Compatibility Conditions for Linear and Nonlinear Elastic Materials and their Discrete Approximations

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- Fault Tolerance
- Four models
  - (NC) Prescribed Nonlinear Strain
  - (ND) Prescribed Length
  - (LC) Prescribed Linearized Strain
  - (LD) Prescribed Elongations
    - Counting Formula
    - Wagon-Wheel Condition
    - Wagon-Wheel Implies Compatibility of (LC)
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This is a preliminary study of compatibility conditions on discrete structures. A structure or truss in $\mathbf{R}^d$ with $d = 2$ or $d = 3$ consists of a finite number of vertices (nodes) connected by straight edges ("bars," or "links") which form a connected graph. We often consider rigid substructures of the hexagonal grid in $\mathbf{R}^2$, *e.g.*,

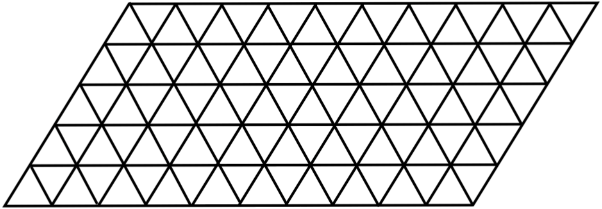


Figure: Truss.

If there is a surplus of edges in the truss, then it has fault tolerance or resilience to damage. Edges may be removed (damaged) without the truss losing rigidity.

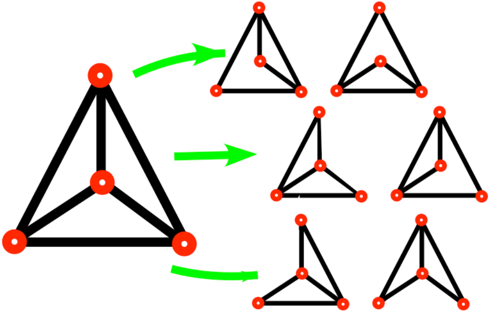


Figure: Over-rigid truss. Removing any edge leaves a rigid truss.

Suppose we specify lengths of edges and try to solve for positions of the vertices. If the truss has more edges than necessary to determine these positions, this over-specification results in fault tolerance, a resilience to damage. But it also requires compatibility conditions in length data. We explore how compatibility conditions as a measure of excess rigidity.

We consider four problems.
(NC) The Prescribed Nonlinear Strain,
(ND) its discrete approximation the Prescribed Length,
(LC) the linearization of (NC), the Prescribed Linearized Strain and
(LD) the linearization of (ND) Prescribed Elongations.

The continuum problems are overdetermined PDE's. The discretized problems are overdetermined equations. Each problem requires Compatibility Conditions on their data to be solvable. The compatibility conditions for (NC), (ND) and (LC) are fairly well understood. The compatibility conditions for (LD) are less well understood and are investigated in this work.

(NC) Prescribed Nonlinear Strain

7. (NC) Prescribed Green Deformation (Nonlinear Strain).

Let $\mathcal{B} \subset \mathbb{E}^2$ be a Euclidean material disk domain with with piecewise smooth boundary and coordinates (X^1, X^2), and $\mathcal{S} \subset \mathbb{E}^2$ the target domain with coordinates (x^1, x^2). A configuration is an in-plane displacement

$$\phi : \mathcal{B} \to \mathcal{S}.$$

Its material (Lagrangian) strain tensor is

$$E[\phi] = \tfrac{1}{2}\left(F_A \bullet F_B - \delta_{ij}\right) = \tfrac{1}{2}\left(\zeta - I\right), \qquad \text{where } {F^a}_A = \frac{\partial \phi^a}{\partial X^A}$$

where ζ is the Green Deformation tensor (Right Cauchy-Green Deformation Tensor).

The equation for configurations ϕ with prescribed Green tensor

$$\text{(NC)} \qquad \zeta = F_A \bullet F_B$$

is just the equation for a mapping to Euclidean space with prescribed pull-back metric. The compatibility condition for (NC) to be soluble is that the prescribed metric ζ have vanishing Riemannian curvature.

Associated to a configuration is the energy of deformation, whose positive definite energy density depends on the nonlinear strain

$$\text{Energy} = \int_{\mathcal{B}} W(E[\phi](x))\, dx.$$

The study of variational problems for energy minimizing configurations is a major theme of nonlinear elasticity.

- Minimizing energy over all configurations ϕ with appropriate boundary conditions results in elliptic systems.
- Equivalently, we may minimize the energy over all elasticities E satisfying the compatibility conditions.

But our focus in this study are the prescribed strain equations and their discretizations.

For surfaces, the compatibility condition is the vanishing of Riemannian Curvature or ${R_1}^2{}_{12} = K$, the Gauss Curvature. Putting $D^2 = \zeta_{11}\zeta_{22} - \zeta_{12}^2$,

$$\begin{aligned} K = {} & \frac{1}{2D^2}\left(-\zeta_{22,11} + 2\zeta_{12,12} - \zeta_{11,22}\right) \\ & + \frac{\zeta_{22}}{4D^4}\left(\zeta_{11,1}\zeta_{22,1} - 2\zeta_{11,1}\zeta_{12,2} + \zeta_{11,2}^2\right) \\ & + \frac{\zeta_{12}}{4D^4}\left(-2\zeta_{12,1}\zeta_{22,1} - 2\zeta_{11,2}\zeta_{12,2} + 4\zeta_{12,1}\zeta_{12,2} - \zeta_{11,2}\zeta_{22,1} + \zeta_{11,1}\zeta_{22,2}\right) \\ & + \frac{\zeta_{11}}{4D^4}\left(\zeta_{11,2}\zeta_{22,2} - 2\zeta_{12,1}\zeta_{22,2} + \zeta_{22,1}^2\right) \end{aligned}$$

$K = 0$ is the integrability condition for solvability of the differential system

$$\text{(NC)} \qquad \zeta = F_A \bullet F_B.$$

(ND) Prescribed Lengths

For a PL triangulation of $\mathcal{B}$, let V_i be its vertices, E_{ij} its edges and T_{ijk} its triangular faces. Its 1-skeleton is a truss approximating the material $\mathcal{B}$. Let the positive numbers L_{ij} be given for each edge such that L_{ij} satisfy the triangle inequalities.

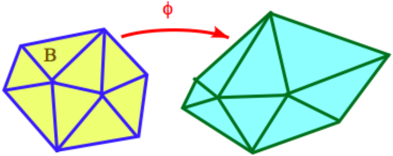


Figure: Polyhedral Embedding.

We seek an embedding of $\mathcal{B}$ with vertices $X_i = \phi(V_i)$ such that realizes the prescribed lengths of edges

$$\text{(ND)} \qquad |X_i - X_j| = L_{ij} \qquad \text{for all edges } ij$$

It is not clear how this discretization is an approximation of (NC). It is not the result of a finite difference scheme nor discrete differential forms.

In what sense does the discretization (ND) approximate (NC)?

The approximation may be understood in A. D. Alexandrov's theory of surfaces with bounded curvature. These are metric spaces $(\mathcal{B}, \rho(x, y))$ where notions of length of curves, geodesics, angle, area and integral curvature make sense, and where the curvature measure is assumed to be bounded above by area.

Examples of Alexandrov spaces include smooth Riemannian surfaces with distance $\rho(x, y)$ induced from the metric and polyhedral surfaces which are piecewise Euclidean surfaces with polygon faces. Alexandrov spaces are in a sense completions of Riemannian surfaces or polyhedral surfaces.

Let $\rho(x, y)$ be the distance between $x, y \in \mathcal{B}$ induced from the metric ζ. For a triangulation $\mathcal{T}$ of $\mathcal{B}$, let $L_{ij} = \rho(V_i, V_j)$ be the induced distance between vertices. Since the distance satisfies the triangle inequality on triangles, the triangles may be realized as Euclidean triangles, and a metric may be defined on each $T \in \mathcal{T}$ by pulling back the Euclidean metric using the affine parameterization. The resulting metric $\zeta_{\mathcal{T}}$ and induced distance $\rho_{\mathcal{T}}$ makes $(\mathcal{B}, \rho_{\mathcal{T}})$ a polyhedral surface.

Theorem (A. D. Alexandrov 1962)

Let $(\mathcal{B}, \rho)$ be an Alexandrov Space with bounded curvature. Let $\mathcal{T}_i$ be a sequence of PL triangulations such that the ρ-diameter of triangles tend uniformly to zero as $i \to \infty$. Let $\rho_{\mathcal{T}_i}$ be the corresponding polyhedral distance. Then $\rho_{\mathcal{T}_i} \to \rho$ uniformly on $\mathcal{B} \times \mathcal{B}$. Moreover, the integral curvature measures $\omega_{\mathcal{T}_i}$

14. Compatibility Condition for the Prescribed Lengths Problem (ND)

The realization problem may be expressed geometrically as follows: find an isometric immersion

$$\phi : (\mathcal{B}, \zeta_{\mathcal{T}}) \to (\mathbb{E}^2, ds^2_{\mathbb{E}^2}).$$

For such immersion to exist, then $\zeta_{\mathcal{T}}$ has to be flat at interior vertices. Suppose V_0 is an interior vertex and $V_1, \ldots, V_n$ are adjacent vertices going around V_0. The angles of adjacent edges may be computed using the cosine law

$$\alpha_i = \cos^{-1}\left(\frac{L^2_{i,i+1} - L^2_{0,i} - L^2_{0,i+1}}{2L_{0,i}L_{0,i+1}}\right)$$

The curvature at the vertex is the angle excess

$$K(V_0) = 2\pi - \textstyle\sum_{i=1}^n \alpha_i$$

The compatibility condition for (ND) is that $K(V_i) = 0$ for all interior verices.

15. Solvability of the Prescribed Length Problem (ND)

By a theorem of Alexandrov, if $K(V_i) = 0$ at all interior vertices, then there exists an isometric immersion $\phi_{\mathcal{T}} : (\mathcal{B}, \zeta_{\mathcal{T}}) \to \mathbb{E}^2$ so that $\zeta_{\mathcal{T}} = \phi_{\mathcal{T}}^*(ds^2_{\mathbb{E}^2})$. (One pastes together triangles in turn and checks that there is a full Euclidean neighborhood surrounding every vertex.)

If a flat prescribed metric ζ is approximated by a polyhedral metric $\zeta_{\mathcal{T}}$, then the curvature at vertices necessarily vanishes, so ζ and $\zeta_{\mathcal{T}}$ are isometric, although the parameterizations may differ slightly.

Theorem

Let $\mathcal{B}$ be a bounded open topological disk in $\mathbb{E}^2$ with polygonal boundary and ζ be a prescribed $\mathcal{C}^2$ positive definite matrix function defined in a neighborhood of $\overline{\mathcal{B}}$ with induced distance ρ. There is a sequence of PL triangulations, $\mathcal{T}_k$ such that the largest ζ-diameter of the triangles of $\mathcal{T}_k$ tends to zero. Let ϕ_k be an isometric immersion of $(\mathcal{B}, \zeta_k)$. Then for $\mathcal{T}_k$ there are rigid motions m_k such that $m_k \circ \phi_k \to \phi : \mathcal{B} \to \mathbb{E}^2$ converges uniformly to a map such that $\rho(x, y) = |\phi(x) - \phi(y)|$ for all x, y. Moreover, $\phi \in \mathcal{C}^1$ and satisfies (NC).

Let $X_i \in \mathbf{R}^d$, $i = 1, \ldots, n$ denote position of the vertex. Let $\{i, j\} \in E$ be pairs of distinct indices connected by a straight edge. $e = \#E$. Suppose that the length L_{ij} is prescribed. Then for each edge $\{i, j\} \in E$, we get an equation, yielding a system of e equations in dn unknowns

$$\text{(ND)} \qquad |X_i - X_j|^2 = (X_i - X_j) \bullet (X_i - X_j) = L_{ij}^2$$

Because rigid motions preserve lengths, rigid motion of a solution is also a solution.

If there is only one solution of (16) up to rigid motion, we say that the truss is rigid. A smooth one-parameter family of solutions is called a flex.

There may be several noncongruent configurations that solve (ND), however they may not allow nontrivial flexes.

(LC) Prescribed Linearized Strain

18. Linearized Strain (LC) and its Compatibility

Linearizing (NC) around $\phi = \mathrm{Id}$ yields the equation of prescribed linearized strain. If we consider a variation $\phi(t)$ for $t \in (-\epsilon, \epsilon)$ with $\phi(0) = \mathrm{Id}$ then an infinitesimal deformation $u : \mathcal{B} \to \mathbb{E}^2$ given by

$$u = \left. \frac{\partial \phi}{\partial t} \right|_{t=0}$$

satisfies equation of prescribed linear strain

$$\text{(LC)} \qquad \frac{1}{2}\left(\frac{\partial u^i}{\partial X_j} + \frac{\partial u^j}{\partial X_i} \right) = \epsilon_{ij}$$

where $\epsilon_{ij} = \epsilon_{ji}$ is the strain field. Were u to exist, since $u : \mathbf{R}^2 \to \mathbf{R}^2$, the strain field satisfies the *continuum compatibility condition* in $\mathcal{B}$,

$$\mathrm{Ink}(\epsilon) = \epsilon_{11,22} - 2\epsilon_{12,12} + \epsilon_{22,11} = 0$$

where $\epsilon_{ij,pq} = \frac{\partial^2 \epsilon_{ij}}{\partial x_p \, \partial x_q}$. Note that $\mathrm{Ink}(\epsilon)$ agrees with the "linear part" of Gauss curvature.

Mechanically, compatibility conditions follows from the requirement that neighboring deformations are consistent and don't overlap. Thus the compatibility condition is a property of a material point.

Of course, the deformation fields that are in kernel of the linear strain operator

$$\text{(LC)} \qquad \frac{1}{2}\left(\frac{\partial u^i}{\partial X_j} + \frac{\partial u^j}{\partial X_i}\right) = 0$$

are Killing Fields, velocity fields whose flow preserves the metric ζ to first order.

(LD) Prescribed Elongations

Now let vertex positions $X_i \in \mathbf{R}^d$ and lengths L_{ij} depend on time t. To deduce the linearized equations, let the structure be deformed from its $t = 0$ position. Differentiating with respect to time, at $t = 0$,

$$\text{(LD)} \qquad (X_i - X_j) \bullet (u_i - u_j) = \lambda_{ij}$$

for all $\{i, j\} \in E$. Here the unknown displacements and prescribed elongations are

$$u_i = \dot{X}_i(0), \qquad \lambda_{ij} = L_{ij}(0)\dot{L}_{ij}(0)$$

We denote the system (LD) with e equations, dn unknowns as

$$Au = \Lambda$$

$\ker A$ denotes the velocities of vertices which preserve the lengths of bars up to first order. $\ker A$ always contain the velocity fields of rigid motions which are r_d dimensional. For $d = 2$ this corresponds to velocities of translations and rotations, so $r_2 = 3$. In $d = 3$, translations and rotations are each 3 dimensional so $r_3 = 6$.

If $\ker A$ only contains velocity fields of rigid motions, then the truss is said to be infintesimally rigid. If $\ker A$ admits other vector fields, then we say the truss is infinitesimally flexible. The system

$$Au = \Lambda$$

has e equations and dn unknowns. To be solvable, the right side must satisfy $\mathcal{C}$ linear compatibility equations, which we denote $B\Lambda = 0$.

James Clerk Maxwell observed that if there are more unknowns than equations, then the truss could not possibly be rigid. There are at most $dn - r_d$ pivot variables and e equations. Thus there are at least

$$\mathcal{M} = e - dn + r_d$$

compatibility conditions. We call $\mathcal{M}$ the Maxwell's Compatibility Count If $\mathcal{M} < 0$ then $\dim \ker A > r_d$ and the structure is infinitesimally flexible. For degenerate systems, it may happen that the structure is infinitesimally flexible but have $\mathcal{M} \geq 0$.

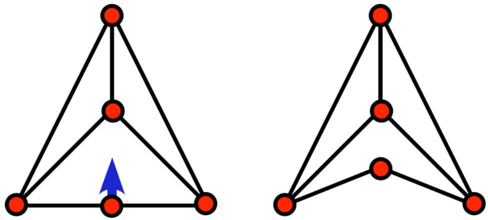


Figure: Both trusses have $e = 7$, $n = 5$ and $\mathcal{M} = 7 - 10 + 3 = 0$.

If the truss is infinitesimally rigid, then $e \geq dn - r_d = \operatorname{rank} A$. To solve

$$Au = \Lambda$$

for u, a general Λ will have to satisfy

$$\mathcal{C} = e - \operatorname{rank} A \geq e - dn + r_d = \mathcal{M}.$$

compatibility equations.

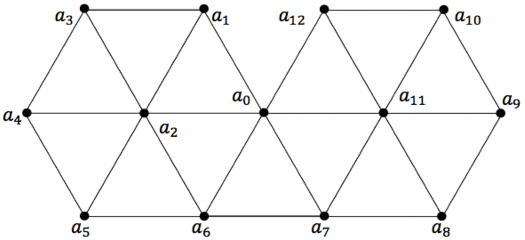
M. F. Thorpe and his collaborators have studied the underdetermined case of $Au = \Lambda$. They studied the onset of flexibility in random structures as network models for solidification of glass.

A. Cherkaev & L. Zhornitskaya and A. Cherkaev, V. Vinogradov & S. Leelavanichkul studied trusses made up of "waiting links" for damage wave propagation and impact protection.

A. Cherkaev, A. Kouznetsov and A. Panchenko looked at still states (no stress) in networks that allow two lengths for each edge in (ND). They also looked at travelling waves bistable lattices (Nonincreasing Hook's Law springs).

Theorem (Counting Formula for Triangulated Trusses)

Let B be a PL domain embedded in the plane with triangular faces and $g+1$ disjoint simple boundary curves. Then generically, the truss consisting of the one-skeleton $B^{(1)}$ has $\mathcal{M}(B^{(1)}) = 3g + v_i$, compatibility conditions where v_i is the number of interior vertices of X.



a_2 and a_{11} are the only interior nodes and $g = 0$ so $\mathcal{M} = 2$. Note that for this truss, one can remove as many as two edges, e.g. a_0a_1 and a_0a_{12}, and keep rigidity. But removing, e.g., the single edge a_6a_7 makes the figure flexible.

Proof. We shall suppose that the truss is a triangulated domain, embedded in the plane and bounded by $g+1$ pairwise disjoint simple closed curves. Let f be the number of triangular faces. The Euler Characteristic for a triangulated domain in the plane is given by the formula

$$\chi = 1 - g = f - e + v.$$

If v_b and v_i denote the number of interior and boundary nodes, and e_b and e_i the number of boundary and interior edges, we have for disjoint simple boundary curves

$$e = e_b + e_i, \qquad v = v_b + v_i, \qquad e_b = v_b, \qquad 3f = e_b + 2e_i \tag{1}$$

Substituting Euler's formula it follows that

$$3\chi = e_b - e_i + 3v_i.$$

Hence the Maxwell Dimension

$$\mathcal{M} = e - 2v + 3 = 3 + e_i - 2v_i - v_b = 3 - 3\chi + v_i = 3g + v_i \geq 0. \qquad \square$$

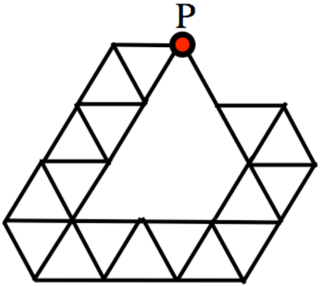


Figure: P is over-determined from two sides giving a compatibility equation.

Compatibility conditions occur in a sub-truss because there are more than two bars attached to a vertex whose elongations have to be consistent.

The number of compatibility conditions $\mathcal{C}$ corresponds to the number of dependent rows in A. Equations correspond to edges of the truss.

Theorem

The number of compatibility conditions $\mathcal{C}$ is the maximal number of edges that can be removed from the truss without losing infinitesimal rigidity.

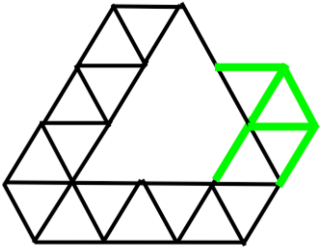


Figure: Removing one green edge will destroy infinitesimal rigidity.

However, not every subset of $\mathcal{C}$ edges can be removed. The truss in the figure has $\mathcal{C} = 1$ but the removal of any one of the green edges results in immediate loss of infinitesimal rigidity.

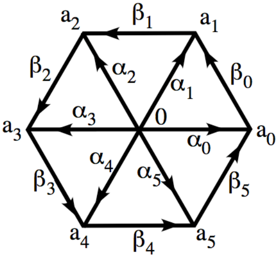


Figure: Smallest sub-truss supporting compatibility equation in hexagonal grid.

We can compute the compatibility condition for the hexagon in two ways. Formulate the equations

$$Au = \Lambda.$$

Gaussian Elimination yields a compatibility (solvability) equation on Λ.

The second method uses plane geometry. If $\alpha_i = |a_i|$ and $\beta_i = |a_{i+1} - a_i|$, where $i = 0, \ldots, 5$ taken mod 6, then by the cosine law the sum of the central angles must be

$$2\pi = \sum_{i=0}^{5} \cos^{-1}\left(\frac{\alpha_{i+1}^2 + \alpha_i^2 - \beta_i^2}{2\alpha_{i+1}\alpha_i}\right)$$

Differentiating

$$0 = -\sum_{i=0}^{5} \frac{\begin{Bmatrix} 2\alpha_{i+1}\alpha_i(2\alpha_{i+1}\dot{\alpha}_{i+1} + 2\alpha_i\dot{\alpha}_i - 2\beta_i\dot{\beta}_i) \\ -(\alpha_{i+1}^2 + \alpha_i^2 - \beta_i^2)(2\alpha_{i+1}\dot{\alpha}_i + 2\alpha_i\dot{\alpha}_{i+1}) \end{Bmatrix}}{4\alpha_{i+1}^2\alpha_i^2\sqrt{1 - \frac{\alpha_{i+1}^2 + \alpha_i^2 - \beta_i^2}{2\alpha_{i+1}\alpha_i}}}$$

For the regular unit hexagon, $\alpha_i = \beta_i = 1$. Hence

$$0 = -\frac{1}{2\sqrt{3}} \sum_{i=0}^{5} \left\{2(2\dot{\alpha}_{i+1} + 2\dot{\alpha}_i - 2\dot{\beta}_i) - (2\dot{\alpha}_i + 2\dot{\alpha}_{i+1})\right\}$$

which reduces to the Wagon Wheel Condition:

$$\mathcal{W} = \sum_{i=0}^{5} \dot{\alpha}_i - \sum_{i=0}^{5} \dot{\beta}_i = 0 \tag{2}$$

For affine hexagons, the compatibility equation is the wagon wheel condition weighted by the respective side lengths

$$0 = \sum_{i=0}^{5} \alpha_i \, \dot{\alpha}_i - \sum_{i=0}^{5} \beta_i \dot{\beta}_i$$

The expansion of both of these in δ has the continuum compatibility condition as the lowest order coefficient.

A general wagon-wheel condition holds for stars (unions of adjacent triangles) about interior vertices of any valence.

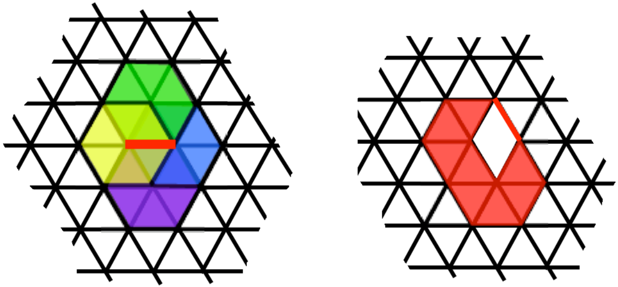


Figure: Compatibility cells around undamaged and damaged edge.

Four independent compatibility hexagons involve a given interior edge in an undamaged hexagonal grid. If damaged (edge removed from the grid) then compatibility regions surrounding the damaged edge are larger. More edges are involved to overdetermine the vertices near the damage so the material is weakened. We expect this to be the crux of a weak formulation, but at the moments we offer only numerical evidence.

33. Relate Compatibility of (LD) to Compatibility of Linearized Strain (LC)

Suppose $D \subset \mathbf{R}^d$ is a domain. Consider the problem determining an infinitesimal deformation $u : D \to \mathbf{R}^d$ by prescribing the strains

$$\text{(LC)} \qquad \frac{1}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right) = \epsilon_{ij}$$

where $\epsilon_{ij} = \epsilon_{ji}$ is a given symmetric strain field. Were such u to exist, because it is a map of Euclidean Spaces the strain field must necessarily satisfy the continuum compatibility condition in D,

$$\epsilon_{ij,pq} - \epsilon_{jp,qi} + \epsilon_{pq,ij} - \epsilon_{qi,jp} = 0$$

for all indices i, j, p, q where $\epsilon_{ij,pq} = \frac{\partial^2 \epsilon_{ij}}{\partial x_p \, \partial x_q}$.

This is the linearized equivalent of saying that the pulled back metric of a map between Euclidean Spaces must have vanishing Riemann curvature.

In $d = 2$ this boils down to one equation

$$\mathrm{Ink}(\epsilon) = \epsilon_{11,22} - 2\epsilon_{12,12} + \epsilon_{22,11} = 0.$$

The infinitesimal deformations equations of a hexagon, $Au = \Lambda$ is a discretization of the continuum equations for prescribed strain

$$(LC) \qquad \frac{1}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right) = \epsilon_{ij}$$

Its compatibility equation approximates continuum compatibility.

Theorem (Expansion of Compatibility Equation for Regular Hexagons)

Let $\mathcal{B}_{3r} \subset \mathbf{R}^2$ be a disk radius $3r$ about 0 and $\mathcal{H} \subset \overline{\mathcal{B}_{2r}}$ be a regular hexagon with side length $\delta \le r$ containing 0. Let $u \in \mathcal{C}^4(\mathcal{B}_{3r}, \mathbf{R}^2)$ be an infinitesimal deformation satisfying the strain equation (LC). The wagon-wheel condition (2) for the u-induced rates of change of distances between vertices of $\mathcal{H}$ has the Taylor expansion about the origin

$$\mathcal{W} = -\frac{3}{4}\left(\epsilon_{11,22} - 2\epsilon_{12,12} + \epsilon_{22,11}\right)\delta^2 + 0 \cdot \delta^3 + \mathrm{o}(\delta^3)$$

as $\delta \to 0$ uniformly in $\overline{\mathcal{B}_{2r}}$ depending on $\|u\|_{\mathcal{C}^4(\overline{\mathcal{B}_{2r}}, \mathbf{R}^2)}$. So if the discrete compatibility condition $\mathcal{W} = 0$ holds for all δ, then the continuum compatibility conditions $\mathrm{Ink}(\epsilon) = 0$ holds.

Proof of the Theorem depends on expressing the rate of change of distance in terms of strains.

Lemma

Let $\mathcal{B}_{3r} \subset \mathbf{R}^2$ be a disk radius $3r$ about the origin and $a_i, a_j \in \mathcal{B}_{3r}$. Let $u \in \mathcal{C}^4(\mathcal{B}_{3r}, \mathbf{R}^2)$ be an infinitesimal deformation with strains given by (34). If $\phi(x,t)$ is a deformation such that $\phi(x,0) = x$ and $\dot\phi(x,0) = u(x)$, then

$$\frac{d}{dt}\bigg|_{t=0} |\phi(a_i,t) - \phi(a_j,t)| = \frac{1}{|a_i - a_j|} \int_0^1 (a_i - a_j)^T \epsilon(\gamma(s))(a_i - a_j)\, ds$$

where $\gamma(s) = a_i + s(a_j - a_i)$ for $0 \le s \le 1$ is a parameterization of the line segment from a_i to a_j.

Proof.

$$
\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} |\phi(a_i,t) - \phi(a_j,t)| \\
&= \left.\frac{d}{dt}\right|_{t=0} \sqrt{(\phi(a_i,t) - \phi(a_j,t))^T(\phi(a_i,t) - \phi(a_j,t))} \\
&= \left.\frac{(\phi(a_i,t) - \phi(a_j,t))^T(\dot{\phi}(a_i,t) - \dot{\phi}(a_j,t))}{\sqrt{(\phi(a_i,t) - \phi(a_j,t))^T(\phi(a_i,t) - \phi(a_j,t))}}\right|_{t=0} \\
&= \frac{(a_i - a_j)^T(u(a_i) - u(a_j))}{\sqrt{(a_i - a_j)^T(a_i - a_j)}} \\
&= \frac{(a_i - a_j)^T(u(a_i) - u(a_j))}{|a_i - a_j|} \\
&= \frac{1}{|a_i - a_j|}\int_0^1 (a_i - a_j)^T \frac{d}{ds} u(\gamma(s))\, ds
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{|a_i - a_j|} \int_0^1 (a_i - a_j)^T \nabla u(\gamma(s))\, \dot\gamma(s)\, ds \\
&= \frac{1}{|a_i - a_j|} \int_0^1 (a_i - a_j)^T \nabla u(\gamma(s))\, (a_i - a_j)\, ds \\
&= \frac{1}{|a_i - a_j|} \int_0^1 (a_i - a_j)^T \epsilon(\gamma(s))\, (a_i - a_j)\, ds
\end{aligned}$$

where $v^T(\nabla u)\, v = \frac{1}{2} v^T(\nabla u + (\nabla u)^T)\, v = v^T \epsilon\, v$, proving the lemma.

To finish, the strains are expressed in Taylor Series about zero. The elongations of the edges of the hexagon are computed by integrating the Taylor Series in their expressions. The twelve elongations are put into the wagon wheel condition and coefficients are collected (using MAPLE!) $\square$

A truss is generic if the number of compatibility conditions equals the Maxwell Count. Equivalently, the truss is generic if and only if it is infinitesimally rigid.

For simplicity, we restrict our attention to subtrusses of the standard hexagonal lattice.

Theorem (Genericity of hexagonal trusses)

Let X be the union of finitely many 2-triangles of the hexagonal lattice. Suppose that the boundary ∂X consists of a $g + 1$ disjoint simple closed curves. Then the truss X is generic: the number of compatibility conditions equals the Maxwell number. Moreover, a basis for the compatibility conditions consits of one condition for each hexagon about an interior vertex and three for each ring-girder around every hole.

$$c = \mathcal{M} = 3g + v_i.$$

The interior vertices may be regarded as material points. The additional compatibility from each hole is a discrtization feature.

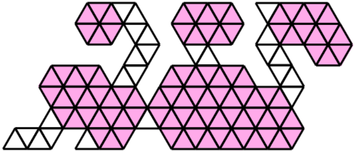


Figure: Decompose Simply Connected Truss into Plates and Girders.

Begin with simply connected trusses. Let $H(V_i)$ denote an open hexagon about an interior vertex. Decompose the union of hexagons about interior vertices into connected components, called plates

$$\coprod P_j = \bigcup_{\text{interior vertex } V_i} H(V_i)$$

The connected components of the remainder are called girders

$$\coprod G_i = X - \overline{\cup_j P_j}$$

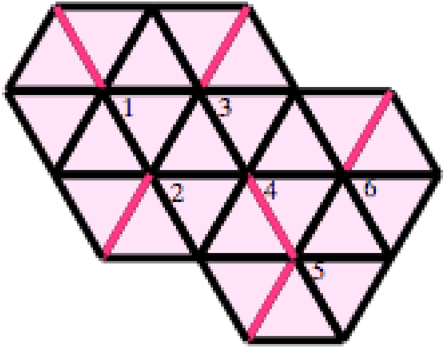


Figure: Order vertices and remove one edge per hexagon, maintaining rigidity.

Argue that each plate is generic. Order the vertices from one end to the other. Remove on edge of each hexagon in turn, maintaining rigidity as you build up the hexagons. Thus v_i edges may be removed.

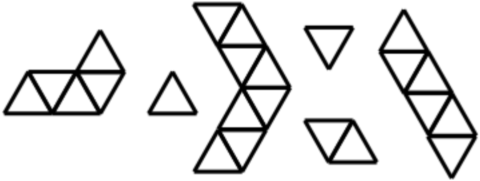


Figure: Girders are statically determined.

Argue that each girder is statically determined: it is rigid but without compatibility conditions. Removing any edge from a girder results in a flexible (hence infinitesimally flexible) structure.

Then argue that a simply connected truss made up of girders and plates is generic. Removing an edge from each hexagon in the plates results in a statically determined truss.

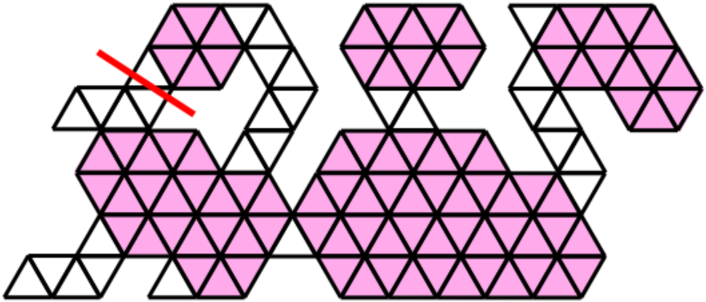


Figure: Taking out a "branch cut" reduces the Maxwell count by three.

For multiply connected domains, argue by induction on the number of holes. Removing a "branch cut" reduces the number compatibility conditions by the number of interior vertices along the cut plus three.

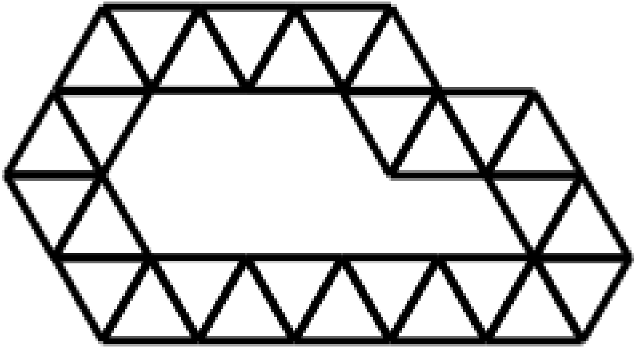


Figure: About each hole is a "ring-girder" which contributes three compatibility conditions.

How much do holes weaken a material? Assume that the material is periodic. Lets compute the large-scale average compatibility condition density for damaged material relative to the undamaged material.

For simplicity, let the basic cell Υ by a $k \times k$ union of hexagons centered on $ae_1 + be_2$ where $a, b = 1, \dots, k$ and $e_1 = (1, 0)$ and $e_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.
Suppose there are h holes per cell and m interior vertices taken by each hole. Assuming that cells are bounded by $h + 1$ pairwise disjoint simple closed curves, let Ω_n be the union consisting of $n \times n$ cells slightly overlapping, centered on $ake_1 + bke_2$ where $a, b = 1, \dots, n$.
The asymptotic compatibility number is

$$AC = \lim_{n \to \infty} \frac{c(\Omega_n)}{\mathrm{A}(\Omega_n)}.$$

The total number of holes is $g = n^2 h$. The total number of interior vertices is

$$v_i = k^2 n^2 - hmn^2$$

The area is base times height minus corner triangles, thus

$$AC = \lim_{n\to\infty} \frac{v_1 + 3g}{\mathrm{A}(\Omega_n)} = \lim_{n\to\infty} \frac{[k^2 n^2 - hmn^2] + 3n^2 h}{nk(nk+1)\frac{\sqrt{3}}{2}} = \frac{k^2 - h(m-3)}{\frac{\sqrt{3}}{2} k^2}.$$

Note that removing a single edge reduces the number of interior vertices by four, but introduces a ring girder which supports three compatibility conditions. Thus $m - 3 \geq 1$ compatibility conditions are lost for each hole.

46. Asymptotic Compatibility Density (LD). - -

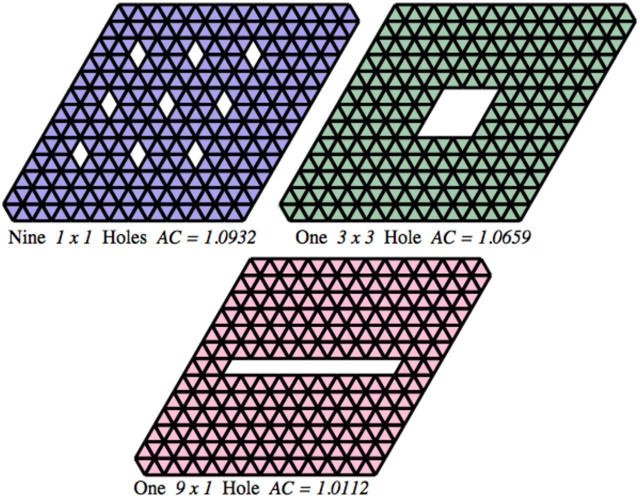


Figure: 13×13 Period Cell with Holes of Area 18 Triangles ($k = 13$, $p = 4$).

$$AC = \frac{k^2 - h(m-3)}{\frac{\sqrt{3}}{2}k^2}, \qquad AC_{\text{many 1-link holes}} = \frac{k^2 - p^2 + 2p - 1}{\frac{\sqrt{3}}{2}k^2}.$$

The asymptotic compatibility depends not just on the total area removed from the cell. Taking out more holes of the same total area has larger AC, a proxy for material resilience.

For example if one link is removed, $m = 4$ and triangle has area 2 triangles. Removing $h = (p-1)^2$ one-link removes $m = 4$ interior vertices per hole and has the same area $2(p-1)^2$ triangles as the $(p-1) \times (p-1)$ rhombus, which removes $h = 1$ hole and $m = p^2$ interior vertices. If the hole is a $(p-1)^2 \times 1$ trapezoid, it also has the same number of triangles, $h = 1$ but removes $m = 2p^2 - 4p + 2$ interior vertices.

$$AC_{\text{rhombus}} = \frac{k^2 - p^2 + 3}{\frac{\sqrt{3}}{2}k^2}, \qquad AC_{\text{trapezoid}} = \frac{k^2 - 2p^2 + 4p - 5}{\frac{\sqrt{3}}{2}k^2}.$$

For simplicity, all edges of the truss have unit length. The infinitesimal deformations u are related to the elongations of the edges via $Au = \Lambda$ where A is a $e \times dn$ matrix of rank r. Let us denote the compatibility conditions $B\Lambda = 0$ where B is an $(e - r) \times e$ matrix. Hooke's Law says the forces $C\Lambda$ along the edges are proportional to the elongations where $C = \operatorname{diag}(c_1, \ldots, c_e)$ is the $e \times e$ diagonal of positive spring constants matrix. $A^T C\Lambda$ are forces at the vertices. $K = A^T CA$ is the stiffness matrix which is nonnegative definite with rank r.

Then the force balance is $A^T C\Lambda = F$ where F is the vector of tractions applied at the e vertices. It has a unique solution if infinitesimal flexes are eliminated by fixing $dn - r$ unknowns.

The equation for balanced forces may be solved for elongations or displacements.

$$A^T C\Lambda = F; \qquad\qquad A^T C\Lambda = F$$
$$B\Lambda = 0. \qquad\qquad \Lambda = Au$$

The analagous equations for linearized elastostatics are in terms of strains ϵ or infinitesimal displacements u are

$$\operatorname{div}\cdot\mathbf{c}\cdot\epsilon = \rho f; \qquad\qquad \operatorname{div}\cdot\mathbf{c}\cdot\epsilon = \rho f$$
$$\nabla\times(\nabla\times\epsilon) = 0. \qquad\qquad \epsilon = \frac{1}{2}(\nabla^T u + \nabla u)$$

where ρ is mass density, f is an external body force and $\mathbf{c}(x)$ is the elasticity tensor.

We tested the whether the localized geometry of compatibility conditions helps to predict the propagation of damage in a grid structure. If some edges are broken in a hexagonal truss under loading, where do the largest (positive or negative) stresses occur? The most highly stressed edges occur next to the damaged edge as we expected.

If we consider a sequence of of trusses under the same loading, starting by breaking an arbitrary edge and then sequentially breaking the most stressed edges one after the other, we see that the damage propagates through the structure by growing from the initially damaged edge.

To eliminate rigid motions, we fixed vertices at the edge of the truss. The stiffness of each edge was initially 1. For simplicity, a damaged edge is not removed, but rather its stiffness is weakened to .001.

To visualize the bending, the computed deformations are added to the undeformed positions for depicting the behavior to loading. Note that this linearized effect is exaggerated and distorted, but gives an intuitive idea of the deformations.

The highest stresses that occurred in the truss were actually next to the pinned nodes at the boundary. Since we were interested in the vicinity of damaged edges, we ignored the stresses near these pinned nodes and removed edges near the damage.

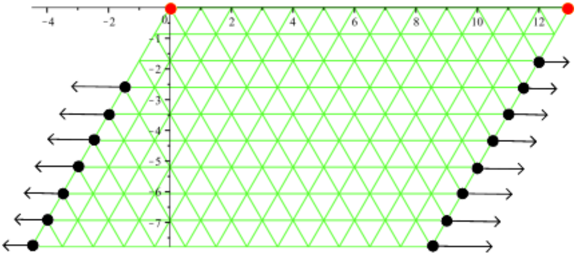


Figure: Pinned Verices and Applied Tractions.

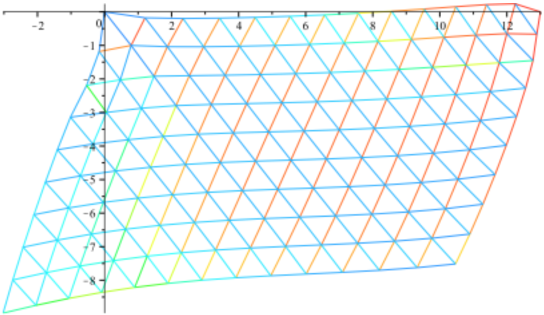


Figure: Deformation of Undamaged Truss. Blue=Compression, Red=Tension

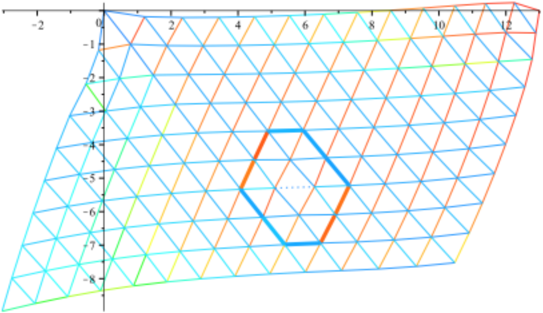


Figure: Artificially Break an Edge. Neighboring Region.

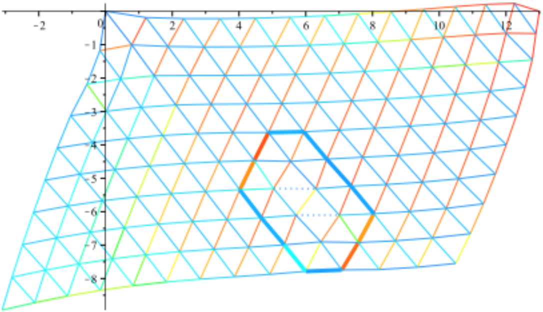


Figure: Resulting Deformation. Flag Most Stressed Neighbors.

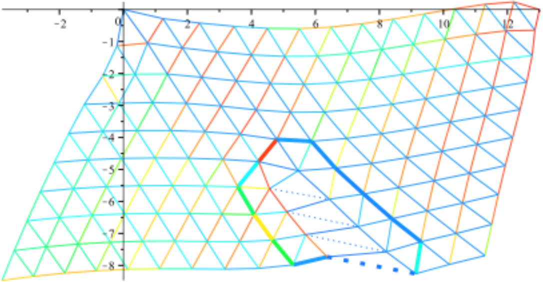


Figure: Break Them with Resulting Deformation. Flag Most Stressed Neighbors.

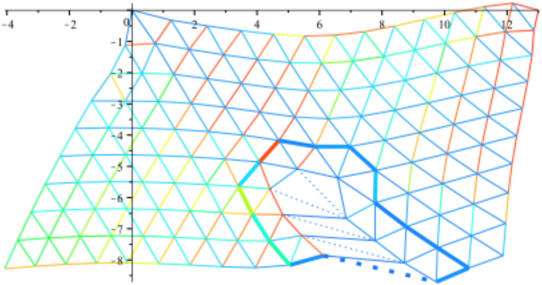


Figure: Break Them with Resulting Deformation. Flag Most Stressed Neighbors..

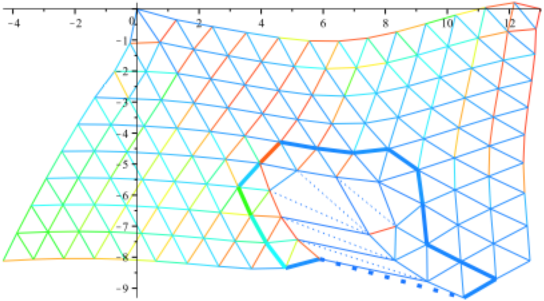


Figure: Break Them with Resulting Deformation. Flag Most Stressed Neighbors..

For a three dimensional "hexagonal" grid system, we try the vertices corresponding to a hexagonal close pack of three space by spheres. One of these is the face-centered cubic lattice

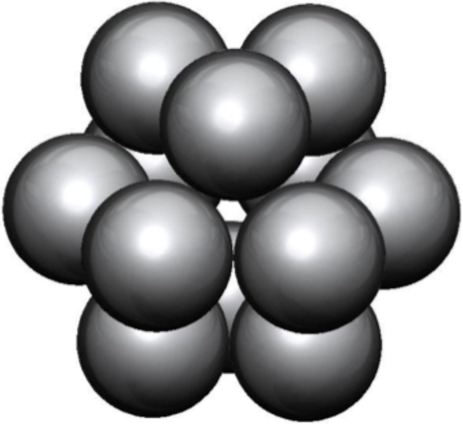


Figure: Face-Centered Cubic Closepack by Spheres.

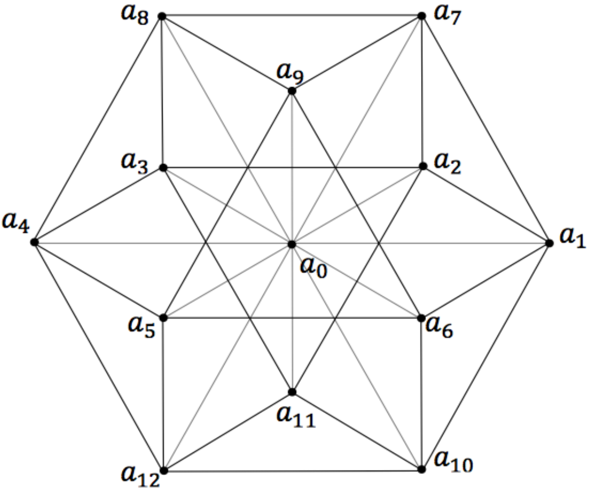


Figure: Connecting Centers of the Closest Neighbors Gives a Cuboctahedron.

There are four compatibility conditions for a cuboctahedron.

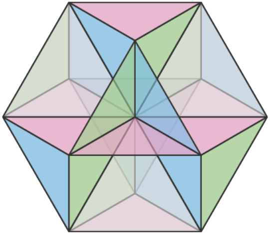


Figure: There is a Wagon-Wheel Condition for Each Planar Hexagon.

This is surprising because there are “three” compatibility conditions for (LC) in $\mathbf{R}^3$. Indeed perturbing locations of the vertices of the cuboctahedron resulted in three compatibility conditions on the perturbed truss.

One gets a three dimensional spherical analog of the wagon wheel condition by adding all four hexagons $2\sum_{i=1}^{12}\dot{\alpha}_i = \sum_{j=1}^{24}\dot{\beta}_j$.

But the regular cubeoctahedron turns out not to be infinitesimally rigid. It allows an infinitesimal flex which twists each of the faces.

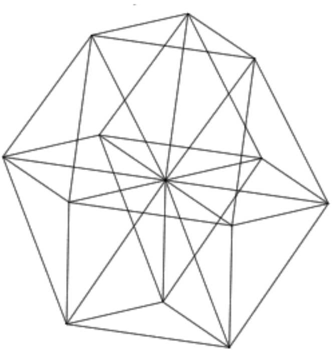


Figure: Cuboctahedron is Infinitesimally Flexible. (Run animated GIF.)

So the matrix in $Ax = \Lambda$ has decreased rank requiring an extra compatibility condition on Λ.

Thanks!