# Heat Equation & Curvature Flow

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The URL for these Beamer Slides: *"Heat Equation and Curvature Flow"*

`http://www.math.utah.edu/~treiberg/CurvFlowSlides.pdf`

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## 4. Outline.

- Success of Ricci Flow Motivates Looking at Elementary Flows.
- Heat Equation on the Circle
  - Separation of Variables.
  - Maximim Principle.
  - Integral Estimates.
    - Wirtinger's inequality.
    - Uniform Convergence of Temperature.
- Curvature Flow of a Plane Curve.
  - Arclength, Tangent Vector, Normal Vector, Curvature.
  - First Variation of Length and Area.
  - Examples of Curvature Flow.
  - Curvature Flow Rounds Out Curves.
    - Maximim Principle.
    - Integral Estimates.
    - Geometric Inequalities.
    - Curvature Flow Reduces Isoperimetric Ratio.
- Some Higher Dimensional Results.

Figure: Richard Hamilton.

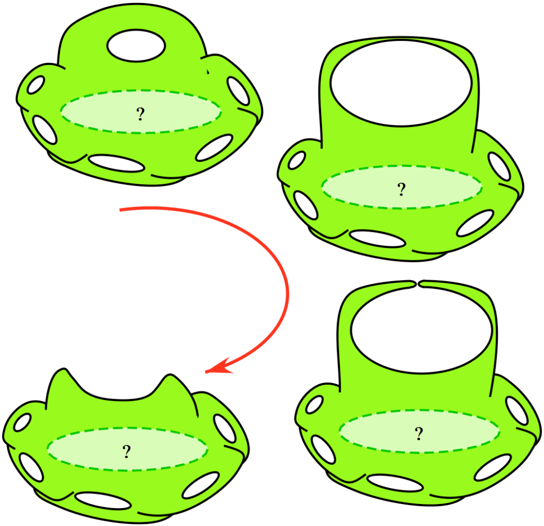
In 1904 Poincaré conjectured that a closed simply connected three manifold $M$ is homeomorphic to the sphere.

In a series of papers beginning 1982, Hamilton perfected the PDE machinery to solve the Poincaré conjecture. Starting from an arbitrary Riemannian metric $g_0$ for $M$, he evolved it according to Ricci Flow

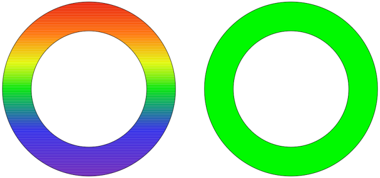
$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}(g), \qquad g(0) = g_0.$$

Since the solution generally encounters singularities, he proposed to intervene with surgery whenever singularities form. In 2003, Perelman found a way to control the topology at singularities enough to say the flow with surgery results in finitely many standard topological maneuvers ending at the sphere. Topological methods applied to standard manifolds finish the argument.

# 6. Schematic of Ricci Flow with Surgery.



The space $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the circle of length $2\pi$. We say that a function $f(\theta) \in \mathcal{C}^k(\mathbb{S}^1)$ if $f(\theta)$ is defined for all $\theta \in \mathbb{R}$, is $k$-times continuously differentiable and is $2\pi$-periodic.



If $g(\theta) \in \mathcal{C}(\mathbb{S}^1)$ is the initial temperature on a thin unit circular rod. Let $u(t,\theta) \in \mathcal{C}^2([0,\infty) \times \mathbb{S}^1)$. The temperature for future times satisfies the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial \theta^2}, && \text{for all } t > 0 \text{ and } \theta \in \mathbb{S}^1. \\ u(0,\theta) &= g(\theta), && \text{for all } \theta \in \mathbb{S}^1. \end{aligned}$$

Figure: J. Fourier 1768–1830.

His 1822 *Théorie analytique de la chaleur* was called "a great mathematical poem" by Kelvin but Lagrange, Laplace and Legendre criticized it for a looseness of reasoning.

We make the ansatz that

$$u(t,\theta) = T(t)\Theta(\theta)$$

for a $2\pi$-periodic $\Theta$. Heat equation becomes

$$T'(t)\Theta(\theta) = T(t)\Theta''(\theta).$$

Separating variables implies that there is a constant $\lambda$ so that

$$\frac{T'(t)}{T(t)} = -\lambda = \frac{\Theta''(\theta)}{\Theta(\theta)}.$$

This results in two equations

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0, \quad \Theta \text{ is } 2\pi\text{-periodic on } \mathbb{R}; \\ T' + \lambda T &= 0, \quad \text{for all } t > 0. \end{aligned}$$

Solutions of the first equation are $2\pi$-periodic if $\lambda = k^2$ for $k \in \mathbb{Z}$

$$\Theta'' + \lambda\Theta = 0$$

which yields the solution $\Theta = A_0$ if $\lambda = 0$ and

$$\Theta(\theta) = A_k \cos k\theta + B_k \sin k\theta$$

if $\lambda = k^2$ for some $k \in \mathbb{N}$ and constants $A_k$, $B_k$.
The corresponding solution of $T' + \lambda T = 0$ is

$$T(t) = e^{-k^2 t}.$$

The PDE is solved by

$$u(t, \theta) = e^{-k^2 t} \left(A_k \cos k\theta + B_k \sin k\theta\right)$$

for $k \in \mathbb{N}$.

By linearity, the solution is obtained by superposition

$$u(t,\theta) = A_0 + \sum_{k=1}^{\infty} e^{-k^2 t}\left(A_k \cos k\theta + B_k \sin k\theta\right). \tag{1}$$

The initial condition is satisfied

$$g(\theta) = u(0,\theta) = \frac{1}{2}A_0 + \sum_{k=1}^{\infty}\left(A_k \cos k\theta + B_k \sin k\theta\right)$$

if one takes the Fourier coefficients of $g(\theta)$. For $k \in \mathbb{Z}_+$,

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\, \cos k\theta \, d\theta, \quad B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\, \sin k\theta \, d\theta$$

## Theorem (Heat Equation Properties)

*Suppose that $g(\theta) \in \mathcal{C}(\mathbb{S}^1)$. Then there is a solution $u \in C([0,\infty) \times \mathbb{S}^1) \cap C^2((0,\infty) \times \mathbb{S}^1)$, given by (1), that satisfies the heat equation*

$$\begin{aligned} u_t &= u_{\theta\theta}, && \text{for } (t,\theta) \in (0,\infty) \times \mathbb{S}^1; \\ u(0,\theta) &= g(\theta), && \text{for all } \theta \in \mathbb{S}^1. \end{aligned}$$

*The solution has the following properties:*

1. *If $u$ and $v$ both satisfy the heat equation, and if $u(0,\theta) < v(0,\theta)$ for all $\theta$ then $u(t,\theta) \le v(t,\theta)$ for all $t \ge 0$ and all $\theta$.*
2. $\min_{\mathbb{S}^1} g \le u \le \max_{\mathbb{S}^1} g$.
3. $\int_{\mathbb{S}^1} (u - \frac{1}{2}A_0)^2 \, d\theta \le e^{-2t} \|g - A_0\|_2^2$ *for all* $t \ge 0$.
4. *There are $c_1, c_2 > 0$ so that $\int_{\mathbb{S}^1} u_\theta^2 \, d\theta \le c_1 e^{-2t}$ for all $t \ge c_2$.*
5. $u(t,\theta) \to \frac{1}{2}A_0$ *uniformly as* $t \to \infty$.

**Theorem (Maximum Principle for Heat Equation on the Circle)**

*Suppose that $u, v \in \mathcal{C}([0,T) \times \mathbb{S}^1) \cap \mathcal{C}^2((0,T) \times \mathbb{S}^1)$ both satisfy the heat equation and initial condition*

$$\begin{aligned} u_t = u_{\theta\theta}, \quad v_t = v_{\theta\theta}, \qquad & \text{on } (0,T) \times \mathbb{S}^1; \\ u \ge v, \qquad & \text{on } \{0\} \times \mathbb{S}^1. \end{aligned}$$

*Then*

$$u \ge v, \qquad \text{on } [0,T) \times \mathbb{S}^1.$$

The idea is that if there is a point $(t_0, \theta_0)$ where solutions first touch $u(t_0,\theta_0) = v(t_0,\theta_0)$ then $u(t_0,\theta) \ge v(t_0,\theta)$ for all $\theta$ for all $\theta$. Hence

$$u_{\theta\theta}(t_0,\theta_0) \ge v_{\theta\theta}(t_0,\theta_0)$$

so $u_t(t_0,\theta_0) \ge v_t(t_0,\theta_0)$ and the solutions move apart.

However, we can't draw this conclusion unless the inequalities are strict.

*Proof.* Choose $\epsilon > 0$. Let $w(t,\theta) = u(t,\theta) - v(t,\theta) + \epsilon t + \epsilon$. Note that $w$ satisfies

$$w_t = u_t - v_t + \epsilon = u_{\theta\theta} - v_{\theta\theta} + \epsilon = w_{\theta\theta} + \epsilon.$$

At $t = 0$ we have $w(0,\theta) > 0$. I claim $w > 0$ for all $(t,\theta)$. If not, there is a first time $t_1 > 0$ where $w = 0$, say at some point $w(t_1,\theta_1) = 0$. Because $w(t,\theta_1) > 0$ for all $0 \leq t < t_1$, we have $w_t(t_1,\theta_1) \leq 0$. Because $w(t_1,\theta) \geq 0$ for all $\theta$, we have $w_{\theta\theta}(t_1,\theta_1) \geq 0$. Plugging into the equation

$$0 \geq w_t(t_1,\theta_1) = w_{\theta\theta}(t_1,\theta_1) + \epsilon \geq 0 + \epsilon > 0$$

which is a contradiction. Thus, $w > 0$ for all $(t,\theta)$, which implies

$$u(t,\theta) - v(t,\theta) > -\epsilon t - \epsilon.$$

But for fixed $(t,\theta)$, by taking $\epsilon > 0$ arbitrarily small, it follows that

$$u(t,\theta) - v(t,\theta) \geq 0. \qquad \square$$

To see (2), let

$$w(t,\theta) = \min_{\mathbb{S}^1} g, \qquad v(t,\theta) = \max_{\mathbb{S}^1} g.$$

Since $w$ and $v$ are constant, they satisfy the heat equation. Since

$$w(0,\theta) \le u(0,\theta) \le v(0,\theta),$$

it follows from (1) that that for all $(t,\theta)$,

$$w(t,\theta) \le u(t,\theta) \le v(t,\theta).$$

Deduce co-evolution of interesting quantities such as average temperature.

To see (3), first notice that the average temperature remains constant in time

$$\frac{d}{dt}\left(\frac{1}{2\pi}\int_{\mathbb{S}^1} u\,d\theta\right) = \frac{1}{2\pi}\int_{\mathbb{S}^1} u_t\,d\theta = \frac{1}{2\pi}\int_{\mathbb{S}^1} u_{\theta\theta}\,d\theta = 0.$$

Thus, for all $t \geq 0$,

$$\frac{1}{2\pi}\int_{\mathbb{S}^1} u(t,\theta)\,d\theta = \frac{1}{2}A_0 = \frac{1}{2\pi}\int_{\mathbb{S}^1} g(\theta)\,d\theta.$$

Also the $\mathcal{L}^2$ deviation decreases. Using Wirtinger's inequality

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}^1}\left[u-\frac{1}{2}A_0\right]^2 d\theta &= 2\int_{\mathbb{S}^1}\left[u-\frac{1}{2}A_0\right] u_t\, d\theta\\ &= 2\int_{\mathbb{S}^1}\left[u-\frac{1}{2}A_0\right] u_{\theta\theta}\, d\theta\\ &= -2\int_{\mathbb{S}^1} u_\theta^2\, d\theta\\ &\le -2\int_{\mathbb{S}^1}\left[u-\frac{1}{2}A_0\right]^2 d\theta\end{aligned}$$

This says $y' \le -2y$ so $y \le y_0 e^{-2t}$ or

$$\int_{\mathbb{S}^1}\left[u-\frac{1}{2}A_0\right]^2 d\theta \le \left(\int_{\mathbb{S}^1}\left[g(\theta)-\frac{1}{2}A_0\right]^2 d\theta\right) e^{-2t}.$$

Wirtinger's Inequality bounds the $L^2$ norm of a function by the $L^2$ norm of its derivative. It is also known as the Poincaré Inequality in higher dimensions. We state stronger hypotheses than necessary.

Figure: Wilhelm Wirtinger 1865–1945.

**Theorem (Wirtinger's inequality)**

*Let $f(\theta)$ be a piecewise $C^1(\mathbb{R})$ function with period $2\pi$ (for all $\theta$, $f(\theta+2\pi) = f(\theta)$). Let $\bar{f}$ denote the mean value of $f$*

$$\bar{f} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta.$$

*Then*

$$\int_0^{2\pi} \left(f(\theta) - \bar{f}\right)^2 d\theta \le \int_0^{2\pi} \left(f'(\theta)\right)^2 d\theta.$$

*Equality holds iff for some constants $a$, $b$,*

$$f(\theta) = \bar{f} + a\cos\theta + b\sin\theta.$$

Idea: express $f$ and $f'$ in Fourier series. Since $f'$ is bounded and $f$ is continuous, the Fourier series converges at all $\theta$

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left\{ a_k \cos k\theta + b_k \sin k\theta \right\}.$$

Fourier coefficients are determined by formally multiplying by $\sin m\theta$ or $\cos m\theta$ and integrating to get

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos m\theta \, d\theta, \qquad b_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin m\theta \, d\theta,$$

hence $2\bar{f} = a_0$. Sines and cosines are complete so Parseval equation holds

$$\int_0^{2\pi} \left(f - \bar{f}\right)^2 = \pi \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2\right). \tag{2}$$

Formally, this is the integral of the square of the series, where after multiplying out and integrating, terms like $\int \cos m\theta \sin k\theta = 0$ or $\int \cos m\theta \cos k\theta = 0$ if $m \neq k$ drop out and terms like $\int \sin^2 k\theta = \pi$ contribute $\pi$ to the sum.

The Fourier Series for the derivative is given by

$$f'(\theta) \sim \sum_{k=1}^{\infty} \{-ka_k \sin k\theta + kb_k \cos k\theta\}$$

Since $f'$ is square integrable, Bessel's inequality gives

$$\pi \sum_{k=1}^{\infty} k^2 \left(a_k^2 + b_k^2\right) \le \int_0^{2\pi} (f')^2. \tag{3}$$

Wirtinger's inequality is deduced form (2) and (3) since

$$\int_0^{2\pi} (f')^2 - \int_0^{2\pi} \left(f - \bar{f}\right)^2 \ge \pi \sum_{k=2}^{\infty} (k^2 - 1) \left(a_k^2 + b_k^2\right) \ge 0.$$

Equality implies that for $k \ge 2$, $(k^2 - 1)\left(a_k^2 + b_k^2\right) = 0$ so $a_k = b_k = 0$, thus $f$ takes the form $f(\theta) = \bar{f} + a\cos\theta + b\sin\theta$. □

By differentiating the equation, we get the evolution equation of heat flux $u_\theta$

$$(u_\theta)_t = (u_t)_\theta = (u_{\theta\theta})_\theta = (u_\theta)_{\theta\theta}$$

$\mathcal{L}^2$ norm of heat flux decreases. Using Wirtinger's inequality

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}^1} u_\theta^2\, d\theta &= 2\int_{\mathbb{S}^1} u_\theta u_{\theta t}\, d\theta \\
&= 2\int_{\mathbb{S}^1} u_\theta u_{\theta\theta\theta}\, d\theta \\
&= -2\int_{\mathbb{S}^1} u_{\theta\theta}^2 d\theta \\
&\le -2\int_{\mathbb{S}^1} u_\theta^2 d\theta
\end{aligned}$$

Thus, for any $0 < t_0 < t$,

$$\int_{\mathbb{S}^1} u_\theta^2(t,\theta)\, d\theta \le \left(\int_{\mathbb{S}^1} u_\theta^2(t_0,\theta)\, d\theta\right) e^{-2(t-t_0)}.$$

# 21. Temperature converges uniformly to its average.

Since $\theta \mapsto u(t,\theta)$ is continuous, there is a point $\theta_0 \in \mathbb{S}^1$ such that $u(t,\theta_0) = \frac{1}{2}A_0$ equals its average. Let $\theta_0 \le \theta_1 < \theta_0 + 2\pi$ be any point on the circle. By Schwarz inequality, for $t \ge c_2$,

$$\begin{aligned}
|u(t,\theta_1) - u(t,\theta_0)|^2 &\le \left| \int_{\theta_0}^{\theta_1} u_\theta(t,\theta)\, d\theta \right|^2 \\
&\le (\theta_1 - \theta_0) \int_{\theta_0}^{\theta_1} u_\theta^2 \, d\theta \\
&\le 2\pi \int_{\mathbb{S}^1} u_\theta^2 d\theta \\
&\le 2\pi c_1 e^{-2t}.
\end{aligned}$$

Thus temperature converges uniformly because for any $\theta_1$ and $t \ge c_2$,

$$\left| u(t,\theta_1) - \frac{1}{2}A_0 \right| \le (2\pi c_1)^{\frac{1}{2}} e^{-t}.$$

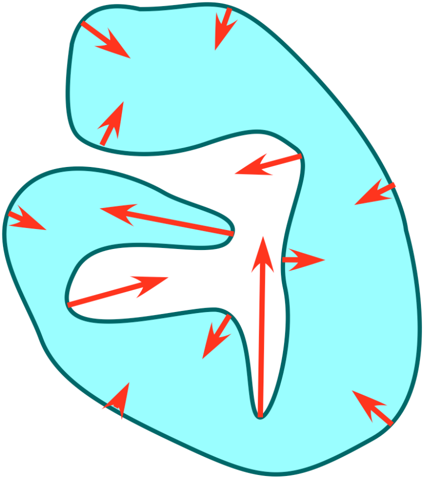


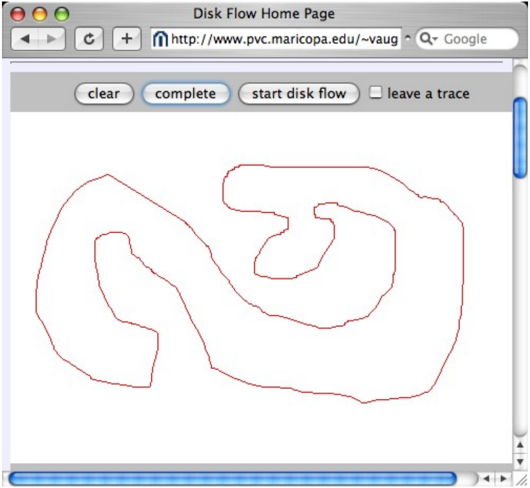
Figure: Deform Curve

Is it possible to continuously deform a curve in such a way that

- The parts that are bent the most are unbent the fastest;
- The curve doesn't cross itself;
- The deformation limits to a circle?

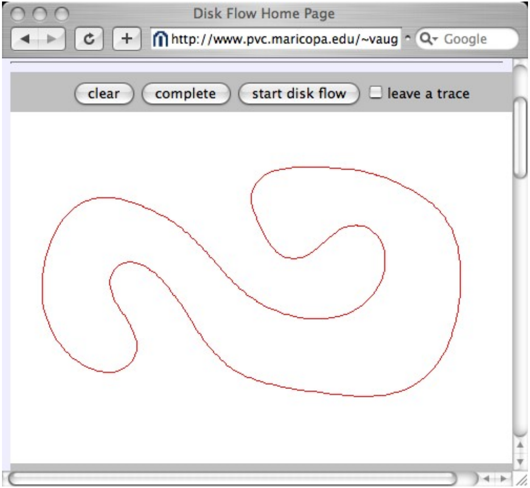
The answer is YES! Method: CURVATURE FLOW Heat Equation!

$$X_t = X_{ss}.$$




Curvature flow applet written by Richard Vaugh, Paradise Valley CC,

`http://www.pvc.maricopa.edu/∼vaughn/java/JDF/df.html`

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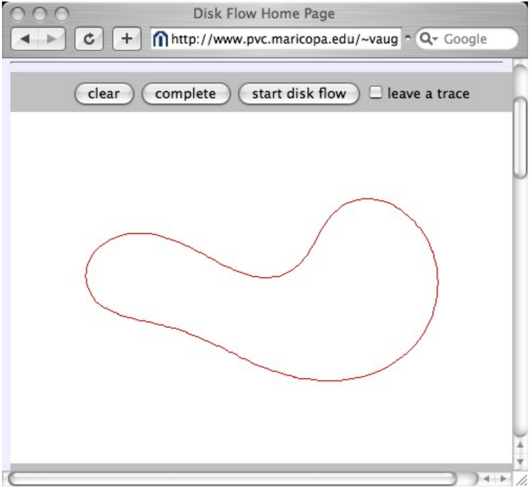
# 23. Vaughn's Applet.

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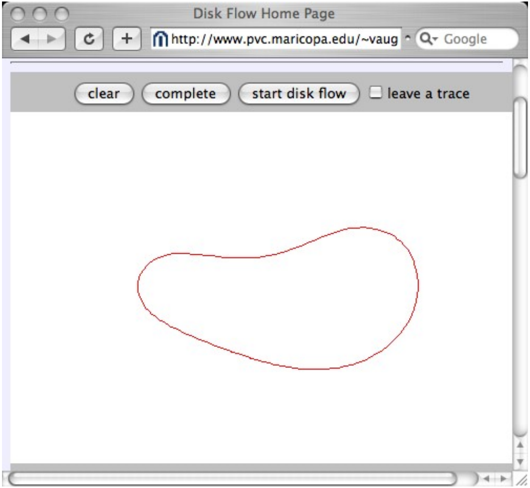


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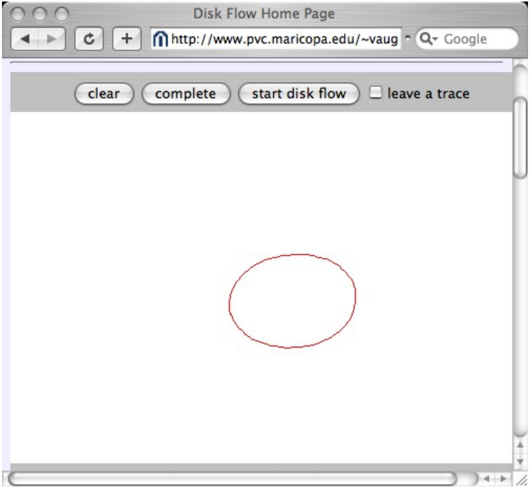


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Let $X(u)$ be a regular smooth closed curve in the plane.
(Regular means $X_u \neq 0$.)

$$X : [0, a] \to \mathbb{R}^2,$$
$$X(0) = X(a);$$
$$X_u(0) = X_u(a).$$

The velocity vector is $X_u$. $s$ is the arclength along the curve $s(u) = \mathrm{L}(X([0, u]))$. The speed of the curve is

$$\frac{ds}{du} = |X_u|$$

The unit tangent and normal vectors are thus

$$T = \frac{X_u}{|X_u|}; \qquad N = \mathcal{R}T = \mathcal{R}\left(\frac{X_u}{|X_u|}\right)$$

where $\mathcal{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the 90° rotation matrix.

The curvature $\kappa$ measures how fast the unit tangent vector turns relative to length along the curve.
Let $\theta$ be the angle of the tangent vector from horizontaL

$$T = (\cos\theta, \sin\theta) \qquad \text{so} \qquad N = \mathcal{R}T = (-\sin\theta, \cos\theta)$$

Then the curvature is

$$\kappa = \frac{d\theta}{ds} \qquad \text{so} \qquad \frac{d}{ds}T = (-\sin\theta, \cos\theta)\frac{d\theta}{ds} = \kappa N$$

and also

$$\frac{d}{ds}N = -\kappa T.$$

For example, for the circle of radius $R$,

$$X(u) = (R\cos u, R\sin u)$$
$$X_u = (-R\sin u, R\cos u)$$
$$\frac{ds}{du} = |X_u| = |(-R\sin u, R\cos u)| = R$$
$$T = \frac{(-R\sin u, R\cos u)}{|(-R\sin u, R\cos u)|} = (-\sin u, \cos u) = (\cos\theta(u), \sin\theta(u))$$

where $\theta(u) = u + \dfrac{\pi}{2}$ so

$$\kappa = \frac{d\theta}{ds} = \frac{du}{ds}\frac{d\theta}{du} = \frac{1}{|X_u|}\cdot 1 = \frac{1}{R}.$$

Equivalently, since $N = \mathcal{R}T = (-\cos u, -\sin u)$,

$$\frac{dT}{ds} = \frac{du}{ds}\frac{dT}{du} = \frac{1}{|X_u|}(-\cos u, -\sin u) = \frac{1}{R}N = \kappa N. \qquad \text{so} \qquad \kappa = \frac{1}{R}.$$

Note that the derivative of the speed gives

$$\frac{d}{du}|X_u| = \frac{d}{du}\sqrt{X_u \cdot X_u} = \frac{1}{2}\frac{1}{|X_u|}2x_u \cdot X_{uu} = \frac{X_u \cdot X_{uu}}{|X_u|}.$$

$\kappa$ determined from the formula

$$\frac{dT}{ds} = \kappa N = \frac{1}{|X_u|}\frac{d}{du}\left(\frac{X_u}{|X_u|}\right) = \frac{1}{|X_u|}\left(\frac{X_{uu}}{|X_u|} - \frac{X_u \frac{d}{du}|X_u|}{|X_u|^2}\right)$$
$$= \frac{1}{|X_u|}\left(\frac{X_{uu}}{|X_u|} - \frac{(X_u \cdot X_{uu})X_u}{|X_u|^3}\right).$$

By the way, this formula decomposes acceleration into tangential and centripetal pieces

$$X_{uu} = \frac{d}{du}|X_u|\,T + \kappa|X_u|^2 N$$

Let's work out how fast length changes when we perturb the curve. The length $\mathrm{L}(X)$ is obtained from the integral

$$L_0 = \int_\Gamma ds = \int_0^a |X_u|\, du.$$

How does the length of $\Gamma_t$ given by $u \mapsto X(t,u)$ change if we deform the curve sideways at a velocity $v$? In a deformation of this kind

$$\frac{\partial}{\partial t} X = vN$$

where $v$ is velocity of deformation and $N$ is the normal to $\Gamma_t$ at $X(t,u)$. We seek the first variation of length which is the derivative of $\mathrm{L}(X)$:

**Theorem (First Variation of Arclength)**

*Suppose that $X(t,u)$ is a smooth family of regular closed curves that are deformed with velocity is $X_t(t,u) = v(t,u)N(t,u)$. Then*

$$\frac{d}{dt} L(X) = -\int_{\Gamma_t} \kappa v \, ds. \tag{4}$$

*Proof.* For arbitrary function $g$, derivatives with respect to arclength

$$g_s = \frac{1}{|X_u|} g_u \qquad \text{and} \qquad ds = |X_u|\, du$$

Differentiating,

$$\begin{aligned}\frac{\partial}{\partial t}|X_u| &= \frac{X_u \cdot X_{ut}}{|X_u|} = \frac{X_u \cdot X_{tu}}{|X_u|} = T \cdot (X_t)_s |X_u| \\ &= T \cdot (vN)_s |X_u| = T \cdot (v_s N - v\kappa T)|X_u| = -\kappa v |X_u|.\end{aligned}$$

Hence

$$\frac{d}{dt}\mathrm{L}(X) = \int \frac{\partial}{\partial t}|X_u|\, du = -\int \kappa v |X_u|\, du = -\int \kappa v\, ds \qquad \square$$

For example, $Y(t,u) = \Big((R-t)\cos u, (R-t)\sin u\Big)$ is a circle of radius $(R-t)$, $N = (-\cos u, -\sin u)$ is the normal for all circles and

$$\frac{dY}{dt} = -(\sin u, \cos u) = -N$$

so $v = 1$ is constant. Since $|Y_u| = R - t$, $ds = (R-t)\,du$ and the curvature of $X$ is $\kappa = \dfrac{1}{R-t}$, the first variation is just

$$\begin{aligned} \frac{d\,\mathrm{L}}{dt} &= -\int_\Gamma \kappa v\,ds = -\int_0^{2\pi} \frac{1}{R-t}\cdot 1\,(R-t)du \\ &= -2\pi = \frac{d}{dt}\,\mathrm{L}(Y) = \frac{d}{dt}2\pi(R-t). \end{aligned}$$

A similar computation gives the first variation of area. Writing $X = (x, y)$, the area $\mathrm{A}(X)$ is obtained from the line integral

$$A_0 = \frac{1}{2}\oint_\Gamma x\,dy - y\,dx = \frac{1}{2}\int_0^a xy_u - yx_u\,du = \frac{1}{2}\int_0^a \mathcal{R}X\cdot X_u\,du$$

since $\mathcal{R}X = (-y, x)$.

How does the area changes if we deform the curve $\Gamma_t$ given by $X(t, u)$ with velocity $v$ in the $N$ direction?

The first variation of area is the time derivative of $\mathrm{A}(X)$:

**Theorem (First Variation of Area.)**

*Suppose that $X(t, u)$ is a smooth family of regular closed curves such that the normal velocity is $X_t(t, u) = v(t, u)N(t, u)$. Then*

$$\frac{d}{dt}A(X) = -\int_{\Gamma_t} v\,ds. \tag{5}$$

*Proof.*

$$
\begin{aligned}
\frac{d}{dt}\mathrm{A}(\Gamma) &= \frac{1}{2}\frac{d}{dt}\int_0^a \mathcal{R}X \cdot X_u \, du \\
&= \frac{1}{2}\int_0^a \mathcal{R}X_t \cdot X_u + \mathcal{R}X \cdot X_{ut} \, du \\
&= \frac{1}{2}\int_0^a \mathcal{R}X_t \cdot X_u + \mathcal{R}X \cdot X_{tu} \, du \\
&= \frac{1}{2}\int_0^a \mathcal{R}X_t \cdot X_u - \mathcal{R}X_u \cdot X_t \, du \\
&= \frac{1}{2}\int_0^a \left[\mathcal{R}(vN) \cdot T - \mathcal{R}T \cdot (vN)\right] |X_u| \, du \\
&= \frac{1}{2}\int_0^a \left[-(vT) \cdot T - N \cdot (vN)\right] ds \\
&= -\int_0^a v \, ds. \qquad \square
\end{aligned}
$$

For example, if $Y$ is the circle $Y(t,u) = \big((R-t)\cos u, (R-t)\sin u\big)$,

$$\frac{dY}{dt} = -(\cos u, \cos v) = -N$$

so $v = 1$ is constant. Then the first variation is just

$$\frac{dA}{dt} = -\int_{\Gamma} 1\, ds = -2\pi(R-t) = \frac{d}{dt}\pi(R-t)^2.$$

Let us assume that we have a family of curves $\Gamma_t$ given by $X(t, u)$ for $t \geq 0$ and $0 \leq u \leq a$. If the curve is moving normally at a velocity $v(t, u) = \kappa(t, u)$, at all points $(t, u)$,

$$\frac{d}{dt} X(t, u) = \kappa(t, u) N(t, u).$$

We say that the family of curves moves by CURVATURE FLOW.

Since $X_s = T$ and $T_s = \kappa N$, the curvature flow satisfies

$$X_t = X_{ss}$$

suggesting that it is a heat equation. Indeed, it is a nonlinear parabolic PDE.

For example, for the family of circles $X(t,u) = \rho(t)(\cos u, \sin u)$, The points move in the normal direction because

$$\frac{d}{dt}X = \rho'(t)(\cos u, \sin u) = -\rho' N$$

Hence the circles flow by curvature if

$$-\rho' = \kappa = \frac{1}{\rho}$$

It follows that

$$\rho(t) = \sqrt{R^2 - 2t}$$

if the initial radius is $\rho(0) = R$.

The circles shrink and completely vanish when $t$ approaches $R^2/2$.

In the jargon, a curve given by $y = f(x)$ is called nonparametric. It is still a parameterized curve $X(u) = (u, f(u))$. Thus

$$X_u = (1, \dot{f}), \qquad |X_u| = \sqrt{1 + \dot{f}^2}, \qquad T = \frac{(1, \dot{f})}{\sqrt{1 + \dot{f}^2}}, \qquad N = \frac{(-\dot{f}, 1)}{\sqrt{1 + \dot{f}^2}}.$$

Computing curvature

$$\frac{dT}{ds} = \frac{1}{|X_u|}\frac{dT}{du} = \frac{(-\dot{f}\ddot{f}, \ddot{f})}{(1 + \dot{f}^2)^2} = \frac{\ddot{f}}{(1 + \dot{f}^2)^{\frac{3}{2}}} \frac{(-\dot{f}, 1)}{\sqrt{1 + \dot{f}^2}} = \kappa N \qquad (6)$$

so the curvature of a nonparametric curve is

$$\kappa = \frac{\ddot{f}}{(1 + \dot{f}^2)^{\frac{3}{2}}}$$

There is no reason to track individual points of the curve as it flows. We could reparameterize as we go and then the trajectory $t \mapsto X(t,u)$ need not be perpendicular to the curve $\Gamma_t$. We need only that the normal projection of velocity be curvature

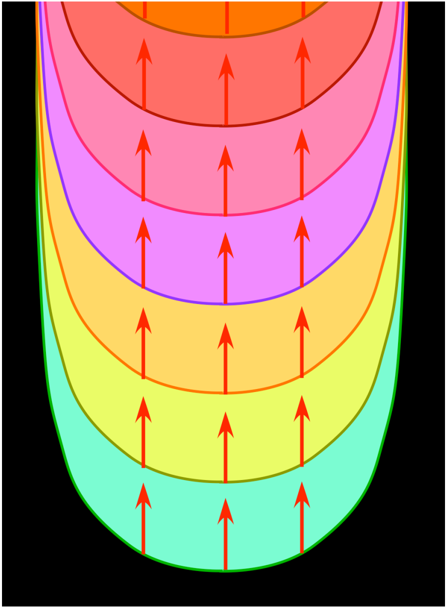
$$N \cdot \frac{\partial X}{\partial t} = \kappa \tag{7}$$

For example, suppose that a fixed curve moves by steady vertical translation. If this is also curvature flow, it is called a soliton. In this case

$$X = \big(u, f(u) + ct\big), \quad X_u = (1, \dot f), \quad |X_u| = \sqrt{1 + \dot f^2}, \quad \kappa = \frac{\ddot f}{(1 + \dot f^2)^{3/2}}$$

Then $\dfrac{\partial X}{\partial t} = (0, c)$. By (7) the equation for a translation soliton satisfies

$$\frac{c}{\sqrt{1 + \dot f^2}} = \frac{\ddot f}{(1 + \dot f^2)^{3/2}} \tag{8}$$



The solution of (8) gives a soliton called the Grim Reaper. (8) becomes

$$c = \frac{\ddot{f}}{a + \dot{f}^2} = \frac{d}{du} \operatorname{Atn} \dot{f}$$

so

$$\operatorname{Atn} \dot{f} = cu + c_1$$
$$\dot{f} = \tan(cu + c_1)$$
$$f = c_2 - \frac{1}{c} \ln \cos(cu + c_1).$$

If $c = 1$ and $c_1 = c_2 = 0$,

$$X(t, u) = \big(u, t - \ln \cos(u)\big)$$

If we change variables according to $v = h(u)$, where $h'$ is nonvanishing, then

$$\frac{\partial}{\partial v} = \frac{\partial u}{\partial v}\frac{\partial}{\partial u} = h'(v)\frac{\partial}{\partial u}.$$

It follows that

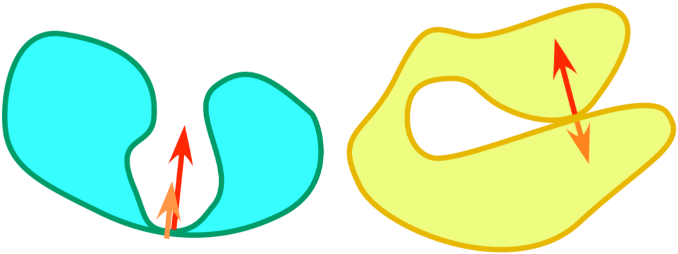
$$\frac{\partial}{\partial s} = \frac{1}{|X_v|}\frac{\partial}{\partial v} = \frac{h'}{|h'X_u|}\frac{\partial}{\partial u} = \frac{\epsilon}{|X_u|}\frac{\partial}{\partial u}$$

where $\epsilon = \pm 1$ according to whether $h' > 0$ or $h' < 0$. Hence $X_{ss}$ is the same regardless of parameterization.

$$\kappa N = X_{ss} = \frac{1}{|X_v|}\frac{\partial}{\partial v}\left(\frac{1}{|X_v|}\frac{\partial}{\partial v}X\right) = \frac{1}{|X_u|}\frac{\partial}{\partial u}\left(\frac{1}{|X_u|}\frac{\partial}{\partial u}X\right).$$

Note: even the orientation of $\Gamma$ doesn't matter.

Initially, $\Gamma_0$ is an embedded curve. The smoothly evolving curve cannot eventually cross itself at distinct points because if they would ever touch, their motion would tend to separate the points.



Suppose at some $t_0 > 0$ the evolving curve first touches at two points $u_1 \neq u_2$ but $X(t_0, u_1) = X(t_0, u_2)$. The touch from inside $\Gamma$ as if the enclosed region is pinched or from the outside as if the regions bend together.

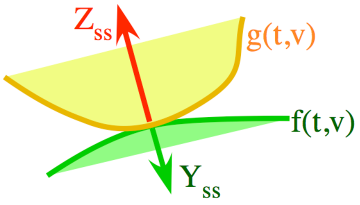



Figure: At Instant of Touching

Let us call $Y(t,u)$ the flow $X$ near $(t_0, u_1)$ and $Z(t,u)$ the flow $X$ near $(t_0, u_2)$. Represent both curves nonparametrivally as graphs over the same variable

$$Y(t,v) = (v, f(t,v)),$$
$$Z(t,v) = (v, g(t,v)).$$

Suppose that $f(t_0, v_0) = g(t_0, v_0)$ and $f(t_1, v) \le g(t_1, v)$ for $v$ near $v_0$ as would be the case at the instant of the first interior touch. It follows that

$$f_v(t_0, v_0) = g_v(t_0, v_0)$$
$$f_{vv}(t_0, v_0) \le g_{vv}(t_0, v_0)$$

It follows from (6) that the flow velocity

$$Y_{ss} = f_{vv} \frac{(-f_v, 1)}{(1 + f_v^2)^2}$$
$$Z_{ss} = g_{vv} \frac{(-g_v, 1)}{(1 + g_v^2)^2}$$

Since the vectors are equal at $(t_0, v_0)$ since $f_v(t_0, v_0) = g_v(t_0, v_0)$.

Because $f_{vv}(t_0, v_0) \le g_{vv}(t_0, v_0)$, the upper curve is moving faster upward than the lower curve: the curves tend to move apart!

This idea can be turned into a proof.

The same argument says that if one curve starts inside another, then they never touch as they flow. Thus the inside blob must extinguish before the shrinking disk does!

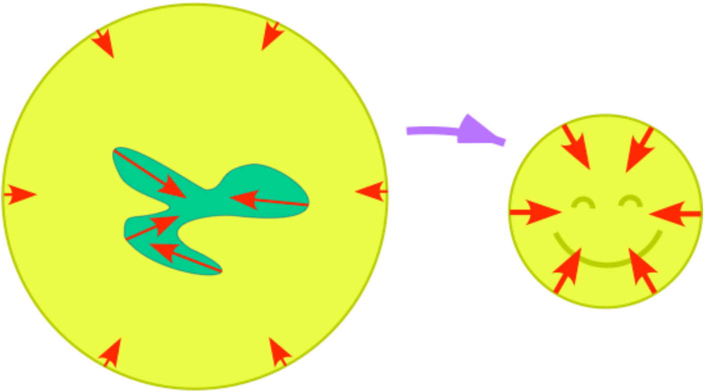


Figure: Flowing inside curve vanishes before the outside curve vanishes.

Also, closed curves inside the Grim Reaper die before it sweeps by!

Let us assume that $\Gamma_t$ are embedded closed curves on the interval $t \in I = [0, T)$. Then the first variation formula (4), (5) tells us that the area and length shrink.

Under curvature flow, the normal velocity is $v = \kappa$. Thus at each instant $t \in I$ we have

$$\frac{dL}{dt} = -\int_{\Gamma_t} \kappa^2 \, ds,$$
$$\frac{dA}{dt} = -\int_{\Gamma_t} \kappa \, ds = -2\pi.$$

The latter integral is the total turning angle ($= 2\pi$) of an embedded closed curve. It follows that

$$\mathrm{A}(\Gamma_t) = A_0 - 2\pi t$$

where $A_0 = \mathrm{A}(\Gamma_0)$ is the area enclosed by the starting curve.

The flow can only exist up to vanishing time $T = \dfrac{A_0}{2\pi}$.

Let $K$ be the region bounded by $\gamma$. The radius of the smallest circular disk containing $K$ is called the circumradius, denoted $R_{\text{out}}$. The radius of the largest circular disk contained in $K$ is the inradius.

$$R_{\text{in}} = \sup\{r : \text{there is } p \in \mathbf{E}^2 \text{ such that } B_r(p) \subseteq K\}$$

$$R_{\text{out}} = \inf\{r : \text{there exists } p \in \mathbf{E}^2 \text{ such that } K \subseteq B_r(p)\}$$

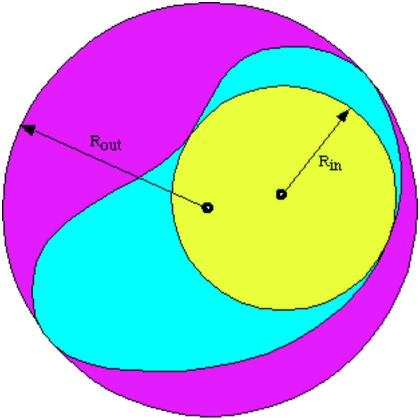


Figure: T. Bonnesen 1873–1935

**Theorem (Bonnesen's Inequality [1921])**

*Let $\Omega$ be a convex plane domain whose boundary has length $L$ and whose area is $A$. Let $R_{in}$ and $R_{out}$ denote the inradius and circumradius of the region $\Omega$. Then*

$$rL \geq A + \pi r^2 \tag{9}$$

*for all $R_{in} \leq r \leq R_{out}$.*

It suffices to show (9) for polygons $\mathcal{P}_n$ and for $R_{\text{in}} < r < R_{\text{out}}$ and then pass (9) to the limit as $\mathcal{P}_n \to \Omega$. Let $B_r(x, y)$ be the closed disk of radius $r$ and center $(x, y)$. Let $E_r$ denote the set of centers whose balls touch $\mathcal{P}_n$.

$$E_r = \left\{(x, y) \in \mathbb{R}^2 : B_r(x, y) \cap \mathcal{P}_n \neq \emptyset\right\}$$

Let $n(x, y)$ denote the number of points in $\partial B_r(x, y) \cap \partial \mathcal{P}_n$. The theorem follows by computing

$$I = \int_{E_r} n(x, y)\, dx\, dy$$

in two ways.

Since $B_r(x, y)$ can neither contain, nor be contained by $\mathcal{P}_n$, except on a set of measure zero, $n$ is finite and

$$n(x, y) \geq 2 \qquad \text{for almost all } (x, y) \in E_r.$$

Hence $E_r$ consists of all points in the interior of $\mathcal{P}_n$ as well as all points within a distance $r$.

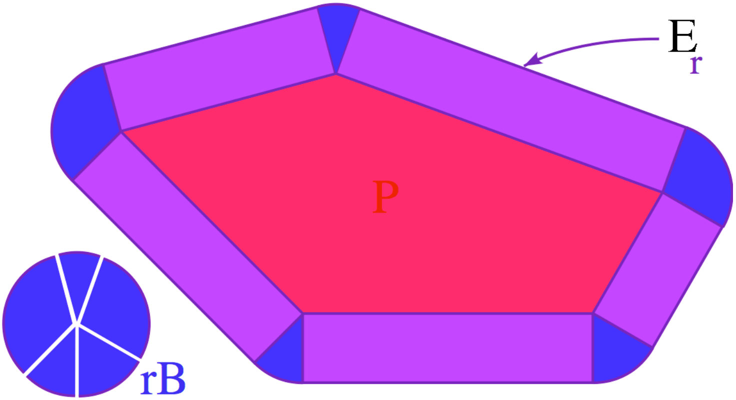
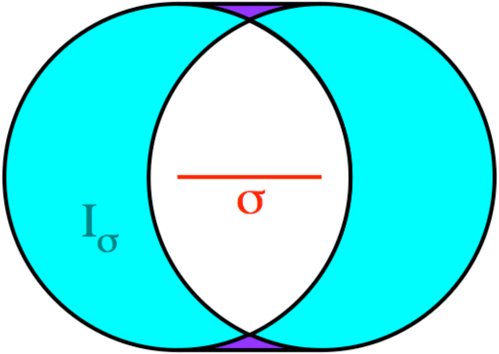



Figure: $E_r$ of polygon has area $\mathrm{A}(P_n) + \mathrm{L}(\partial P_n)r + \pi r^2$.

Thus

$$2A + 2Lr + 2\pi r^2 \le I.$$

It follows from $R_{\text{in}} < r < R_{\text{out}}$ that $B_r$ touches $\mathcal{P}_n$ if and only if it touches $\partial\mathcal{P}_n$. Let $\sigma$ be one of the boundary line segments. Let $n_\sigma(x, y)$ be the number of points in $\sigma \cap \partial B_r(x, y)$. The set $I_\sigma$ of centers of circles touching $\sigma$ is the union of circles with centers on $\sigma$.




$$\int_{\sigma\cap\partial B_r(x,y)\neq\emptyset} n_\sigma \, dx \, dy = 4r \, \mathrm{L}(\sigma).$$

Hence

$$I = \int_{E_r} n \, dx \, dy = \sum_{\sigma} \int_{\sigma \cap B_r(x,y) \neq \emptyset} n_\sigma \, dx \, dy = \sum_{\sigma} 4r \, \mathrm{L}(\sigma) = 4r \, \mathrm{L}(\partial \mathcal{P}_n).$$

Putting both computations together

$$2A + 2Lr + 2\pi r^2 \leq I \leq 4rL$$

or (9),

$$A + \pi r^2 \leq rL. \qquad \square$$

The distance of the tangent line to the origin is called the support function

$$p = -X \cdot N = -X \cdot \frac{\mathcal{R} X_u}{|X_u|}.$$

Hence, the area is

$$A = -\frac{1}{2} \int_\Gamma X \cdot \mathcal{R} X_u \, du = \frac{1}{2} \int_\Gamma p \, ds$$

The length is

$$L = \int_\Gamma T \cdot T \, ds = \int_\Gamma T \cdot X_s \, ds = -\int_\Gamma T_s \cdot X \, ds = -\int_\Gamma \kappa N \cdot X \, ds = \int_\Gamma \kappa p \, ds$$

Figure: M. Gage

**Theorem (Gage's Inequality [1983])**

*Let $\Omega$ be a $\mathcal{C}^2$ convex plane domain whose boundary $\Gamma$ has length $L$ and whose area is $A$. Then*

$$\frac{\pi L}{A} \le \int_\Gamma \kappa^2 \, ds. \tag{10}$$

*Proof.* By Schwarz's Inequality,

$$L^2 = \left( \int_\Gamma \kappa p \, ds \right)^2 \le \int_\Gamma \kappa^2 ds \int_\Gamma p^2 \, ds$$

(10) follows if $\Omega$ can be moved so

$$\int_\Gamma p^2 \, ds \le \frac{AL}{\pi}. \tag{11}$$

*Proof.* First we assume $\Omega$ is symmetric about the origin. Then

$$R_{\text{in}} \le p \le R_{\text{out}}.$$

Integrate Bonnesen's Inequality

$$L \int_\Gamma p\, ds \ge A \int_\Gamma ds + \pi \int_\Gamma p^2\, ds$$
$$2AL \ge AL + \pi \int_\Gamma p^2\, ds$$

which is (11) for symmetric domains.
Now assume $\Omega$ is any convex domain. We claim that $\Omega$ can be bisected by a line that divides the area into equal parts and cuts both boundary points at parallel tangents.

The claim depends on the Intermediate Value Theorem. Let $g(s)$ be the continuous function such that $s < g(s) < s + L$ which gives the place along $\Gamma$ such that the line through the points $X(s)$ and $X(g(s))$ bisects the area. Hence $g(g(s)) = s + L$. Consider the continuous function

$$h(s) = \left[T(s) \times T(g(s))\right] \cdot Z$$

where $Z$ is the positively oriented normal to the plane. $h(s) = 0$ iff $T(s) = -T(g(s))$.
Observe $h(0) = -h(g(0))$. If $h(0) = 0$ then $s_0 = 0$ determines the line. Otherwise, by IVT, there is an $s_0 \in (0, g(0))$ where $h(s_0) = 0$ and $s_0$ determines the line.

Let $L$ be the line segment from $X(s_0)$ to $X(g(s_0))$. Move the curve so the midpoint of $L$ is the origin.

Let $\gamma_1$ and $\gamma_2$ be the sides of $\Gamma$ split by $L$.

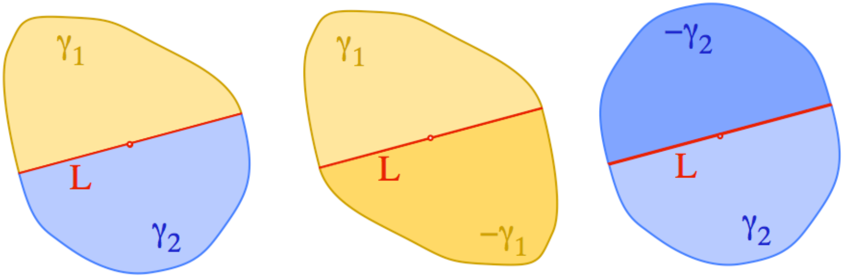



Figure: Cut and Reglue along Line $L$ that Halves the Area and Touches the Curve at Parallel Tangents.

Erasing $\gamma_2$ for the moment, reflect $\gamma_1$ through the origin to form a closed, convex curve, that is symmetric with respect to the origin. Note that the tangents at the endpoints of $\gamma_1$ must be parallel for $\gamma_1 \cup (-\gamma_1)$ to be convex. We apply (11)

$$2 \int_{\gamma_1} p^2 \, ds \leq \frac{2AL_1}{\pi}.$$

where $2L_1$ is the length of $\gamma_1 \cup (-\gamma_1)$. Similarly with $\gamma_2 \cup (-\gamma_2)$,

$$2 \int_{\gamma_2} p^2 \, ds \leq \frac{2AL_2}{\pi}.$$

Adding these two inequalities yields

$$2 \int_{\Gamma} p^2 \, ds \leq \frac{2AL_1 + 2LA_2}{\pi} = \frac{2AL}{\pi}$$

which is (11) for general convex domains. $\square$

For any closed curve, the isoperimetric inequality says that the boundary cannot be shorter than that of a circle with the same area

$$L^2 \geq 4\pi A.$$

**Theorem (Gage [1983])**

*Let $\Gamma_t$ be a $\mathcal{C}^2$ family of curves flowing by curvature $X_t = X_{ss}$ for $0 \leq t < T$ starting from a convex $\Gamma_0$. Then*

1. *The isoperimetric ratio decreases:* $\frac{d}{dt}\left(\frac{L^2}{4\pi A}\right) \leq 0.$
2. *If the flow survives until $T = \frac{1}{2\pi}A_0$ (it does by the Gage and Hamilton Theorem) and $A \to 0$ as $t \to T$, then*

$$\frac{L^2}{4\pi A} \to 1 \qquad \text{as } t \to T.$$

*Hence, the curves become circular in the sense that*

$$\frac{(R_{out}(t) - R_{in}(t))^2}{4R_{out}^2} \leq \frac{L^2}{4\pi A} - 1 \to 0$$

(1) is a computation with the first variation formulas and Gage's Inequality

$$\frac{d}{dt}\left(\frac{L^2}{4\pi A}\right) = \frac{2ALL' - L^2A'}{4\pi A^2}$$
$$= \frac{L}{2\pi A}\left(-\int_{\Gamma_t} \kappa^2\, ds + \frac{\pi L}{A}\right)$$
$$\leq 0.$$

(2) is similar, using a stronger form of Gage's Inequlity.

(3) follows from a the Strong Isoperimetric Inequality of Bonnesen and $A \leq \pi R_{\text{out}}^2$

$$\frac{\pi^2 \left(R_{\text{out}}(t) - R_{\text{in}}(t)\right)^2}{4\pi^2 R_{\text{out}}^2} \leq \frac{L^2 - 4\pi A}{4\pi A} = \frac{L^2}{4\pi A} - 1.$$

**Theorem (Strong Isoperimetric Inequality of Bonnesen)**

*Let $\Omega$ be a convex planar domain with boundary length $L$ and area $A$. Let $R_{in}$ and $R_{out}$ denote the inradius and circumradius of the $\Omega$. Then*

$$L^2 - 4\pi A \ge \pi^2 (R_{out} - R_{in})^2. \tag{12}$$

*Proof.* Consider the quadratic function $f(s) = \pi s^2 - Ls + A$. By Bonnesen's inequality, $f(s) \le 0$ for all $R_{\text{in}} \le s \le R_{\text{out}}$. Hence these numbers are located between the zeros of $f(s)$, namely

$$R_{\text{out}} \le \frac{L + \sqrt{L^2 - 4\pi A}}{2\pi}$$

$$\frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} \le R_{\text{in}}.$$

Subtracting these inequalities gives

$$R_{\text{out}} - R_{\text{in}} \le \frac{\sqrt{L^2 - 4\pi A}}{\pi},$$

which is (12). □

**Theorem (M. Gage & R. Hamilton [1986], M. Grayson [1987])**

*Suppose that $\Gamma_0 \in \mathcal{C}^2$ is an embedded curve in the plane with bounded curvature and encloses area $A_0$. Then there is a unique smooth family of embedded curves independent of parametrization $\Gamma_t$ that satisfies $X_t = X_{ss}$ on $[0, T_\infty)$, where $T_\infty = \frac{1}{2\pi} A_0$. The solution has the following properties:*

1. *If $\Gamma_0$ is nonconvex, there is a time $t_1 \in (0, T_\infty)$ where $\Gamma_{t_1}$ is convex.*
2. *If $\Gamma_{t_1}$ is ever convex but possibly with straight line segments, then $\Gamma_t$ will be strictly convex for $t > t_1$.*
3. *The flow shrinks to a round point in the sense that*
   - $\dfrac{R_{in}}{R_{out}} \to 1$ *as* $t \to T_\infty$;
   - $\dfrac{\min_{\Gamma_t} \kappa}{\max_{\Gamma_t} \kappa} \to 1$ *as* $t \to T_\infty$;
   - *Curvature stays bounded on compact intervals* $[0, T_\infty - \epsilon]$;
   - $\max_{\Gamma_t} |\partial_\theta^\alpha \kappa| \to 0$ *as* $t \to T_\infty$ *for all* $\alpha \geq 1$, *where* $T = (\cos\theta, \sin\theta)$.

**Theorem**

*Let $\Gamma_t$ be a $\mathcal{C}^2$ family of curves flowing by curvature $X_t = X_{ss}$ for $t \in [0, T)$. Then*

1. *If $g(t, u) \in \mathcal{C}^2$ is any function, then $g_{st} = g_{ts} + \kappa^2 g_s$;*
2. *$T_t = \kappa_s N$ and $N_t = -\kappa_s T$;*
3. *$\kappa_t = \kappa_{ss} + \kappa^3$.*

Compute (1) using $g_{tu} = g_{ut}$ and $X_{tu} = X_{ut}$ (but $g_{st} \neq g_{ts}$):

$$
\begin{aligned}
g_{st} = \left(\frac{g_u}{|X_u|}\right)_t &= \frac{g_{tu}}{|X_u|} - \frac{X_u \cdot X_{tu}}{|X_u|^3} g_u \\
&= g_{ts} - T \cdot (X_t)_s \, g_s \\
&= g_{ts} - T \cdot (\kappa N)_s \, g_s \\
&= g_{ts} - T \cdot (\kappa_s N - \kappa^2 T) \, g_s \\
&= g_{ts} + \kappa^2 g_s .
\end{aligned}
$$

To see (2a), use (1) and $X_t = \kappa N$,

$$T_t = X_{st} = X_{ts} + \kappa^2 X_s = (\kappa N)_s + \kappa^2 T = \kappa_s N - \kappa^2 T + \kappa^2 T = \kappa_s N.$$

To see (2b), as the rotation $\mathcal{R}$ doesn't depend on $t$,

$$N_t = (\mathcal{R}T)_t = \mathcal{R}(T_t) = \mathcal{R}(\kappa_s N) = -\kappa_s T.$$

Finally, to see (3), differentiate the defining equation for curvature $T_s = \kappa N$, using (2b)

$$T_{st} = (\kappa N)_t = \kappa_t N + \kappa N_t = \kappa_t N - \kappa \kappa_s T$$

Then using (1) and (2a)

$$T_{st} = T_{ts} + \kappa^2 T_s = (\kappa_s N)_s + \kappa^3 N = \kappa_{ss} N - \kappa \kappa_s T + \kappa^3 N.$$

Equating gives (3)

$$\kappa_t = \kappa_{ss} + \kappa^3.$$

**Corollary**

*Let $\Gamma_t$ be a $\mathcal{C}^2$ family of curves flowing by curvature $X_t = X_{ss}$ for $t \in [0, T)$. Suppose that $\Gamma_0$ is convex. Then $\Gamma_t$ is convex for all $t \in [0, T)$.*

*Proof idea.* The curve is convex iff it's curvature is nonnegative. Hence $\kappa(0, s) \geq 0$ for all $s$. Apply the maximum principle to $\kappa$ which satisfies

$$\kappa_t = \kappa_{ss} + \kappa^3.$$

If $t_0 \in [0, T)$ is the first time $\kappa$ is zero, say at $s_0$, then we have

$$\kappa(t_0, s_0) = 0 \qquad \text{and} \qquad \kappa(t_0, s) \geq 0 \qquad \text{for all } s.$$

Thus $\kappa_{ss}(t_0, s_0) \geq 0$ and

$$\kappa_t(t_0, s_0) = \kappa_{ss}(t_0, s_0) + \kappa^3(t_0, s_0) \geq 0 + 0$$

so the curvature is increasing and cannot dip below the $x$-axis. $\square$

**Theorem (G. Huisken, [1984].)**

*Suppose that $M_0^n \subset \mathcal{R}^{n+1}$ is a $\mathcal{C}^2$ embedded closed convex hypersurface in Euclidean Space with bounded curvature. Then there is a unique smooth family of embedded hypersurfaces, independent of parametrization, $M_t$, that satisfy $X_t = \Delta X = nHN$ on a maximal interval $[0, T_\infty)$. The solution has the following properties:*

1. *$M_t$ will be strictly convex for $t > 0$.*
2. *The flow shrinks to a round point in the sense that*
   - $\dfrac{R_{in}}{R_{out}} \to 1$ *as* $t \to T_\infty$;
   - $\dfrac{\min_{\Gamma_t} \kappa_1}{\max_{\Gamma_t} \kappa_n} \to 1$ *as* $t \to T_\infty$, *where the* $\kappa_1 \le \cdots \le \kappa_n$ *are principal curvatures of* $M_t$ *(eigenvalues of the second fundamental form* $h_{ij}$*.);*
   - *The principal curvatures stay bounded on compact intervals* $[0, T_\infty - \epsilon]$;
   - $\max_{\Gamma_t} |\partial_\theta^\alpha h_{ij}| \to 0$ *as* $t \to T_\infty$ *for all* $\alpha \ge 1$, *where* $\theta = N \in \mathbb{S}^n$.

**Theorem (R. Hamilton, [1982].)**

*Suppose that* $(M_0^n, g_0)$ *be a smooth compact Riemannian Manifold and* $g_0$ *a metric with bounded sectional curvature. Then there is a unique smooth family of metrics* $g_t$*, that satisfy*

$$g_t = -2\,Ric(g)$$

*on a maximal interval* $[0, T_\infty)$*. The solution has the following properties:*

1. *If* $(M^3, g_0)$ *has positive sectional curvature, then it will have positive curvature for* $t > 0$*.*
2. *The flow shrinks to a manifold of constant positive sectional curvature in the sense that*
   - $\dfrac{\min_{\Gamma_t} Sect(g)}{\max_{\Gamma_t} Sect(g)} \to 1$ *as* $t \to T_\infty$*, where the* $Sect(g)$ *is the sectional curvature of* $g$*;*
   - *The sectional curvatures stay bounded on compact intervals* $[0, T_\infty - \epsilon]$*;*
   - $\max_{\Gamma_t} (T - t)\,|\nabla^\alpha Riem(g)| \to 0$ *as* $t \to T_\infty$ *for all* $\alpha \geq 1$*.*

# 66. Convex curves stay convex.

# 67. Convex curves stay convex.

Thanks!