{2}_{1}h_{22}}{4h^{2}_{12}}\right]^{2} - 4(h_{11}h_{22}-h^{2}_{12})y^{2}_{1}y^{1}_{2}}{4h^{2}_{12}}
$$

If $y^{2}_{1}y^{1}_{2} \leq 0$ then $\text{det}(y_{ij}) \leq 0$ and equal to zero if $y^{1}_{2} = y^{2}_{1} = 0$ so $y^{1}_{1} = y^{2}_{2} = 0$ by (5). If $y^{2}_{1}y^{1}_{2} \geq 0$ then

$$
\text{det}(y_{ij}) = -(y^{1}_{1})^{2} - y^{2}_{1}y^{1}_{2} \leq 0
$$

also and equal to zero implies $y^{1}_{1} = y^{2}_{2} = 0$ and, say, $y^{1}_{2} = 0$. Then $y^{2}_{1} = 0$ by (5). The case $y^{2}_{1} = 0$ is similar.
Theorem (Liebmann, 1899)

Strictly convex ovaloids are infinitesimally rigid.

Proof (Blaschke, 1921).

Assume the origin is in the interior of $M$ and $e_3$ is the inner normal. It follows that

$$\xi^3 = X \cdot e_3 < 0.$$ 

$M$ is closed and oriented so by Stokes Theorem and Blaschke's Formula

$$0 = \int_M d\theta = \int_M 2\xi^3 \det(y^i_j) \, dA$$

Since $M^2$ is strictly convex, $h_{ij}$ is positive definite so by the Algebraic Lemma, $\det(y^i_j) \leq 0$ and the integrand in nonnegative. Since the integral equals zero, the determinant must vanish, hence all $y^i_j = 0$ everywhere on $M^2$ by the Lemma. Hence $Y$ is constant.
Consider the translation field
\[ T = Z - Y \times X. \]
Differentiating, since \( Y \) satisfies \( dZ = Y \times dX \) so
\[ dT = dZ - dY \times X - Y \times dX = -dY \times X \]
which vanishes because \( Y \) is constant. Thus \( Z = T + Y \times X \) is an infinitesimal congruence and \( M \) is infinitesimally rigid.
In 1930, Stefan Cohn-Vossen found that by cutting an arbitrarily small nonconvex groove into a surface of revolution, one could create an infinitesimally flexible surface. He also found that by gluing together four cones, one could get an infinitesimally flexible flying saucer surface. The radial distance as a function of $x$ is given by the even piecewise linear function, depicted here.
37. Cohn-Vossen’s Flexible Flying Saucer.
37. Cohn-Vossen’s Flexible Flying Saucer.
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37. Cohn-Vossen’s Flexible Flying Saucer.
38. Cohn-Vossen’s Flexible Flying Saucer. Stop!
In 1952, E. Rembs found that for countably many values in \( \frac{1}{3} < c^2 < \frac{1}{2} \), the nonconvex surfaces of revolution (7) admit nontrivial infinitesimal deformations.

He used separation of variables and Fourier series.

Equation of section as \( c \) varies

\[
(x^2 + r^2)^2 + 2c^2(x^2 - r^2) = 1 - 2c^2 \tag{7}
\]

\( c = 0 \) is sphere. \( c^2 = \frac{1}{2} \) is revolution of lemniscate.
The deformations $Z$ of Rembs were given by (finite) Fourier Series. Hence both the surface and the deformation are **analytic** functions (those given by convergent power series.) Thus the existence of non-trivial isometric deformations is not an artifact of lack of smoothness as in the Barrel Example, Fig. 2

In the barrel surface, the derivatives of $z$-component of $X$ would be dead zero when computed in the flat part so the power series would consist of the constant term only. Thus barrel surface cannot be analytic because its continuation to $M$ would remain constant.

For small $\tau > 0$ the two surfaces

$$X \pm \tau Z$$

are **analytic** and **isometric** (have the same metric)

$$ds^2_{+\tau} = ds^2_{-\tau} = dX \cdot dX + \tau^2 dZ \cdot DZ$$

but, as observer by Rembs, **not congruent**.
A Collander Surface is a smooth convex surface $M$ with $m$ smooth boundary curves, “holes,” $C_1, \ldots, C_m$ such that $K > 0$ away from the boundary curves and for each $k = 1 \ldots, m$ there is a plane $\pi_k$ such that $C_k$ is a strictly convex curve in $\pi_k$ and $M$ makes a first order tangency to $\pi_k$ along $C_k$. 
Theorem (Rembs 1919)

*Collander surfaces are infinitesimally rigid.*

Proof.

The proof is as in the Ovaloid Theorem. For a deformation field and its corresponding bending field $Y$, it suffices to show that the Blaschke's integrand vanishes. This time, Stokes' Theorem has boundary terms

$$\sum_{i=1}^{m} \int_{C_i} \theta = \int_M d\theta = \int_M 2\xi^3 \det(y^i_j) \, dA$$

As $\xi^3 \det(y^i_j) \geq 0$ by Lemma 3., it suffices to show that each

$$\int_{C_i} \theta \leq 0 \quad (8)$$

Fix $i$ and take the orientation of $C_i$ as part of $\partial M$. Thus if we take a local frame for $M$ such that $M$ is convex toward $e_3$ and $e_1$ is tangent to $C_i$, then $e_2$ must point into $M$. 
Since $e_3$ is the normal to $\pi_3$ along $C_i$ it is constant, so along $C_i$,

$$0 = \nabla_{e_1} e_3 = \omega_3^j(e_1)e_j = -h_{jk}\omega^k(e_1)e_j = -h_{j1}e_j$$

It follows that $h_{11} = h_{12} = h_{21} = 0$ along $C_i$. Equation (4) tells us

$$0 = -y_{12}^1 h_{11} + 2y_{11}^1 h_{12} + y_{21}^2 h_{22} = y_{21}^2 h_{22}$$

along $C_i$. Now we also assumed that $M$ makes first order contact, so that $h_{22} > 0$ on $C_i$. Hence on $C_i$,

$$y_{21}^2 = 0.$$ 

The covariant derivative of a vector field, $dY = y^A_j \omega^j = dy^A + y^B \omega_B^A$. Since $y^3_{11} = 0$ everywhere, then along $C_i$,

$$0 = y^3_{11} = dy^3(e_1) + y^j \omega^3_j(e_1) = e_1 y^3 + y^j h_{j1} = \frac{dy^3}{ds}.$$ 

Thus $y^3 = \text{const.}$ along $C_i$. 
Proof of the Rigidity of Collander Surfaces. -

\[ \theta = (X, Y, dY) = \xi^A y^B y^i_j \omega^j (e_A, e_B, e_j) \]

\[ = \xi^3 y^k y^i_j \omega^j (e_3, e_k, e_j) + y^3 \xi^k y^i_j \omega^j (e_k, e_3, e_j) \]

\[ = \xi^3 (y^1 y^2_j - y^2 y^1_j) \omega^j + y^3 (\xi^2 y^1_j - y^1 y^2_j) \omega^j \]

On \( C_i \) where \( y^{21} = 0 \) and \( y^3 = \text{const.} \) we have

\[ \theta = -\xi^3 y^2 y^1_1 \omega^1 + y^3 \xi^2 y^1_1 \omega^1 + (\cdots) \omega^2. \]

The second term is an exact differential so integrates to zero on \( C_i \).

Indeed, since \( C_i \) is planar, there is a constant vector field \( V_i \) on three space such that \( e_3 = V_i \) on \( C_i \). Since \( dT = X \times dY \) we have on \( C_i \),

\[ d(V_i \cdot T) = V_i \cdot dT = (V, X, dY) = \xi^A y^i_j (e_3, e_A, e_i) \omega^j \]

\[ = (\xi^1 y^2_j - \xi^2 y^1_j) \omega^j = -\xi^2 y^1_1 \omega^1 + (\cdots) \omega^2. \]
As $\xi^3 < 0$ is constant, the proof reduces to show each

$$\int_{C_i} y^2 y^1 y^1 \omega^1 \leq 0.$$ 

The area enclosed by $C_i$ may be computed as follows. Let $V$ be the constant vector field such that $e_i = V$ along $C_i$. Observe that

$$d(V, X, dX) = (V, dX, dX) = (e_3, e_i, e_j) \omega^i \wedge \omega^j = 2 \omega^1 \wedge \omega^2$$

so that since $C_i$ is oriented to the outside surface,

$$A(C_i) = -\frac{1}{2} \int_{C_i} (V, X, dX) > 0.$$ 

The integrand can be written

$$\begin{align*} 
(V, X, dX) &= \xi^A \xi^B i \omega^i (e_3, e_A, e_B) \\
&= (\xi^1 \xi^2 i - \xi^2 \xi^1 i) \omega^i = -\xi^2 \xi^1 \omega^1 + (\cdots) \omega^2
\end{align*}$$

This simplifies because $dX = \xi^A i \omega^i e_A = \omega^j e_j$ implies $\xi^i j = \delta^i j$. 
Now, $C_i$ is convex away from $e_2$ so

$$\nabla_{e_1} e_1 = \omega_1^A(e_1)e_A = \omega_1^2(e_1)e_2 + \omega_1^3(e_1)e_3 = \omega_1^2(e_1)e_2 + h_{11}e_3 = -\kappa e_2$$

where the curvature of the plane curve $C_i$ is $\kappa = \omega_2^1(e_1) > 0$. But

$$1 = \xi^1_1 = d\xi^1(e_1) + \xi^i\omega_i^1(e_1) = e_1\xi^1 + \xi^2\omega^1_2 = \frac{d}{ds}\xi^1 + \kappa\xi^2$$

$$0 = \xi^2_1 = d\xi^2(e_1) + \xi^i\omega_i^2(e_1) = e_1\xi^2 + \xi^1\omega^1_2 = \frac{d}{ds}\xi^2 - \kappa\xi^1$$

Similarly, using $y^2_1 = h_{11} = h_{12} = 0$ along $C_i$

$$y^1_1 = dy^1(e_1) + y^A\omega_A^1(e_1) = e_1y^1 + y^2\omega^1_2(e_1) + y^3h_{11}$$

$$= \frac{d}{ds}y^1 + \kappa y^2$$

$$0 = y^2_1 = dy^2(e_1) + y^A\omega_A^2(e_1) = e_1y^2 + y^1\omega^1_2(e_1) + y^3h_{12}$$

$$= \frac{d}{ds}y^2 - \kappa y^1$$
The preceding formulas simplify if we parameterize $C_i$ using the angle of the $e_1$ relative to some fixed vector in $\pi_i$. Taking

\[ t = e_1 = (\cos \vartheta, -\sin \vartheta) \quad n = e_2 = (\sin \vartheta, \cos \vartheta) \]

then $\xi^2 = X \cdot n$. $\xi^2 < 0$ everywhere if the origin is inside $C_i$. Denote derivatives with respect to $\vartheta$ by dot. Thus $\dot{t} = -n$ and $\dot{n} = t$. Since $\dot{X}$ is parallel to $t$, differentiating $\xi^2 = X \cdot n$,

\[ \dot{\xi}^2 = \dot{X} \cdot n + X \cdot \dot{t} = X \cdot t \]

hence the position vector is $X = \dot{\xi}^2 t + \xi^2 n$. Differentiating

\[ \dot{X} = \ddot{\xi}^2 t - \dot{\xi}^2 n + \dot{\xi}^2 n + \xi^2 t = (\ddot{\xi}^2 + \xi^2) t \]

It follows that

\[ \frac{1}{\kappa} = \frac{ds}{d\vartheta} = \ddot{\xi}^2 + \xi^2, \quad \frac{d}{ds} = \frac{1}{\kappa} \frac{d}{d\vartheta}, \quad \omega^1 = \frac{1}{\kappa} d\vartheta. \]
From (11) and (12) we see that

\[ \dot{\xi}^2 = \frac{1}{\kappa} \frac{d}{ds} \xi^2 = \xi^1; \quad \ddot{\xi}^2 = \frac{1}{\kappa} \frac{d}{ds} \xi^1 = -\xi^2 + \frac{\xi^1_1}{\kappa} \]

\[ \dot{y}^2 = \frac{1}{\kappa} \frac{d}{ds} y^2 = y^1; \quad \ddot{y}^2 = \frac{1}{\kappa} \frac{d}{ds} y^1 = -y^2 + \frac{y^1_1}{\kappa} \]

Finally, the integrands have the simplified form

\[ J_1 = \int_{C_i} (V, X, dX) = -\int_{C_i} \xi^2 \xi^1_1 \omega^1 = -\int_0^{2\pi} \xi^2 (\xi^2 + \ddot{\xi}^2) d\vartheta < 0; \]

\[ J_2 = \int_{C_i} (V, X, dY) = -\int_{C_i} \xi^2 y^1_1 \omega^1 = -\int_0^{2\pi} \xi^2 (y^2 + \ddot{y}^2) d\vartheta = 0; \]

\[ J_3 = \int_{C_i} (V, Y, dx) = -\int_{C_i} y^2 \xi^1_1 \omega^1 = -\int_0^{2\pi} y^2 (\xi^2 + \ddot{\xi}^2) d\vartheta; \]

\[ J_4 = \int_{C_i} (V, Y, dY) = -\int_{C_i} y^2 y^1_1 \omega^1 = -\int_0^{2\pi} y^2 (y^2 + \ddot{y}^2) d\vartheta. \]

Integration by parts shows \( J_3 = J_2 = 0 \). It remains to show that \( J_4 \geq 0 \).
We conclude the proof using a basic inequality we assume for now.

**Lemma (Wirtinger’s Inequality)**

Let $f(\theta)$ be a smooth function with period $2\pi$ i.e., $f(\theta + 2\pi) = f(\theta)$ for all $\theta$. Suppose $f$ has mean value zero

$$0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \, d\theta.$$

Then

$$\int_0^{2\pi} f(\theta)^2 \, d\theta \leq \int_0^{2\pi} \dot{f}(\theta)^2 \, d\theta.$$

**Proof of the Collander Theorem, Continued.**

Since $\xi_1^1 = 1$, it follows from (10),

$$2A(C_1) = -\int_0^{2\pi} \xi^2 \, d\vartheta > 0.$$
Thus there is a constant \( c \) so that \( f = c\xi^2 + y^2 \) satisfies

\[
\int_{0}^{2\pi} f \, d\vartheta = 0.
\]

From Wirtinger’s Inequality, we have

\[
0 \leq \int_{0}^{2\pi} \left( \dot{f}^2 - f^2 \right) \, d\vartheta
= -\int_{0}^{2\pi} f (f + \ddot{f}) \, d\vartheta
= c^2 J_1 + cJ_2 + cJ_3 + J_4
\]

Now \( J_1 < 0 \) and \( J_2 = J_3 = 0 \) implies \( J_4 \geq 0 \). The same argument holds for each \( C_i \) completing the proof of the Collander Theorem.
Wirtinger’s Inequality bounds the $L^2$ norm of a function by the $L^2$ norm of its derivative. It is also known as the Poincaré Inequality in higher dimensions. We state stronger hypotheses than necessary.

**Theorem (Wirtinger’s inequality)**

Let $f(\theta)$ be a piecewise $C^1(\mathbb{R})$ function with period $2\pi$ (for all $\theta$, $f(\theta + 2\pi) = f(\theta)$). Let $\bar{f}$ denote the mean value of $f$

$$\bar{f} = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \, d\theta.$$  

Then

$$\int_0^{2\pi} (f(\theta) - \bar{f})^2 \, d\theta \leq \int_0^{2\pi} (f'(\theta))^2 \, d\theta.$$ 

Equality holds iff for some constants $a$, $b$,

$$f(\theta) = \bar{f} + a \cos \theta + b \sin \theta.$$
Proof of Wirtinger’s Inequality.

Proof.

Idea: express $f$ and $f'$ in Fourier series. Since $f'$ is bounded and $f$ is continuous, the Fourier series converges at all $\theta$

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \{a_k \cos k\theta + b_k \sin k\theta\}$$

where the Fourier coefficients are determined by formally multiplying by $\sin m\theta$ or $\cos m\theta$ and integrating to get

$$a_m = \frac{1}{\pi} \int_{0}^{2\pi} f(\theta) \cos m\theta \, d\theta, \quad b_m = \frac{1}{\pi} \int_{0}^{2\pi} f(\theta) \sin m\theta \, d\theta,$$

hence $2\bar{f} = a_0$. Sines and cosines are complete so Parseval’s equation holds

$$\int_{0}^{2\pi} (f - \bar{f})^2 = \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2). \quad (13)$$

Formally, this is the integral of the square of the series, where after multiplying out and integrating, terms like $\int \cos m\theta \sin k\theta = 0$ or $\int \cos m\theta \cos k\theta = 0$ if $m \neq k$ drop out and terms like $\int \sin^2 k\theta = \pi$ contribute $\pi$ to the sum.
53. Proof of Wirtinger's Inequality. -

The Fourier Series for the derivative is given by

\[ f'(\theta) \sim \sum_{k=1}^{\infty} \{ -ka_k \sin k\theta + kb_k \cos k\theta \} \]

Since \( f' \) is square integrable, Bessel's inequality gives

\[ \pi \sum_{k=1}^{\infty} k^2 (a_k^2 + b_k^2) \leq \int_0^{2\pi} (f')^2. \]  \hspace{1cm} (14)

Wirtinger's inequality is deduced from (13) and (14) since

\[ \int_0^{2\pi} (f')^2 - \int_0^{2\pi} (f - \overline{f})^2 \geq \pi \sum_{k=2}^{\infty} (k^2 - 1) (a_k^2 + b_k^2) \geq 0. \]

Equality implies that for \( k \geq 2 \), \((k^2 - 1) (a_k^2 + b_k^2) = 0 \) so \( a_k = b_k = 0 \), thus \( f \) takes the form \( f(\theta) = \overline{f} + a \cos \theta + b \sin \theta \).
Thanks!