Deforming Surfaces

Andrejs Treibergs

University of Utah

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http://www.math.utah.edu/~treiberg/DeformingSurfacesSlides.pd

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4. Outline.

- Deformation of a Surface
  - Length Preserving Deformations
  - Equation of Infinitesimal Deformation
  - Rigid Motion
- Example of Infinitesimal Deformation
- Geometric Preliminaries
- Infinitesimal Rigidity of Ovaloids Theorem
  - Bending Vector Field.
  - Blaschke's Integral Formula.
  - Algebraic Lemma about Determinants
- Example of Infinitesimally Flexible Surfaces
  - Cohn-Vossen's Flying Saucer.
  - Rembs' Lemniscate Surface.
- Infinitesimal Rigidity of Collanders
  - Wirtinger's Inequality.

This is one of those topics that is usually discussed at the end of a text on curves and surfaces, and not often reached in a modern abbreviated treatment. Let me try to skip over many technicalities and state some results.

We imagine a two dimensional surface in three space as the image of a smooth vector function $M_0^2 = X_0(W)$ where

$$X_0 : W \to \mathbf{R}^3$$

is a smooth map on $W \subset \mathbf{R}^2$, an open subset. We imagine deforming the surface $M_\tau^2 = X(W, \tau)$ by a smooth one-parameter family of maps

$$X : W \times (-\varepsilon, \varepsilon) \to \mathbf{R}^3$$

such that $X(u, v; 0) = X_0(u, v)$ for all points $(u, v) \in W$.

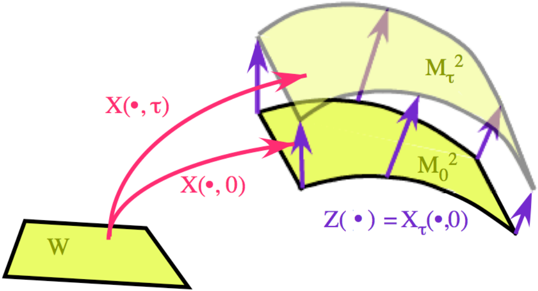


Figure 1: Deformation of Surface

The deformation vector field is the velocity of the deformation

$$Z(\bullet) = \frac{dX}{d\tau}(\bullet; 0).$$

An infinitesimal isometric deformation is a deformation that preserves lengths on the surface up to first order.

If $\gamma(\sigma) = (u(\sigma), v(\sigma)) \in W$ is a curve for $\sigma \in [a, b]$, then its length in M^2_τ of γ is

$$L(\gamma, \tau) = \int_a^b ds_\tau = \int_a^b \left| \frac{d}{d\sigma} X(\gamma(\sigma), \tau) \right| \, d\sigma$$

where $d\gamma[1] = \gamma' = (u', v')$,an element of arclength, is by chain rule

$$\begin{aligned} ds_\tau &= |dX \circ d\gamma[1]| \, d\sigma = \left| dX(\gamma(\sigma);\tau)[u'(\sigma), v'(\sigma)] \right| \, d\sigma \\ &= \begin{pmatrix} X_u(\gamma(\sigma);\tau) \bullet X_u(\gamma(\sigma);\tau)\, u'(\sigma)^2 \\ +2X_u(\gamma(\sigma);\tau) \bullet X_v(\gamma(\sigma);\tau)\, u'(\sigma)v'(\sigma) \\ +X_v(\gamma(\sigma);\tau) \bullet X_v(\gamma(\sigma);\tau)\, v'(\sigma)^2 \end{pmatrix}^{\frac{1}{2}} d\sigma \\ &= \Big((dX \bullet dX)[\gamma'(\sigma), \gamma'(\sigma)] \Big)^{\frac{1}{2}} d\sigma \end{aligned}$$

which, for short is written $ds^2_\tau = dX \bullet dX$.

Note that for a regular surface, dX is full rank so that $dX \bullet dX$ is a symmetric, positive definite quadratic form called the metric form.

To preserve the length of curves up to first order we must have for every curve $\gamma : [a, b] \to W$,

$$0 = \frac{d}{d\tau}\bigg|_{\tau=0} L(\gamma) = \int_a^b \frac{(dX \bullet dZ)[\gamma'(\sigma), \gamma'(\sigma)]}{\Big((dX \bullet dX)[\gamma'(\sigma), \gamma'(\sigma)]\Big)^{\frac{1}{2}}}\, d\sigma$$

which implies that the quadratic form

$$dX \bullet dZ = 0. \tag{1}$$

This is the equation for infinitesimal isometric deformation for unknown vector field Z.

9. Equation in Local Coordinates.

The infinitesimal deformation equation (1) is shorthand for the more cumbersome, but perhaps more familiar form

$$\begin{aligned} 0 &= X_u \bullet Z_u \\ 0 &= X_u \bullet Z_v + X_v \bullet Z_u \\ 0 &= X_v \bullet Z_v \end{aligned}$$

where the vector functions

$$X, Z : W \to \mathbf{R}^3$$

give the position and deformation field in a local coordinates.

This is a system of three linear first order partial differential equations for the three unknown components of the vector function Z.

If a surface moves as a rigid body, then the lengths of all curves on M_τ^2 are preserved. Rigid motions are the composition of of rotations and translations. The deformation due to a rigid motion may be written in terms of smooth rotation matrix $R(\tau)$ and translation vector $T(\tau)$:

$$X(\bullet;\tau) = R(\tau)X_0(\bullet) + T(\tau)$$

such that $R(0) = I$ and $T(0) = 0$. The deformation vector becomes

$$Z(\bullet) = \left.\frac{d}{d\tau}\right|_{\tau=0} X(\bullet,\tau) = R'(0)X_0(\bullet) + T'(0) = A \times X_0(\bullet) + B.$$

Multiplying by the skew symmetric matrix $R'(0)$ (why?) is the same as taking a cross product with the rotation vector
$A = (R'_{32}(0), R'_{13}(0), R'_{21}(0))$.
The translation vector $B = T'(0)$. Such a deformation is called an infinitesimal congruence. Observe that an infinitesimal congruence is an infinitesimal isometric deformation:

$$dX \bullet dZ = dX \bullet [A \times dX] = 0.$$

11. Why Derivative of Rotation $R'(0)$ is Skew.

$R(t)$ is a smooth family of rotations matrices with $R(0) = I$. Since R preserves Euclidean inner product,

$$RV \bullet RW = V \bullet W$$

for all vectors V, W. Hence

$$R^T RV \bullet W = V \bullet W$$

which implies

$$R^T R = I.$$

Differentiating with respect to t,

$$(R')^T R + R^T R' = 0$$

Since $R(0) = I$, at $t = 0$,

$$R' + R = 0$$

so $R'(0)$ is skew.

Thus there are numbers a, b, c so that

$$R'(0)V = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} av_2 + bv_3 \\ -av_1 + cv_3 \\ -bv_1 - cv_3 \end{pmatrix}$$

$$= \begin{vmatrix} i & j & k \\ -c & b & -a \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{pmatrix} -c \\ b \\ -a \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = A \times V$$

The vector V is the rotation vector. It is parallel to the axis of the rotation. Its magnitude gives the angular velocity.

A surface M^2 is infinitesimally rigid is every vector field Z on M^2 satisfying the infinitesimal rigidity equation (1)

$$dX \bullet dZ = 0$$

is an infinitesimal congruence. If there is a solution Z which is not an infinitesimal congruence, then we call it an infinitesimal flex. An infinitesimal deformation gives a corresponding deformation, the simple flex,

$$Y = X_0 + \tau Z$$

which preserves the metric to first order at $\tau = 0$ because $dY \bullet dY =$

$$dX \bullet dX + 2\tau dX \bullet dY + \tau^2 dY \bullet dY = dX \bullet dX + \tau^2 dY \bullet dY.$$

Note that both $Y = X \pm \tau Z$ have the same metric forms, *i.e.*, are isometric.

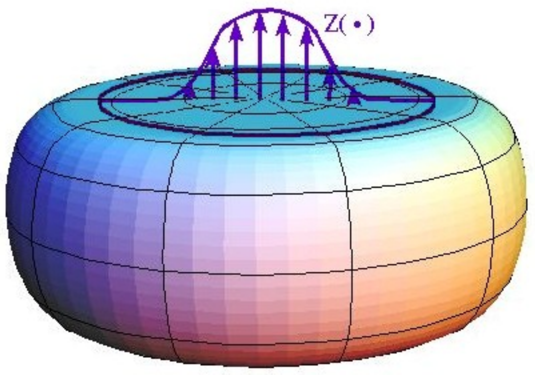


Figure 2: Barrel Example: Surface has isometric Deformation Supported in Flat Region

Suppose that the surface X_0 has a flat planar region F, whose normal vector is $(0,0,1)$. Let $\psi : F \to \mathbb{R}$ be a smooth a "bump function" such that $\psi \geq 0$ and $\psi = 0$ near ∂F and off F. Then

$$Z = (0, 0, \psi)$$

which is zero off F is an infinitesimal deformation. (dX and dZ are perpendicular.)

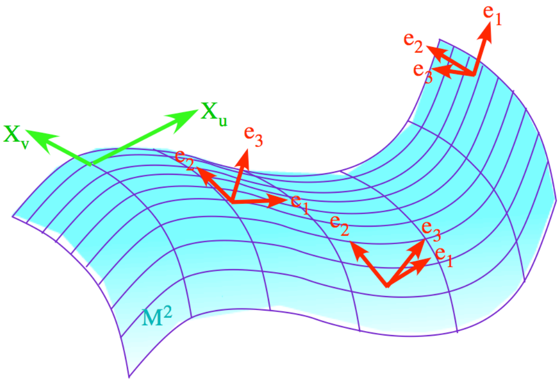


Figure 3: Moving Frame

Locally, a surface is $X : W \to \mathbf{R}^3$.
Tangent plane is spanned by basis $\{X_u, X_v\}$.

At each point replace by smoothly varying orthonormal tangent vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$. Unit normal is

$$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2.$$

Define dual basis of one-forms $\{\omega^A\}$ by $\omega^A(\mathbf{e}_B) = \delta^A{}_B$. So

$$dX = \sum_{i=1}^{3} \omega^i \mathbf{e}_i$$

and metric is

$$ds^2 = (\omega^1)^2 + (\omega^2)^2.$$

Extrinsic Geometry deals with how M sits in its ambient space.

Near $P \in M$, the surface may be parameterized as the graph over its tangent plane, where $f(u_1, u_2)$ is the "height" above the tangent plane

$$X(u_1, u_2) = P + u_1 \mathbf{e}_1(P) + u_2 \mathbf{e}_2(P) + f(u_1, u_2)\mathbf{e}_3(P). \tag{2}$$

So $f(0) = 0$ and $Df(0) = 0$. The Hessian of f at 0 gives the shape operator at P. It is also called the Second Fundamental Form.

$$h_{ij}(P) = \frac{\partial^2 f}{\partial u_i \, \partial u_j}(0)$$

The Mean Curvature and Gaussian Curvature at P are

$$H(P) = \frac{1}{2}\operatorname{tr}(h_{ij}(P)), \qquad K(P) = \det(h_{ij}(P)).$$

A one-form on $\mathbf{R}^3$ is an expression of the form

$$\alpha = f\,dx + g\,dy + h\,dz$$

where f, g, h are smooth functions on $\mathbf{R}^3$. One-forms may be integrated along a curve Γ given by

$$\gamma : [a, b] \to \mathbf{R}^3 \qquad \text{where} \qquad \gamma(t) = \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix}$$

by the formula

$$\int_\Gamma \alpha = \int_a^b \Big(f(\gamma(t))u'(t) + g(\gamma(t))v'(t) + h(\gamma(t))w'(t) \Big)\, dt.$$

The differential d (or gradient) of a function ϕ on $\mathbf{R}^3$ is a one form

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy + \frac{\partial \phi}{\partial z}\, dx.$$

A two-form on $\mathbf{R}^3$ is an expression of the form

$$\beta = f\,dx\wedge dy + g\,dx\wedge dz + h\,dy\wedge dz$$

where f, g, h are smooth functions on $\mathbf{R}^3$. Two-forms may be integrated along a surface $M = X(U)$ given by

$$X : U \to \mathbf{R}^3 \qquad \text{where} \qquad X(s,t) = \begin{pmatrix} u(s,t) \\ v(s,t) \\ w(s,t) \end{pmatrix}$$

and $U \subset \mathbf{R}^2$ is smooth bounded domain by $\displaystyle\int_M \beta =$

$$\iint_U \left(f(X(s,t))\frac{\partial(u,v)}{\partial(s,t)} + g(X(s,t))\frac{\partial(u,w)}{\partial(s,t)} + h(X(s,t))\frac{\partial(v,w)}{\partial(s,t)} \right)\,ds\,dt$$

where the Jacobean is

$$\frac{\partial(u,v)}{\partial(s,t)} = \frac{\partial u}{\partial s}(s,t)\frac{\partial v}{\partial t}(s,t) - \frac{\partial u}{\partial t}(s,t)\frac{\partial v}{\partial s}(s,t)$$

An example of a two-form is the area form for the surface M. In terms of the orthonormal coframe it is given by

$$\beta = dA = \omega^1 \wedge \omega^2$$

which is everywhere positive on positively oriented surfaces M. Its integral over any subset $N \subset M$ gives the area of the subset

$$\mathrm{A}(N) = \int_N \omega^1 \wedge \omega^2.$$

The exterior derivative d (or curl) of a one-form

$$\alpha = f\,dx + g\,dy + h\,dz$$

on $\mathbf{R}^3$ is a two-form

$$d\alpha = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right) dx \wedge dz + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz.$$

Stokes Theorem in this notation is

$$\int_\Sigma d\,\alpha = \int_{\partial\Sigma} \alpha$$

where $\partial\Sigma$ is the collection of oriented boundary curves of the surface Σ. The right side is zero if the surface is closed (has no boundary such as a sphere or torus.)

Upper case Roman indices run over $A, B, C, \ldots = 1, 2, 3$ and lower case run over $i, j, k, \ldots = 1, 2$. Einstein Convention: repeated lower and upper indices are assumed to be summed.

Because $\mathbf{e}_A \cdot \mathbf{e}_A = 1$, taking the directional derivative $d\mathbf{e}_A \cdot \mathbf{e}_A = 0$, so that $d\mathbf{e}_A \perp \mathbf{e}_A$ and we may express the rate of rotation of the frame

$$d\mathbf{e}_A = {\omega_A}^B \mathbf{e}_B.$$

${\omega_A}^B$ are called connection forms. Differentiation of $\mathbf{e}_A \cdot \mathbf{e}_B = \delta_{AB}$ implies ${\omega_A}^B$ is skew and satisfies

$$d\omega^A = \omega^B \wedge {\omega_B}^A. \tag{3}$$

Also, differentiating the normal defines $d\mathbf{e}_3 = {\omega_3}^i \mathbf{e}_i$. Moreover

$${\omega_3}^i = -h_{ij}\omega^j$$

recovers the second fundamental form.

An ovaloid is a $\mathcal{C}^3$ closed surface M^2 which is the boundary of a bounded convex domain of three space. It is strictly convex if the second fundamental form h_{ij} with respect to the inner normal is positive definite.

Theorem (Liebmann, 1899)

Strictly convex ovaloids are infinitesimally rigid.

This question was asked by Jellet in 1854, who was unable to prove it.

In 1835 Minding posed a related problem: if two ovaloids are isometric, must they be congruent? For round spheres, this was proved by Liebmann and Hilbert in 1903. The first proof for ovaloids was given by Weyl in 1915.

See my 2012 USAC Colloquium Slides "Geometry of Bending Surfaces", `http://www.math.utah.edu/ treiberg/BendingSlides.pdf`

Let X be local coordinates in the neighborhood of a point of a surface $M^2 \subset \mathbf{R}^3$. Let Z be an infinitesimal deformation. Then

$$dX \bullet dZ = 0.$$

This implies that there is a globally defined vector field Y, the bending vector, such that

$$dZ = Y \times dX.$$

Intuitively, this is because dZ is perpendicular to dX.

Thus, the differential of an infinitesimal deformation dZ gives a local rotation whose axis Y varies from point to point.

Because if we choose a right-handed orthonormal frame on M such that $\mathbf{e}_1$ and $\mathbf{e}_2$ are tangent so $\mathbf{e}_3$ is normal to M. Let $\{\omega^A\}$ be the dual coframe so $dX = \omega^i\,\mathbf{e}_i$ and $Z = \zeta^A\mathbf{e}_A$ (Einstein convention!) Then the covariant derivative is in coordinates $dZ = \zeta^B{}_j\,\omega^j\,\mathbf{e}_B$. Substituting,

$$\begin{aligned}
0 &= \left(\omega^i\,\mathbf{e}_i\right) \bullet \left(\zeta^B_j\,\omega^j\,\mathbf{e}_B\right) \\
&= \left(\delta_{iB}\zeta^B{}_j\right)\,\omega^i \otimes \omega^j \\
&= (\zeta^1{}_1)\,\omega^1 \otimes \omega^1 + (\zeta^1{}_2 + \zeta^2{}_1)\,\omega^1 \otimes \omega^2 + (\zeta^2{}_2)\,\omega^2 \otimes \omega^2
\end{aligned}$$

so $\zeta^1{}_1 = \zeta^2{}_2 = \zeta^2{}_1 + \zeta^1{}_2 = 0$. Hence $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ and so on implies

$$\begin{aligned}
dZ &= \zeta^1{}_2\,\omega^2\,\mathbf{e}_1 + \zeta^2{}_1\,\omega^1\,\mathbf{e}_2 + \left(\zeta^3{}_1\,\omega^1 + \zeta^3{}_2\,\omega^2\right)\,\mathbf{e}_3 \\
&= \left(\zeta^3{}_2\,\mathbf{e}_1 - \zeta^3{}_1\,\mathbf{e}_2 + \zeta^2{}_1\,\mathbf{e}_3\right) \times \left(\omega^j\,\mathbf{e}_j\right) = Y \times dX.
\end{aligned}$$

The desired $Y = y^A\,\mathbf{e}_A = \zeta^3{}_2\,\mathbf{e}_1 - \zeta^3{}_1\,\mathbf{e}_2 + \zeta^2{}_1\,\mathbf{e}_3$.

Theorem (Liebmann, 1899)

Strictly convex ovaloids are infinitesimally rigid.

Figure 4: Wilhelm Blaschke 1885–1962

Proof (Blaschke, 1921).

Here is the basic idea of Blaschke's amazing proof using an integral formula.

□

The idea is to show that the local bending vector is a constant. In other words, its derivative dY should vanish.

Blaschke finds a one-form θ such that

$$d\theta = \Phi(X, dX, dY)\,\omega^1 \wedge \omega^2$$

where $\Phi \le 0$ is a nonpositive function if the surface is convex and $\Phi = 0$ implies $dY = 0$. Then by Stokes Theorem on the ovaloid M,

$$0 = \int_M d\theta = \int_M \Phi(X, dX, dY)\,\omega^1 \wedge \omega^2$$

whch implies $\Phi = 0$ at all points so $dY = 0$.

27. Prepare the Proof: Differentiate dZ.

Differentiating $dZ = Y \times dX$ we find

$$0 = dY \times dX$$

which means tangent planes are parallel to each other. Hence the covariant derivative may be written

$$dY = {y^i}_j \, \omega^j \mathbf{e}_i$$

such that ${y^1}_1 + {y^2}_2 = 0$.

Because, if we write the covariant derivative $dY = {y^A}_i \omega^i \mathbf{e}_A$ we get

$$\begin{aligned} 0 &= ({y^A}_i\, \omega^i \mathbf{e}_A) \times (\omega^j \mathbf{e}_j) \\ &= \left({y^A}_i\, \omega^i \wedge \omega^j\right)\, \mathbf{e}_A \times \mathbf{e}_j \\ &= \omega^1 \wedge \omega^2 \left\{({y^1}_1 + {y^2}_2)\, \mathbf{e}_1 \times \mathbf{e}_2 - {y^3}_2\, \mathbf{e}_3 \times \mathbf{e}_1 + {y^3}_1\, \mathbf{e}_3 \times \mathbf{e}_2\right\} \end{aligned}$$

so that ${y^1}_1 + {y^2}_2 = {y^3}_1 = {y^3}_2 = 0$.

Differentiating $dY = {y^i}_j \, \omega^j \mathbf{e}_i$,

$$\begin{aligned} 0 &= d{y^i}_j \wedge \omega^j \mathbf{e}_i + {y^i}_j \, \omega^k \wedge {\omega_k}^j \mathbf{e}_i - {y^i}_j \, \omega^j \wedge \omega_i^B \mathbf{e}_B \\ &= \left(d{y^i}_j - {y^i}_k \, {\omega_j}^k + {y^k}_j \, {\omega_k}^i \right) \omega^j \mathbf{e}_i - {y^i}_j h_{ik} \, \omega^j \wedge \omega^k \mathbf{e}_3 \end{aligned}$$

This gives a first order system of PDE's for ${y^i}_j$. The $\mathbf{e}_3$ coefficient

$$0 = \left({y^i}_1 h_{i2} - {y^i}_2 h_{i1} \right) \omega^1 \wedge \omega^2$$

Or using ${y^1}_1 + {y^2}_2 = 0$.

$$0 = -{y^1}_2 h_{11} + 2{y^1}_1 h_{12} + {y^2}_1 h_{22} \tag{4}$$

Using the determinant of a matrix whose three column vectors are given, we consider the one-form defined globally on M^2,

$$\theta = \det(X, Y, dY)$$

Differentiating using ${y^3}_j = {y^1}_1 + {y^2}_2 = 0$ and $\xi^A = X \bullet \mathbf{e}_A$,

$$\begin{aligned} d\theta &= \det(dX, Y, dY) + \det(X, dY, dY) \\ &= \det\left(\omega^i \mathbf{e}_i,\ y^B \mathbf{e}_B,\ {y^k}_j \omega^j \mathbf{e}_k\right) + \det\left(\xi^A \mathbf{e}_A,\ {y^i}_j \omega^j \mathbf{e}_i,\ {y^k}_\ell \omega^\ell \mathbf{e}_k\right) \\ &= -y^3\left({y^2}_2 + {y^1}_1\right)\omega^1 \wedge \omega^2 + \xi^3\left({y^1}_j {y^2}_\ell - {y^2}_j {y^1}_\ell\right)\omega^j \wedge \omega^\ell \\ &= 2\xi^3\left({y^1}_1 {y^2}_2 - {y^2}_1 {y^1}_2\right)\omega^1 \wedge \omega^2 \end{aligned}$$

Lemma

Assume h_{ij} is symmetric and positive definite and ${y^i}_j$ satisfies

$$0 = -{y^1}_2 h_{11} + 2{y^1}_1 h_{12} + {y^2}_1 h_{22}; \qquad {y^1}_1 + {y^2}_2 = 0. \tag{5}$$

Then

$$\det({y^i}_j) = {y^1}_1 {y^2}_2 - {y^2}_1 {y^1}_2 \le 0$$

and is equal to zero only if all ${y^i}_j = 0$.

Positive definite means $h_{11} > 0$, $h_{22} > 0$ and $\det(h_{ij}) > 0$.

There are many proofs. We give an elementary argument.

Proof.

In case $h_{12} = 0$ we have

$$0 = -y^1{}_2 h_{11} + y^2{}_1 h_{22} \tag{6}$$

so $y^1{}_2$ and $y^2{}_1$ have the same sign so

$$\det(y^i{}_j) = -(y^1{}_1)^2 - y^2{}_1 y^1{}_2 \leq 0.$$

If equal to zero then $y^1{}_1 = -y^2{}_2 = 0$ and, say, $y^1{}_2 = 0$ which imples $y^2{}_1 = 0$ by (6). The case $y^2{}_1 = 0$ is similar. □

In case $h_{12} \neq 0$ we have by (5), $y^1{}_1 = \frac{y^1{}_2 h_{11} - y^2{}_1 h_{22}}{2h_{12}}$. Hence

$$\begin{aligned} \det(y^i{}_j) &= -\left(\frac{y^1{}_2 h_{11} - y^2{}_1 h_{22}}{2h_{12}}\right)^2 - y^1{}_2 y^2{}_1 \\ &= -\frac{\left[y^1{}_2 h_{11} - y^2{}_1 h_{22}\right]^2 + 4h_{12}^2 y^2{}_1 y^1{}_2}{4h_{12}^2} \\ &= -\frac{\left[y^1{}_2 h_{11} + y^2{}_1 h_{22}\right]^2 - 4(h_{11}h_{22} - h_{12}^2) y^2{}_1 y^1{}_2}{4h_{12}^2} \end{aligned}$$

If $y^2{}_1 y^1{}_2 \leq 0$ then $\det(y^i{}_j) \leq 0$ and equal to zero if $y^1{}_2 = y^2{}_1 = 0$ so $y^1{}_1 = y^2{}_2 = 0$ by (5). If $y^2{}_1 y^1{}_2 \geq 0$ then

$$\det(y^i{}_j) = -(y^1{}_1)^2 - y^2{}_1 y^1{}_2 \leq 0$$

also and equal to zero implies $y^1{}_1 = y^2{}_2 = 0$ and, say, $y^1{}_2 = 0$. Then $y^2{}_1 = 0$ by (5). The case $y^2{}_1 = 0$ is similar. $\square$

Theorem (Liebmann, 1899)

Strictly convex ovaloids are infinitesimally rigid.

Proof (Blaschke, 1921).

Assume the origin is in the interior of M and $\mathbf{e}_3$ is the inner normal. It follows that

$$\xi^3 = X \bullet \mathbf{e}_3 < 0.$$

M is closed and oriented so by Stokes Theorem and Blaschke's Formula

$$0 = \int_M d\theta = \int_M 2\xi^3 \det({y^i}_j)\, dA$$

Since M^2 is strictly convex, h_{ij} is positive definite so by the Algebraic Lemma, $\det({y^i}_j) \le 0$ and the integrand in nonnegative. Since the integral equals zero, the determinant must vanish, hence all ${y^i}_j = 0$ everywhere on M^2 by the Lemma. Hence Y is constant. $\square$

Consider the translation field

$$T = Z - Y \times X.$$

Differentiating, since Y satisfies $dZ = Y \times dX$ so

$$dT = dZ - dY \times X - Y \times dX = -dY \times X$$

which vanishes because Y is constant. Thus $Z = T + Y \times X$ is an infinitesimal congruence and M is infinitesimally rigid. □

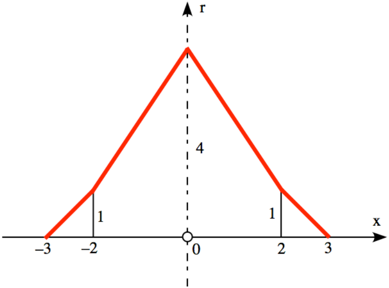
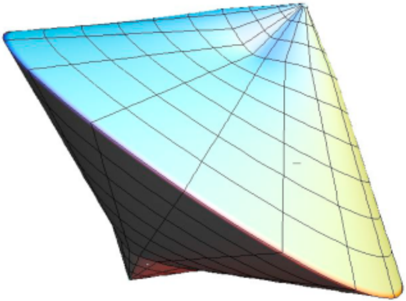


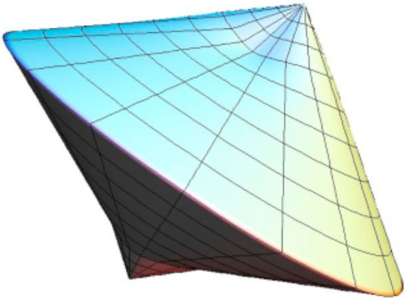
Figure 5: Section of Cohn-Vossen's Infinitesimally Flexible Surface of Rotation

In 1930, Stefan Cohn-Vossen found that by cutting an arbitrarily small nonconvex groove into a surface of revolution, one could create an ifinitesimally flexible surface. He also found that by gluing together four cones, one could get an infinitesimally flexible flying saucer surface. The radial distance as a function of x is given by the even piecewise linear function, depicted here.

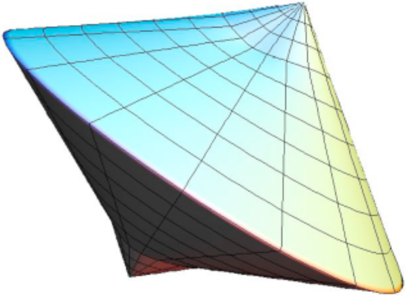
37. Cohn-Vossen's Flexible Flying Saucer.



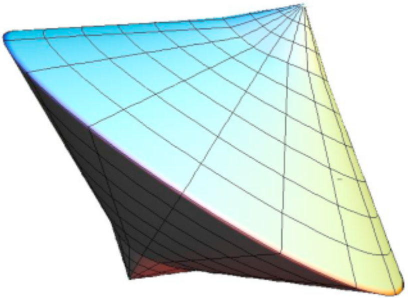
37. Cohn-Vossen's Flexible Flying Saucer.



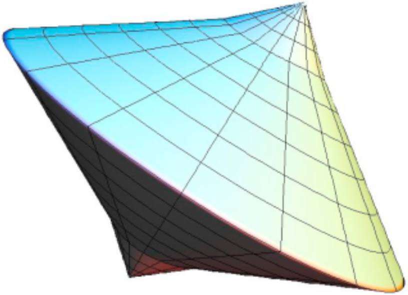
37. Cohn-Vossen's Flexible Flying Saucer.



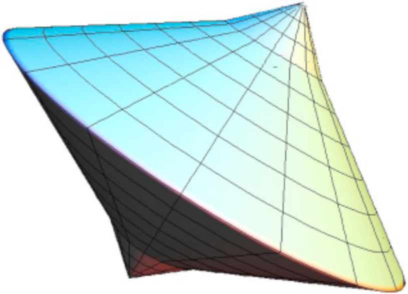
37. Cohn-Vossen's Flexible Flying Saucer.



37. Cohn-Vossen's Flexible Flying Saucer.



37. Cohn-Vossen's Flexible Flying Saucer.



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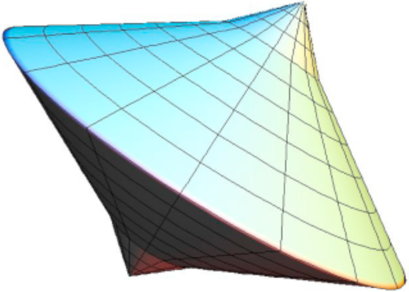
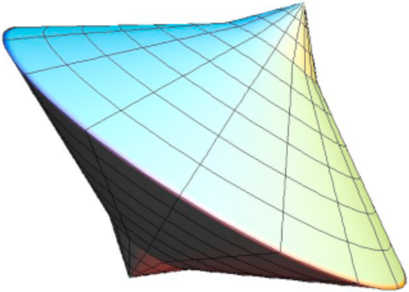
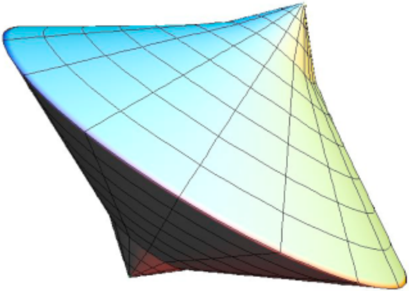


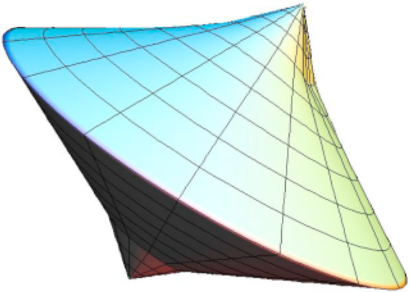
37. Cohn-Vossen's Flexible Flying Saucer.

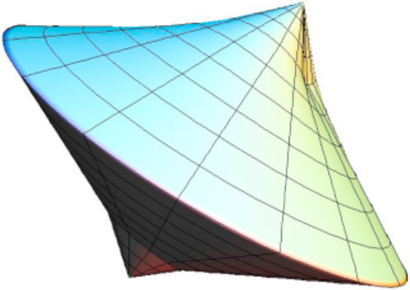


37. Cohn-Vossen's Flexible Flying Saucer.

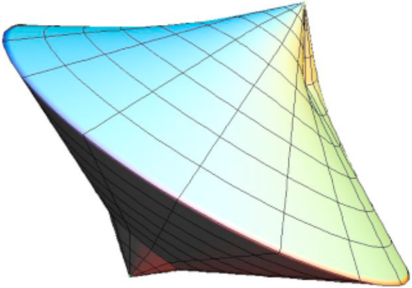


37. Cohn-Vossen's Flexible Flying Saucer.





37. Cohn-Vossen's Flexible Flying Saucer.



38. Cohn-Vossen's Flexible Flying Saucer. Stop!

In 1952, E. Rembs found that for countably many values in $\frac{1}{3} < c^2 < \frac{1}{2}$, the nonconvex surfaces of revolution (7) admit nontrivial infinitesimal deformations.

He used separation of variables and Fourier series.

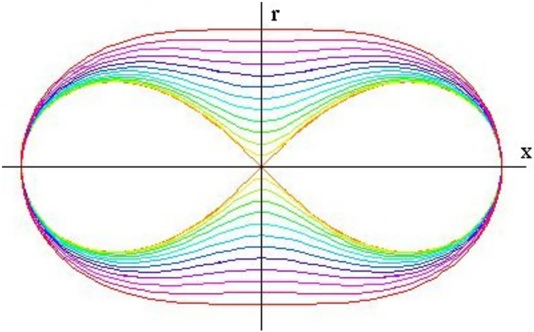


Figure 6: Sections of Rembs Surfaces

Equation of section as c varies

$$(x^2 + r^2)^2 + 2c^2(x^2 - r^2) = 1 - 2c^2 \qquad (7)$$

$c = 0$ is sphere. $c^2 = \frac{1}{2}$ is revolution of lemniscate.

40. Application: Isometric Non-Congruent Analytic Surfaces.

The deformations Z of Rembs were given by (finite) Fourier Series. Hence both the surface and the deformation are analytic functions (those given by convergent power series.) Thus the existence of non-trivial isometric deformations is not an artifact of lack of smoothness as in the Barrel Example, Fig. 2

In the barrel surface, the derivatives of z-component of X would be dead zero when computed in the flat part so the power series would consist of the constant term only. Thus barrel surface cannot be analytic because its continuation to M would remain constant.

For small $\tau > 0$ the two surfaces

$$X \pm \tau Z$$

are analytic and isometric (have the same metric)

$$ds^2_{+\tau} = ds^2_{-\tau} = dX \bullet dX + \tau^2 \, dZ \bullet DZ$$

but, as observer by Rembs, not congruent.

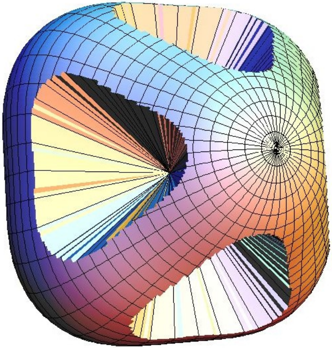


Figure 7: Collander Surface: Convex Surface with Planar Boundary Components

A Collander Surface is a smooth convex surface M with m smooth boundary curves, "holes," $C_1, \ldots, C_m$ such that $K > 0$ away from the boundary curves and for each $k = 1 \ldots, m$ there is a plane π_k such that C_k is a strictly convex curve in π_k and M makes a first order tangency to π_k along C_k.

Theorem (Rembs 1919)

Collander surfaces are infinitesimally rigid.

Proof.

The proof is as in the Ovaloid Theorem. For a deformation field and its corresponding bending field Y, it suffices to show that the Blaschke's integrand vanishes. This time, Stokes' Theorem has boundary terms

$$\sum_{i=1}^{m} \int_{C_i} \theta = \int_M d\theta = \int_M 2\xi^3 \det(y^i{}_j)\, dA$$

As $\xi^3 \det(y^i{}_j) \geq 0$ by Lemma 3., it suffices to show that each

$$\int_{C_i} \theta \leq 0 \tag{8}$$

Fix i and take the orientation of C_i as part of ∂M. Thus if we take a local frame for M such that M is convex toward $\mathbf{e}_3$ and $\mathbf{e}_1$ is tangent to C_i, then $\mathbf{e}_2$ must point into M.

□

Since $\mathbf{e}_3$ is the normal to π_3 along C_i it is constant, so along C_i,

$$0 = \nabla_{\mathbf{e}_1}\mathbf{e}_3 = {\omega_3}^j(\mathbf{e}_1)\mathbf{e}_j = -h_{jk}\omega^k(\mathbf{e}_1)\mathbf{e}_j = -h_{j1}\mathbf{e}_j$$

It follows that $h_{11} = h_{12} = h_{21} = 0$ along C_i. Equation (4) tells us

$$0 = -{y^1}_2 h_{11} + 2{y^1}_1 h_{12} + {y^2}_1 h_{22} = {y^2}_1 h_{22}$$

along C_i. Now we also assumed that M makes first order contact, so that $h_{22} > 0$ on C_i. Hence on C_i,

$${y^2}_1 = 0.$$

The covariant derivative of a vector field, $dY = {y^A}_j\omega^j = dy^A + y^B\omega_B^A$. Since ${y^3}_1 = 0$ everywhere, then along C_i,

$$0 = {y^3}_1 = dy^3(\mathbf{e}_1) + y^j{\omega_j}^3(\mathbf{e}_1) = \mathbf{e}_1 y^3 + y^j h_{j1} = \tfrac{dy^3}{ds}.$$

Thus $y^3 = \text{const.}$ along C_i. □

$$\begin{aligned}\theta = (X, Y, dY) &= \xi^A y^B y^i{}_j \omega^j (\mathbf{e}_A, \mathbf{e}_B, \mathbf{e}_j) \\ &= \xi^3 y^k y^i{}_j \omega^j (\mathbf{e}_3, \mathbf{e}_k, \mathbf{e}_j) + y^3 \xi^k y^i{}_j \omega^j (\mathbf{e}_k, \mathbf{e}_3, \mathbf{e}_j) \\ &= \xi^3 \left(y^1 y^2{}_j - y^2 y^1{}_j \right) \omega^j + y^3 \left(\xi^2 y^1{}_j - y^1 y^2{}_j \right) \omega^j\end{aligned}$$

On C_i where $y^2{}_1 = 0$ and $y^3 = \text{const.}$ we have

$$\theta = -\xi^3 y^2 y^1{}_1 \omega^1 + y^3 \xi^2 y^1{}_1 \omega^1 + (\cdots)\omega^2.$$

The second term is an exact differential so integrates to zero on C_i. Indeed, since C_i is planar, there is a constant vector field V_i on three space such that $\mathbf{e}_3 = V_i$ on C_i. Since $dT = X \times dY$ we have on C_i,

$$\begin{aligned}d(V_i \bullet T) = V_i \bullet dT = (V, X, dY) &= \xi^A y^i{}_j (\mathbf{e}_3, \mathbf{e}_A, \mathbf{e}_i) \omega^j \\ &= \left(\xi^1 y^2{}_j - \xi^2 y^1{}_j \right) \omega^j = -\xi^2 y^1{}_1 \omega^1 + (\cdots)\omega^2. \qquad (9)\end{aligned}$$

As $\xi^3 < 0$ is constant, the proof reduces to show each

$$\int_{C_i} y^2 {y^1}_1 \, \omega^1 \le 0.$$

The area enclosed by C_i may be computed as follows. Let V be the constant vector field such that $\mathbf{e}_i = V$ along C_i. Observe that

$$d(V, X, dX) = (V, dX, dX) = (\mathbf{e}_3, \mathbf{e}_i, \mathbf{e}_j)\omega^i \wedge \omega^j = 2\omega^1 \wedge \omega^2$$

so that since C_i is oriented to the outside surface,

$$\mathrm{A}(C_i) = -\tfrac{1}{2} \int_{C_i} (V, X, dX) > 0.$$

The integrand can be written

$$\begin{aligned} (V, X, dX) &= \xi^A {\xi^B}_i \omega^i (\mathbf{e}_3, \mathbf{e}_A, \mathbf{e}_B) \\ &= (\xi^1 {\xi^2}_i - \xi^2 {\xi^1}_i)\omega^i = -\xi^2 {\xi^1}_1 \, \omega^1 + (\cdots)\omega^2 \end{aligned} \tag{10}$$

This simplifies because $dX = {\xi^A}_i \omega^i \mathbf{e}_A = \omega^j \mathbf{e}_j$ implies ${\xi^i}_j = {\delta^i}_j$.

Now, C_i is convex away from $\mathbf{e}_2$ so

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1 = {\omega_1}^A(\mathbf{e}_1)\mathbf{e}_A = {\omega_1}^2(\mathbf{e}_1)\mathbf{e}_2 + {\omega_1}^3(\mathbf{e}_1)\mathbf{e}_3 = {\omega_1}^2(\mathbf{e}_1)\mathbf{e}_2 + h_{11}\mathbf{e}_3 = -\kappa\mathbf{e}_2$$

where the curvature of the plane curve C_i is $\kappa = {\omega_2}^1(\mathbf{e}_1) > 0$. But

$$\begin{aligned} 1 = {\xi^1}_1 &= d\xi^1(\mathbf{e}_1) + \xi^i{\omega_i}^1(\mathbf{e}_1) = \mathbf{e}_1\xi^1 + \xi^2{\omega_2}^1 = \tfrac{d}{ds}\xi^1 + \kappa\xi^2 \\ 0 = {\xi^2}_1 &= d\xi^2(\mathbf{e}_1) + \xi^i{\omega_i}^2(\mathbf{e}_1) = \mathbf{e}_1\xi^2 + \xi^1{\omega_1}^2 = \tfrac{d}{ds}\xi^2 - \kappa\xi^1 \end{aligned} \tag{11}$$

Similarly, using ${y^2}_1 = h_{11} = h_{12} = 0$ along C_i

$$\begin{aligned} {y^1}_1 &= dy^1(\mathbf{e}_1) + y^A{\omega_A}^1(\mathbf{e}_1) = \mathbf{e}_1 y^1 + y^2{\omega_2}^1(\mathbf{e}_1) + y^3 h_{11} \\ &= \tfrac{d}{ds}y^1 + \kappa y^2 \\ 0 = {y^2}_1 &= dy^2(\mathbf{e}_1) + y^A{\omega_A}^2(\mathbf{e}_1) = \mathbf{e}_1 y^2 + y^1{\omega_1}^2(\mathbf{e}_1) + y^3 h_{12} \\ &= \tfrac{d}{ds}y^2 - \kappa y^1 \end{aligned} \tag{12}$$

$\square$

The preceding formulas simplify if we parameterize C_i using the angle of the $\mathbf{e}_1$ relative to some fixed vector in π_i. Taking

$$t = \mathbf{e}_1 = (\cos\vartheta, -\sin\vartheta) \qquad n = \mathbf{e}_2 = (\sin\vartheta, \cos\vartheta)$$

then $\xi^2 = X \bullet n$. $\xi^2 < 0$ everywhere if the origin is inside C_i. Denote derivatives with respect to ϑ by dot. Thus $\dot{t} = -n$ and $\dot{n} = t$. Since $\dot{X}$ is parallel to t, differentiating $\xi^2 = X \bullet n$,

$$\dot{\xi}^2 = \dot{X} \bullet n + X \bullet t = X \bullet t$$

hence the position vector is $X = \dot{\xi}^2\, t + \xi^2\, n$. Differentiating

$$\dot{X} = \ddot{\xi}^2\, t - \dot{\xi}^2\, n + \dot{\xi}^2\, n + \xi^2\, t = (\ddot{\xi}^2 + \xi^2)t$$

It follows that

$$\frac{1}{\kappa} = \frac{ds}{d\vartheta} = \ddot{\xi}^2 + \xi^2, \qquad \frac{d}{ds} = \frac{1}{\kappa}\frac{d}{d\vartheta}, \qquad \omega^1 = \frac{1}{\kappa}d\vartheta.$$

From (11) and (12) we see that

$$\dot{\xi}^2 = \frac{1}{\kappa}\frac{d}{ds}\xi^2 = \xi^1; \qquad \ddot{\xi}^2 = \frac{1}{\kappa}\frac{d}{ds}\xi^1 = -\xi^2 + \frac{{\xi^1}_1}{\kappa}$$
$$\dot{y}^2 = \frac{1}{\kappa}\frac{d}{ds}y^2 = y^1; \qquad \ddot{y}^2 = \frac{1}{\kappa}\frac{d}{ds}y^1 = -y^2 + \frac{{y^1}_1}{\kappa}$$

Finally, the integrands have the simplified form

$$J_1 = \int_{C_i}(V, X, dX) = -\int_{C_i}\xi^2{\xi^1}_1\omega^1 = -\int_0^{2\pi}\xi^2(\xi^2 + \ddot{\xi}^2)\,d\vartheta < 0;$$
$$J_2 = \int_{C_i}(V, X, dY) = -\int_{C_i}\xi^2{y^1}_1\omega^1 = -\int_0^{2\pi}\xi^2(y^2 + \ddot{y}^2)\,d\vartheta = 0;$$
$$J_3 = \int_{C_i}(V, Y, dx) = -\int_{C_i}y^2{\xi^1}_1\omega^1 = -\int_0^{2\pi}y^2(\xi^2 + \ddot{\xi}^2)\,d\vartheta;$$
$$J_4 = \int_{C_i}(V, Y, dY) = -\int_{C_i}y^2{y^1}_1\omega^1 = -\int_0^{2\pi}y^2(y^2 + \ddot{y}^2)\,d\vartheta.$$

Integration by parts shows $J_3 = J_2 = 0$. It remains to show that $J_4 \geq 0$. □

We conclude the proof using a basic inequality we assume for now.

Lemma (Wirtinger's Inequality)

Let $f(\theta)$ be a smooth function with period 2π i.e., $f(\theta+2\pi)=f(\theta)$ for all θ. Suppose f has mean value zero

$$0=\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta.$$

Then

$$\int_0^{2\pi} f(\theta)^2\,d\theta \le \int_0^{2\pi} \dot{f}(\theta)^2\,d\theta.$$

Proof of the Collander Theorem, Confinued.

Since ${\xi^1}_1=1$, it follows from (10),

$$2A(C_1)=-\int_0^{2\pi} \xi^2\,d\vartheta>0.$$

□

Thus there is a constant c so that $f = c\xi^2 + y^2$ satisfies

$$\int_0^{2\pi} f \, d\vartheta = 0.$$

From Wirtinger's Inequality, we have

$$\begin{aligned} 0 &\le \int_0^{2\pi} \dot{f}^2 - f^2 \, d\vartheta \\ &= -\int_0^{2\pi} f(f + \ddot{f}) \, d\vartheta \\ &= c^2 J_1 + cJ_2 + cJ_3 + J_4 \end{aligned}$$

Now $J_1 < 0$ and $J_2 = J_3 = 0$ implies $J_4 \ge 0$. The same argument holds for each C_i completing the proof of the Collander Theorem. $\square$

Wirtinger's Inequality bounds the L^2 norm of a function by the L^2 norm of its derivative. It is also known as the Poincaré Inequality in higher dimensions. We state stronger hypotheses than necessary.

Theorem (Wirtinger's inequality)

Let $f(\theta)$ be a piecewise $C^1(\mathbb{R})$ function with period 2π (for all θ, $f(\theta + 2\pi) = f(\theta)$). Let $\bar{f}$ denote the mean value of f

$$\bar{f} = \tfrac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta.$$

Then

$$\int_0^{2\pi} \left(f(\theta) - \bar{f}\right)^2 \, d\theta \le \int_0^{2\pi} \left(f'(\theta)\right)^2 \, d\theta.$$

Equality holds iff for some constants a, b,

$$f(\theta) = \bar{f} + a\cos\theta + b\sin\theta.$$

Proof.

Idea: express f and f' in Fourier series. Since f' is bounded and f is continuous, the Fourier series converges at all θ

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \{a_k \cos k\theta + b_k \sin k\theta\}$$

where the Fourier coefficients are determined by formally multiplying by $\sin m\theta$ or $\cos m\theta$ and integrating to get

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos m\theta \, d\theta, \qquad b_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin m\theta \, d\theta,$$

hence $2\bar{f} = a_0$. Sines and cosines are complete so Parseval's equation holds

$$\int_0^{2\pi} \left(f - \bar{f}\right)^2 = \pi \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2\right). \tag{13}$$

Formally, this is the integral of the square of the series, where after multiplying out and integrating, terms like $\int \cos m\theta \sin k\theta = 0$ or $\int \cos m\theta \cos k\theta = 0$ if $m \neq k$ drop out and terms like $\int \sin^2 k\theta = \pi$ contribute π to the sum.

The Fourier Series for the derivative is given by

$$f'(\theta) \sim \sum_{k=1}^{\infty} \{-ka_k \sin k\theta + kb_k \cos k\theta\}$$

Since f' is square integrable, Bessel's inequality gives

$$\pi \sum_{k=1}^{\infty} k^2 \left(a_k^2 + b_k^2\right) \le \int_0^{2\pi} (f')^2. \tag{14}$$

Wirtinger's inequality is deduced form (13) and (14) since

$$\int_0^{2\pi} (f')^2 - \int_0^{2\pi} \left(f - \bar{f}\right)^2 \ge \pi \sum_{k=2}^{\infty} (k^2 - 1) \left(a_k^2 + b_k^2\right) \ge 0.$$

Equality implies that for $k \ge 2$, $(k^2 - 1)\left(a_k^2 + b_k^2\right) = 0$ so $a_k = b_k = 0$, thus f takes the form $f(\theta) = \bar{f} + a\cos\theta + b\sin\theta$.

Thanks!