Notes from a talk by: Dr. Andrej Cherkaev on March 9th, 2010
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Solving Problems Without Solutions

Part 1 Notes by Charles Cox

One might first ask, "Why study applied math?" As an answer, the speaker suggests:

Applied math relates to many different and otherwise independent research topics. It deals with "real" and socially demanding problems for which solutions have lasting significance. It relates most readily to the fields of: physics, biology, astronomy, material science, geo-science, informatics, and finance. Regardless of the broad area to which math can be applied, the methods of application themselves are effectively unified.

There is great intellectual freedom within applied mathematics. There are no *a priori* rules and frames and the methods are diverse. We can choose to apply methods of: differential and integral equations, statistics, modeling, differential geometry, scientific computing, analysis, and others to tackle our chosen problems.

There are novel and diverse methods constantly emerging and being tested. Some of these are: game theory, pseudo-differential operators, nonconvex variational problems, computational methods, and optimization theory.

Most importantly, there are problems which, as stated, do not have solutions. Regardless, the applied mathematician must find some meaningful way to provide an answer.

Part 2

It is important, then, to ask the right kind of question so that an answer can be obtained. Examples of such questions are:

1) Why does wood in the trunks of some trees grow in a spiral pattern?

2) How can one see through a wall without destroying it?

3) What is the shape of the optimal dome with respect to certain definitions of optimality?

4) What properties will ensure that an automobile bumper absorbs a maximal amount of energy without being damaged?

5) How does one model photovoltaic processes theoretically?

6) Can the effects of climate change be predicted and modeled?

7) How does one design a band-gap cable to transport signals without aberrations?

Part 3

We look first at how to describe the spiraling pattern of woods in certain trees' trunks. The problem is solved by developing a theory of stresses in helicoidal cylinders that is now also used for cables. In short, spiraling provides an advantage for the tree in terms of strength to resist damage from high winds, heavy snows, or irregular rooting. It also increases the distance water must travel through the tree for photosynthesis. We expect that an optimal angle of spiraling exists and should be selected for in the evolution of the tree. Through the theory, as developed using the methods of applied mathematics, such an optimum angle can be deduced to meet certain criterion and can be shown to be consistent with natural data.

Part 4

We now look at the problem of the optimal hole in an elastic plane. If one puts a single hole in a membrane which has uniform load at infinite distance from the hole, it is trivial to see that the optimal shape of the hole is a circle. For uneven loads the obvious adjustment is to make it an ellipse with its small axis parallel to the direction of minimal loading. A crack-like or "line" hole is optimal is the load is applied on only one direction. In the case of multiple holes we can use techniques of complex analysis to map the many holes to a single-hole problem which can be solved. A problem arises, however, when one considers loads of differing signs.

What if the membrane is stretched in one direction but compressed in the other? Conditions for an optimal shape require piece-wise constant tension that keeps its absolute value constant everywhere on the boundary and jumps in certain points of the contour. Of course, this means that the optimal shape has edges of some kind. They can be found to be like slightly deformed rectangles such that each edge has a slight curvature and therefore a slightly larger than right-angle joint with its neighboring edge.

Part 5

We now look at some classical logical paradoxes to emphasize certain themes in applied mathematics.

What is the largest natural number? For the largest natural number N, we know it must satisfy, for all $n \in \mathbb{N}$, that $n^2 \geq n$. But N is the largest so $N^2 = N$ which implies that $N = 1$. This is clearly a problem. We've shown that the largest natural number is also the smallest. The problem is that we assumed that such an element exists. A way around this problem is to consider either regularization of the problem, or relaxation of its constraints. To regularize the problem would be to consider only integers less than some chosen maximum. To relax the constraints we could consider, perhaps, transfinite numbers. Either method removes the apparent paradox and contradiction.

Another illuminating example is the problem to find a function $f(x)$ with the constraint that $\int_0^{\pi} f(x)\,\mathrm{d}x = 1$ such that $\int_0^{\pi} f(x)\sin(x)\ \mathrm{d}x$ is maximal. Clearly $\int_0^{\pi} f(x)\sin(x)\ \mathrm{d}x \leq 1$ but how do we determine which function maximizes the integral? In fact such a problem has no solution unless we either regularize the problem by requiring that $f(x) \leq M$ for all x or relaxing the constraints to include distributions such as the delta function.

The problem of differentiating $|x|$ leads to a similar set of possible solutions. We either smooth the function around $x = 0$ or we introduce the idea of a subderivative at that point.

Each of these paradoxes/problems was addressed by either regularization or relaxation. Generally speaking the process of regularization is to add constraints to the problem to prevent it form reaching some undesirable limiting case. The problem with this approach is that the new solution will almost always depend heavily on the chosen regularization and such a constraint may be rather artificial. Relaxation effectively increases the generality of the potential solution to the problem, usually making the solution much more complex. Successive relaxation may even lead to new fields of study for an otherwise simplistic problem.

One can now use these principles to solve the following problem: What surface, connecting two parallel circles in $\mathbb{R}^3$, will have a minimal surface area? Again, the problem can be approached by either requiring that this surface has some minimal radius along its axis, or allowing limiting cases.

End of lecture